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CONCERNING $\int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2}\, dx_1 \cdots dx_k$ AND A TAYLOR SERIES METHOD*

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Abstract. The integral of the title equals the mean distance m_k from the origin of a point uniformly distributed over the k-dimensional unit hypercube I^k. Closed form expressions are given for $k = 1, 2$ and 3, while for general k, $m_k \simeq (k/3)^{1/2}$. Using *interalia* methods from geometry, Cauchy–Schwarz inequalities and Taylor series expansions, several inequalities and an asymptotic series for m_k are established. The Taylor series method also yields a slowly convergent infinite series for m_k and can be applied to more general problems including the mean distance between two points independently distributed at random in I^k.

1. Introduction. This note arises from wanting an expression for the mean distance from the origin,

$$(1.1)\qquad m_k = E\left(\sum_{i=1}^{k} X_i^2\right)^{1/2} = \int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2}\, dx_1 \cdots dx_k$$

of a point $(X_1, \cdots, X_k)$ uniformly distributed in the unit hypercube I^k in R^k (see, for example, Anderssen and Bloomfield (1975)). We were not able to find any reference to the evaluation of m_k for which we have found the following closed form expressions for $k = 1, 2$ and 3 by the method of our concluding section:

$$(1.2)\qquad \begin{aligned} m_1 &= .5,\\ m_2 &= \frac{2}{3}\int_0^1 (1 + r^2)^{1/2}\, dr = \{2^{1/2} + \log(1 + 2^{1/2})\}/3,\\ m_3 &= \frac{1}{2}\int_0^{\pi/4} [\{1 + \sec^2\theta\}^{3/2} - 1]\, d\theta\\ &= 3^{1/2}/4 + \log\{(1 + 3^{1/2})/2^{1/2}\} - \pi/24. \end{aligned}$$

In the sequel, we start by outlining some simple limits and inequalities for m_k and detail a method for computing m_k exactly, giving numerical results in Table 1 for $k \leqq 10$. An asymptotic series for m_k is given which adequately supplements Table 1. This series is illustrated together with asymptotically tight upper and lower bounds.

The method used for these more detailed results stems from Taylor series and, as such, it is one of the few explicit studies of finding the expectation of a random variable Y from the moments of the random variable X underlying the definition of Y as some function $Y = f(X)$. The method can be applied immediately to studying certain other fractional moments of $X_1^2 + \cdots + X_k^2$, or to studying, for example, the mean distance M_k between two points distributed uniformly and independently in the unit hypercube: from the geometric probability viewpoint, this latter problem is a natural companion to finding m_k.

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TABLE 1

k	m_k	M_k
1	.5	.33333
2	.76519572	.52141
3	.96059196	.66167
4	1.12189962	.77766
5	1.26240664	.87853
6	1.38857409	.96895
7	1.50408610	1.05159
8	1.61127356	1.12817
9	1.71172160	1.19985
10	1.80656663	1.26748

2. Simple results concerning m_k. Throughout, $X_1, \cdots, X_k$ will denote random variables (r.v.'s) that are independently and identically distributed (i.i.d.) like the r.v. X which has mean $\mu = EX$ and variance $\sigma^2 = \operatorname{var} X$. Define

$$Y_k^2 = X_1^2 + \cdots + X_k^2, \qquad Y_k \geqq 0. \tag{2.1}$$

In this section, assume X to have the rectangular distribution on (0, 1), so that $\mu = \frac{1}{2}$, $\sigma^2 = \frac{1}{12}$ and $EX^2 = \frac{1}{3}$. Then $m_k = EY_k$.

LEMMA 1.

$$(k/4)^{1/2} \leqq m_k \leqq (k/3)^{1/2}.$$

Proof. The Cauchy–Schwarz inequality proves that $EY_k \leqq (EY_k^2)^{1/2} = (k/3)^{1/2}$. For the other inequality, write $D_0 = Y_k$, and $D_1 = (\sum_{i=1}^k (1 - X_i)^2)^{1/2}$ for the distances of $(X_1, \cdots, X_k)$ from the origin and from the vertex of the hypercube opposite the origin, respectively. Then D_0 and D_1 are r.v.'s that have the same distribution, and $D_0 + D_1 \geqq k^{1/2}$ by the triangular inequality; hence, $m_k = ED_0 \geqq k^{1/2}/2$.

Observe that the Cauchy–Schwarz inequality can be applied to $D_0 + D_1$, for which $ED_0^2D_1^2 = k/30 + k(k-1)/9$. Then

$$2m_k \leqq (2ED_0^2 + 2ED_0D_1)^{1/2} \leqq (2k/3 + 2(ED_0^2D_1^2)^{1/2})^{1/2},$$

whence the upper bound, (tighter than in Lemma 1), in the next lemma.

LEMMA 2.

$$m_k \leqq (k/3)^{1/2}[\{1 + (1 - 7/10k)^{1/2}\}/2]^{1/2}.$$

The method used to derive the lower bound in Lemma 1 can be used to tighten that inequality to yield (for example) Lemma 3.

LEMMA 3.

$$m_k \geqq 2^{-k} \sum_{j=0}^{k} \binom{k}{j} (k + 3j)^{1/2}/3.$$

Proof. Divide I^k into 2^k equal hypercubes, and consider the hypercube containing $(X_1, \cdots, X_k)$. $(X_1, \cdots, X_k)$ is uniformly distributed within this hypercube, so its mean distance from any vertex is $m_k/2$. The vertex of the hypercube furthest from the origin is at distance $d_j \equiv (k + 3j)^{1/2}/2$ for some $j = 0, \cdots, k$, and there are $\binom{k}{j}$ hypercubes having this distance d_j. Then by the triangular inequality,

$$m_k \geqq 2^{-k} \sum_{j=0}^{k} \binom{k}{j} \{(k + 3j)^{1/2}/2 - m_k/2\},$$

from which the asserted inequality follows.

Asymptotically, this lower bound $\sim (5k/18)^{1/2}$.

The method can be generalized by dividing I^k into p^k equal hypercubes for some larger integer p, leading to a $(p - 1)$-fold summation involving a multinomial coefficient in place of $\binom{k}{j}$. This gives a bound that is asymptotically $(k/3)^{1/2}\{(p + \frac{1}{2})/(p + 1)\}^{1/2}$. The method can also yield upper bounds, but none of them is asymptotically tight as are the upper bounds in Lemmas 1 and 2.

Another lower bound,

$$m_k \geqq 2^{-k} \sum_{j=0}^{k} \binom{k}{j} (k + 8j)^{1/2}/4, \tag{2.2}$$

can be proved via the fact that the median of a triangle is shorter than the average length of the two adjacent sides, and hence that the mean distance from the origin of a point uniformly distributed at random over a set which excludes the origin and possesses a center distant d_C from the origin, is at least as large as d_C. This lower bound $\sim (5k/16)^{1/2}$ for large k, and it is always tighter than that of Lemma 3. It is illustrated in Table 2.

TABLE 2

Errors $(m_{\text{approx}} - m_k)$ *in approximations to* m_k

k	Asymptotic (up to k^{-3})	(2.2)	$(4.5)_{L,3}$	$(4.5)_{U,3}$
1	−.02802	.0	−.08745	.01646
2	−.00268	−.01636	−.02512	.00612
3	−.00042	−.02451	−.01685	.00289
4	$-.0_468$	−.03041	−.01085	.00157
5	$-.0_52$	−.03535	−.00727	.00095
6	$.0_59$	−.03971	−.00510	.00061
7	$.0_59$	−.04367	−.00373	.00042
8	$.0_58$	−.04731	−.00281	.00030
9	$.0_56$	−.05070	−.00218	.00023
10	$.0_54$	−.05389	−.00173	.00018
15	$.0_512$	−.06765	−.00069	$.0_464$
20	$.0_64$	−.07908	−.00035	$.0_431$
large	$.018k^{-7/2}$	$-.018k^{1/2}$	$-.72k^{-5/2}$	$.055k^{-5/2}$

Observe that the methods underlying both (2.2) and Lemma 3 amount to approximating quadratures.

Yet another lower bound on m_k is obtained by finding the mean distance from the origin of a point uniformly distributed in the orthant of unit volume of a hypersphere in R^k. This leads to

$$m_k \geqq (2/\pi^{1/2})\{\Gamma(1 + k/2)\}^{1/k}/(1 + k^{-1}), \tag{2.3}$$

but since the right-hand side here $\sim\{k/(\pi e/2)\}^{1/2} = (k/4.27)^{1/2}$, this bound can be of use only for small values of k. In fact, it is sharper than the bounds of Lemma 1 for $k \leqq 45$, of Lemma 3 for $k \leqq 9$, and of (2.2) only for $k = 2$.

Regarding $X_1, X_2, \cdots$ as an infinite sequence of i.i.d.r.v.'s, we have the next lemma.

LEMMA 4. $0 \leqq 3Y_k/k \leqq 3$ *and* $3Y_k^2/k \to 1$ $(k \to \infty)$ *with probability one.*

Proof. The boundedness of $3Y_k/k$ is obvious, and the convergence follows from the strong law of large numbers applied to the sample mean $(X_1^2 + \cdots + X_k^2)/k$ of i.i.d.r.v.'s.

It follows from Lemma 4 that

$$m_k/(k/3)^{1/2} \to 1, \qquad k \to \infty. \tag{2.4}$$

3. Exact results for m_k. Lemma 4 and (2.4) suggest writing

$$m_k = (k/3)^{1/2}E(1 + Z_k)^{1/2}, \tag{3.1}$$

where $Z_k = \{\sum_{i=1}^k (X_i^2 - 1/3\}/\{k/3\}$, for which $EZ_k = 0$, and $-1 \leqq Z_k \leqq 2$. A direct Taylor series expansion of $(1 + z)^{1/2}$ is of course valid only for $|z| \leqq 1$, so taking expectations in such an expression cannot be justified (and, indeed, it does not yield any exact result), though it will be seen in § 4 that it yields asymptotic formulae for m_k.

Instead, we write for any fixed finite $k = 1, 2, \cdots$ and any $\alpha > 0$,

$$Y_k = (\alpha k)^{1/2}(1 + \zeta_\alpha)^{1/2}, \tag{3.2}$$

$$\zeta_\alpha \equiv \left\{\sum_{i=1}^k (\alpha^{-1}X_i^2 - 1)\right\}\Big/k. \tag{3.3}$$

Then $-1 \leqq \zeta_\alpha \leqq \alpha^{-1} - 1$, so for all $\alpha \geqq \frac{1}{2}$, the binomial expansion of $(1 + \zeta_\alpha)^{1/2}$ yields a power series that is uniformly absolutely convergent in ζ_α (see around (4.1)). By taking expectations we thus obtain Lemma 5.

LEMMA 5. *For* $\alpha \geqq \frac{1}{2}$,

$$m_k = (\alpha k)^{1/2} \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} E\zeta_\alpha^r. \tag{3.4}$$

To compute the expectations used in (3.4), recall that ζ_α is the mean of the sum of k i.i.d. bounded r.v.'s,

$$\zeta_\alpha \equiv \zeta_{\alpha,k} = S_k/k = (U_1 + \cdots + U_k)/k,$$

say, so writing $u_j = EU_1^j$, we have

$$E\zeta_{\alpha,k+1}^r = \sum_{j=0}^r \binom{r}{j} E\zeta_{\alpha,k}^{r-j}\{k/(k+1)\}^{r-j}u_j/(k+1)^j. \tag{3.5}$$

Integration by parts yields the reduction formula for $r = 1, 2, \cdots$,

$$u_r = E\{(X_1^2 - \alpha)/\alpha\}^r = \int_0^1 (\alpha^{-1}x^2 - 1)^r \, dx$$
$$= \{(\alpha^{-1} - 1)^r - 2ru_{r-1}\}/(2r + 1),$$

while $u_0 = 1$. Note that while the function $k^r E\zeta_{\alpha,k}^r/r! = ES_k^r/r!$ leads to a convolution free of a multiplier (cf. (3.5)), the slow convergence of (3.4) (see below) necessitates the use of a large number of terms in (3.4) which if computed otherwise than as above can lead to errors from overflow and underflow.

An alternative approach giving ES_k^r in terms of k and EU_1^s $(s = 1, \cdots, r)$ is also possible, but the formulae become very complex as r and k increase: the coefficients for $r = 2(1)12$ can be found in Table 1.1.r of David et al. (1966).

For $\alpha \geqq \frac{1}{2}$, the equation

$$(3.6) \qquad E\zeta_{\alpha,k}^r = \int_{-1}^{\alpha^{-1}-1} u^r f_k(u) \, du,$$

where $f_k(\cdot)$ is the probability density function of $\zeta_{\alpha,k}$, can be used to study the moment further. First, $f_k(u)$ has its density concentrated around $\mu = \alpha^{-1}/3 - 1$ by the central limit theorem. Next, it can be shown that for $-1 < u < (\alpha k)^{-1} - 1$, $f_k(u)$ equals

$$k(\alpha k\pi/4)^{k/2}(1 + u)^{(k-2)/2}/\{2\Gamma(1 + k/2)\}.$$

So for given (large) r, the contribution to $E\zeta_{\alpha,k}^r$ from the density near -1 increases with increasing α and decreases with increasing k. On these grounds, then, we may anticipate that (3.4) is best when α is least ($\alpha = \frac{1}{2}$) and worst for small k: this was in fact found to be the case when using (3.4) with $\alpha = 1, \frac{1}{2}, \frac{1}{3}$ (for this last value the series diverges, obviously!), and $k = 2(1)10$. It should be noted that since for $k \geqq 2$, $f_k(u)$ is continuous and uniformly bounded within $(-1, \alpha^{-1} - 1)$, (3.6) shows that $E\zeta_{\alpha,k}^r$ is about $O(r^{-1})$, and since $\binom{\frac{1}{2}}{r} = O(r^{-3/2})$, convergence of the series in (3.4) is ultimately about that of $\sum r^{-5/2}$.

The entries for m_k in Table 1 were computed by summing the series at (3.4) via (3.5) using about 190 terms, working to 11 digits, and extrapolating from the sum to n terms $\equiv m_k(n) \simeq m_k(\infty) + an^{-3/2} + bn^{-5/2}$ where necessary.

4. Asymptotic bounds for m_k. The argument below may partly obscure the simple intuitive idea underlying the method, namely, that we should bound the function $(1 + z)^{1/2}$ over the range of z-values of interest by a polynomial of smallish degree, and then get bounds in m_k by estimating $E(1 + Z_k)^{1/2}$. Approximations to $EY = Ef(X)$ via Taylor series expansions of f about the mean $\mu = EX$ are familiar (see, e.g., Anscombe (1948), Kendall and Stuart (1963, p. 232)): precise arguments are not so common.

Write $(1 + z)^{1/2}$ as a Taylor series expansion with the Lagrange integral expression for the remainder, namely,

$$(1 + z)^{1/2} = \sum_{r=0}^{n-1} \binom{\frac{1}{2}}{r} z^r + \binom{\frac{1}{2}}{n} z^n \int_0^1 n(1 - u)^{n-1}(1 + zu)^{-n+1/2} \, du.$$

For positive even integers n, the remainder term here, which we write as $\binom{\frac{1}{2}}{n} z^n R_n(z)$, say, is of constant sign for all $z > -1$, so defining for $-1 \leqq \xi < \eta$

$$
\begin{aligned}
&c_n(\xi, \eta) = \inf_{\xi \leqq z \leqq \eta} R_n(z), \qquad C_n(\xi, \eta) = \sup_{\xi \leqq x \leqq \eta} R_n(z), \\
&\binom{\frac{1}{2}}{2n} z^{2n} C_{2n}(\xi, \eta) \geqq (1 + z)^{1/2} - \sum_{r=0}^{2n-1} \binom{\frac{1}{2}}{r} z^r \geqq \binom{\frac{1}{2}}{2n} z^{2n} c_{2n}(\xi, \eta), \\
&\qquad \xi \leqq z \leqq \eta.
\end{aligned} \tag{4.1}
$$

Putting $\xi = -1$ and $\eta = \alpha^{-1} - 1$ for any $\alpha \geqq \frac{1}{2}$, we obtain more detail for a proof of Lemma 5.

The nonnegativity and monotonic behavior of $R_n(z)$ enable the bounds just given to be evaluated as $c_n(\xi, \eta) = R_n(\eta)$, $C_n(\xi, \eta) = R_n(\xi)$. Putting $z = Z_k \equiv \zeta_{1/3}$, $\alpha = \frac{1}{3}$, $\xi = -1$, $\eta = 2$, we find on taking expectations that

$$
\begin{aligned}
\binom{\frac{1}{2}}{2n} EZ_k^{2n} R_{2n}(-1) &\geqq m_k(3/k)^{1/2} - \sum_{r=0}^{2n-1} \binom{\frac{1}{2}}{r} EZ_k^r \\
&\geqq \binom{\frac{1}{2}}{2n} E(Z_k^{2n}) R_{2n}(2).
\end{aligned} \tag{4.2}
$$

For fixed n, the fact that α is so chosen that $E\zeta_{1/3} = EZ_k = 0$ shows that

$$EZ_k^{2n} = O(k^{-n}) = EZ_k^{2n-1}, \qquad k \to \infty.$$

Similarly, since $E|Z_k^{2n+1}| \leqq (E|Z_k^{4n+2}|)^{1/2} = O(k^{-n-(1/2)})$, we obtain Lemma 6.

LEMMA 6. *The series*

$$m_k = (k/3)^{1/2} \sum_{r=0}^{n} \binom{\frac{1}{2}}{r} EZ_k^r + O(k^{-n/2}) \tag{4.3}$$

is an asymptotic expansion ($k \to \infty$) *of* m_k.

Let it be emphasized that for any α other than $\alpha = \frac{1}{3}$, the error in the analogue of (4.3) for any given finite n is bounded away from 0 as $k \to \infty$. Evaluation of (4.3) leads to the asymptotic expansion

$$
\begin{aligned}
m_k = (k/3)^{1/2}(1 - 1/10k - 13/280k^2 - 101/2800k^3 - 37533/1232000k^4) \\
+ O(k^{-9/2}).
\end{aligned} \tag{4.4}
$$

Computation reveals that using (4.4) with terms up to k^{-3} yields 6 correct digits for $k \geqq 10$ (some errors for this approximation are in Table 2), and with terms up to k^{-4}, one has 6 digits for $k \geqq 9$ and 7 digits for $k \geqq 15$.

Lemma 6 leaves open the question of closeness of fit of the asymptotic series. While the second inequality in (4.2) yields an upper bound on m_k, the following lemma can be applied to give a lower bound on m_k better than that obtained from the first inequality in (4.2).

LEMMA 7. *Let the function f have a $(2n+2)$ order derivative $f^{(2n+2)}(x) \leqq 0$ for $\xi \leqq x \leqq \eta$, where $\xi < 0 < \eta$, $f^{(2n+2)}$ integrable on (ξ, η). Then the function*

$$S_{2n}(x; a, b) \equiv \sum_{r=0}^{2n-1} (x^r/r!) f^{(r)}(0) + \{x^{2n}/(2n)!\}(a + bx),$$

where a and b are defined as the solution of

$$S_{2n}(\xi; a, b) = f(\xi), \qquad S_{2n}(\eta; a, b) = f(\eta),$$

satisfies

$$S_{2n}(x; a, b) \leqq f(x), \qquad \xi \leqq x \leqq \eta,$$

with equality at ξ, η, and 0 ($2n$ times).

Proof. Using the general form of the Taylor series expansion,

$$\begin{aligned}\{f(x) - S_{2n}(x; a, b)\}/\{x^{2n}/(2n)!\} \\ &= 2n \int_0^1 (1-u)^{2n-1} f^{(2n)}(xu)\, du - a - bx \\ &\equiv A_{2n}(x).\end{aligned}$$

By definition, $A_{2n}(\xi) = A_{2n}(\eta) = 0$, and $A_{2n}(0)$ is defined by continuity. Since $A''_{2n}(x) \leqq 0$ $(\xi \leqq x \leqq \eta)$, the conclusion follows by a convexity argument.

We apply the lemma to $(1 + Z_k)^{1/2}$, with $\xi = -1$, $\eta = 2$, and obtain Lemma 8.

LEMMA 8. *Let A_{2n}, B_{2n} be the solution of*

$$\sum_{r=0}^{2n-1} \binom{\frac{1}{2}}{r} (-1)^r + A_{2n} - B_{2n} = 0,$$

$$\sum_{r=0}^{2n-1} \binom{\frac{1}{2}}{r} 2^r + (A_{2n} + 2B_{2n}) 2^{2n} = 3^{1/2}.$$

Then

$$\begin{aligned} A_{2n}EZ_k^{2n} + B_{2n}EZ_k^{2n+1} &\leqq m_k(3/k)^{1/2} - \sum_{r=0}^{2n-1} \binom{\frac{1}{2}}{r} EZ_k^r \\ &\leqq (A_{2n} + 2B_{2n})EZ_k^{2n}. \end{aligned} \tag{4.5}$$

Computation of these lower and upper bounds for $n = 1, 2, 3$ shows that, except for the lower bounds with $k = 1$ and $n = 2, 3$, the bounds for $n = 3$ are closer than for $n = 2$, which are closer than for $n = 1$, and that the upper bound is much closer to m_k than is the lower bound. The lower and upper bounds for $n = 3$ are illustrated in Table 2 as $(4.5)_{L,3}$ and $(4.5)_{U,3}$, respectively. For values of $k \geqq 3$, these bounds are better than any of those of § 2.

5. Other applications: Mean distance between two points in I^k. Some of the methods of the preceding sections readily carry over to the computation of $E(\sum X_i^2)^\gamma$ for certain other γ. The binomial theorem method is simple to use, and the appropriate $(\gamma \neq \frac{1}{2})$ analogues of the triangular and Cauchy–Schwarz inequalities may be considered for further inequalities.

Taking $X = (X_1, \cdots, X_k)$ and $Y = (Y_1, \cdots, Y_k)$ to be two points distributed at random in I^k, it is also easy to see how the Taylor series methods of §§ 3 and 4 apply to finding

$$ED_{XY} \equiv E|X - Y| = M_k. \tag{5.1}$$

Rather more work is needed for the analogues of Lemma 2 and the rest of Lemma 1. Observe first that D_{XY} has the same distribution as $D_{X'Y'}$, where X', Y' are determined from X, Y by means of

$$\begin{aligned} X' &= (\min(X_1, Y_1), \cdots, \min(X_k, Y_k)), \\ Y' &= (\max(X_1, Y_1), \cdots, \max(X_k, Y_k)), \end{aligned} \tag{5.2}$$

and that then $D_{0X'} = (\sum (X_i'^2)^{1/2}$, $D_{X'Y'}$, $D_{Y'1} = (\sum (1 - Y_i')^2)$ are three exchangeable random variables. Since $D_{0X'} + D_{X'Y'} + D_{Y'1} \geqq k^{1/2}$, we obtain

$$M_k \geqq k^{1/2}/3 \tag{5.3}$$

(this inequality is sharp for $k = 1$). Of course, the Cauchy–Schwarz inequality gives

$$M_k \leqq (k/6)^{1/2} = (ED_{XY}^2)^{1/2}, \tag{5.4}$$

and since

$$ED_{0X'}^2 D_{Y'1}^2 = k/90 + k(k - 1)/36 = (k/6)^2(1 - 3/5k),$$

the analogue of Lemma 2, giving a bound sharper than (5.4), is

$$M_k \leqq (k/6)^{1/2}[\{1 + 2(1 - 3/5k)^{1/2}\}/3]^{1/2}. \tag{5.5}$$

Explicitly, we can find

$$M_1 = 1/3, \qquad M_2 = 4\int_0^1 dx \int_0^1 (1 - x)(1 - y)(x^2 + y^2)^{1/2}\, dy$$
$$= \{(2^{1/2} + 2)/5 + \log(1 + 2^{1/2})\}/3 = .521405433 \cdots.$$

Table 1 gives M_k for $k = 1(1)10$.

6. Taylor series approximations. The work of §§ 3 and 4, in giving an example of the computation of the mean $EY = Ef(X)$ of a function of a r.v. X, shows that while expansion of $f(\cdot)$ about $\mu = EX$ may not be appropriate for giving EY exactly, a workable approximation may nevertheless be still obtainable. When some derivatives of f are of constant sign, such expansion about μ may yield bounds sharper than expanding about some other point. The approximation

$$EY \simeq f(\mu) + \sigma^2 f''(\mu) \quad \text{with } Y = (\textstyle\sum X_i^2)^{1/2}, \quad f(u) = u^{1/2},$$

yields

$$m_k \simeq (k/3)^{1/2}(1 - 1/10k)$$

which, having an error that is $O(k^{-3/2})$, is asymptotically sharper than the upper bound from (4.5) with $n = 1$, but not with $n = 2$.

7. Further results concerning m_k. There remains a method for computing m_k that does not have an analogue for M_k. It depends on the fact that m_k equals the mean distance from the origin of a point distributed at random in a hyperpyramid with vertex at the origin and base one of the faces of I^k with $(1, \cdots, 1)$ as a vertex. Then

$$m_k = E(1 + X_1^2 + \cdots + X_{k-1}^2)^{1/2}/(1 + k^{-1}). \tag{7.1}$$

It is left as an exercise to verify that the following are the respective analogues of Lemma 1, Lemma 2, and (2.2):

$$m_k \leqq \{(k + 2)/3\}^{1/2}/(1 + k^{-1}), \tag{7.2}$$

$$\begin{aligned} k^{-1/2} &\leqq (1 + k^{-1})m_k + m_{k-1} \\ &\leqq (k/3)^{1/2}\{2 + k^{-1} + 2(1 + 3/10k - 13/10k^2)^{1/2}\}^{1/2}, \end{aligned} \tag{7.3}$$

$$m_k \geqq 2^{-k+1} \sum_{j=0}^{k-1} \binom{k-1}{j} (15 + k + 8j)^{1/2}/4(1 + k^{-1}). \tag{7.4}$$

Of rather more importance is the relation

$$\begin{aligned} m_k\{2(k + 1)\}^{1/2}/k &= E\left\{1 + \sum_{i=1}^{k-1} (2X_i^2 - 1)/(k + 1)\right\}^{1/2} \\ &= \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} \{(k - 1)/(k + 1)\}^r E\zeta^r_{.5,k-1} \end{aligned} \tag{7.5}$$

which, being an infinite series converging geometrically fast, is better than (3.4) with $\alpha = .5$, especially for smaller values of k. The entries for m_k in Table 1 have been verified by this method which required only 20 to 30 terms to give convergence in the ninth digit.

REFERENCES

R. S. ANDERSSEN AND P. BLOOMFIELD (1975), *Properties of the random search in global optimization*, J. Optimization Theory Appl., 16, No. 5-6, September, in press.

F. J. ANSCOMBE (1948), *The transformation of Poisson, binomial, and negative binomial data*, Biometrika, 35, pp. 246–254.

F. N. DAVID, M. G. KENDALL AND D. E. BARTON (1966), *Symmetric Functions and Allied Tables*, Cambridge University Press, Cambridge.

M. G. KENDALL AND A. STUART (1963), *The Advanced Theory of Statistics*, vol. I, Griffin, London.