THE FIRST 200,000,001 ZEROS OF RIEMANN's ZETA FUNCTION

by

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1. INTRODUCTION

This paper contains a description of extensive computations carried out by Brent at the Department of Computer Science of the Australian National University (Canberra) and by van de Lune, te Riele and Winter at the Mathematical Centre (Amsterdam, The Netherlands). The main results will appear in 1982 in Mathematics of Computation. The details of the computations by van de Lune, te Riele and Winter have been described in the Mathematical Centre Report NW 113/81 [12].

Riemann's zeta function is the meromorphic function $\zeta\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$, which, for $\mathrm{Re}(s) > 1$, may be represented explicitly by

$$\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}, \quad (s = \sigma + it).$$

It is well-known (see TITCHMARSH [17, Chapters II and X]) that

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s)$$

is an entire function of order 1, satisfying the functional equation

$$\xi(s) = \xi(1-s),$$

so that

$$\Xi(z) := \xi(\tfrac{1}{2} + iz), \quad (z \in \mathbb{C}),$$

being an even entire function of order 1, has an infinity of zeros. The Riemann Hypothesis is the statement that all zeros of $\Xi(z)$ are real, or, equivalently, that all non-real zeros of $\zeta(s)$ lie on the "critical" line $\sigma = \frac{1}{2}$. Since $\zeta(\bar{s}) = \overline{\zeta(s)}$ we may restrict ourselves to the half plane $t > 0$. To this day, Riemann's Hypothesis has neither been proved nor disproved.

Numerical investigations related to this unsolved problem were initiated by Riemann himself and later on continued more systematically by the writers listed below (including their progress).

Investigator	Year	The first n complex zeros of $\zeta(s)$ are simple and lie on $\sigma = \frac{1}{2}$
GRAM [6]	1903	$n = 15$
BACKLUND [1]	1914	$n = 79$
HUTCHINSON [7]	1925	$n = 138$
TITCHMARSH [16]	1935/6	$n = 1,041$

Those listed above utilized the Euler-Maclaurin summation formula and performed their computations by hand or desk calculator whereas those listed below applied the Riemann-Siegel formula in conjunction with electronic computing devices.

LEHMER [10,11]	1956	$n = 25,000$
MELLER [13]	1958	$n = 35,337$
LEHMAN [9]	1966	$n = 250,000$
ROSSER, YOHE & SCHOENFELD [15]	1968	$n = 3,500,000$
BRENT [2]	1979	$n = 81,000,001$

An excellent explanatory account of most of these computations may be found in EDWARDS [4].

In this paper (which presupposes the knowledge of BRENT [2]) we report on extensive computations by which the first named author has extended his former result to $n = 156,800,001$ and by which the remaining three authors (LR & W, for short) have extended this bound to $n = 200,000,001$. Independently of Brent, LR & W have also checked the range $[g_{81,000,000}, g_{120,000,000})$.

In practice, the numerical verification of the Riemann hypothesis in a given range consists of separating the zeros of the well-known real function Z(t) (see formula (2.6) of BRENT [2] or formula (3.1) in Section 3 of this paper), or, equivalently, of finding sufficiently many sign changes of Z(t). Our programs (aiming at a fast separation of these zeros) are based, essentially, on the modification of LEHMER's [11] method introduced by ROSSER et al. [15]. However, LR & W have developed a more efficient strategy of searching for sign changes of Z(t) in Gram blocks of length $L \geq 2$. Brent's average number of Z-evaluations, needed to separate a zero from its predecessor, amounts to about 1.41 (compare BRENT [2]) whereas LR & W have brought this figure down to about 1.21. It may be noted here that in the most recent version of the program of LR & W this figure has been reduced further to about 1.185. From the statistics in Section 4 it follows that in the range $[g_{156,800,000}, g_{200,000,000})$ this average number of Z-evaluations could not have been reduced below 1.135 by any program which evaluated $Z(t)$ at all Gram points. We also note that about 98 percent of the running time of the LR & W - program was spent on evaluating $Z(t)$. This program was executed on a CDC CYBER 175 computer and ran about ten times as fast as the UNIVAC 1100/42 program of Brent. This is roughly what could be expected, given the relative speeds of the different machines.

2. THE STRATEGY FOR FINDING SUFFICIENTLY MANY SIGN CHANGES OF Z(t)

We recall some definitions. Let $\theta(t)$ be the real continuous function defined by

$$(2.1) \qquad \theta(t) = \arg[\pi^{-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\,it)], \quad \theta(0) = 0.$$

The j-th Gram point g_j is defined as the unique number satisfying $\theta(g_j) = j\pi$ ($j = -1,0,1,2,\ldots$). A Gram point g_j is called good if $(-1)^j Z(g_j) > 0$ and bad otherwise. A Gram block of length L (≥ 1) is an interval $B_j = [g_j, g_{j+L})$ such that g_j and g_{j+L} are good Gram points and $g_{j+1},\ldots,g_{j+L-1}$ are bad Gram points. An interval $[g_j, g_{j+1})$ is called a Gram interval. A Gram block B_j of length L is said to satisfy "Rosser's rule" if $Z(t)$ has at least L zeros in B_j.

The strategy of Brent for finding the required number of sign changes of $Z(t)$ is based on this rule. LR & W refined this strategy in order to reduce the number of Z-evaluations as much as they could. This will be described here in some detail.

In order to reduce the number of Z-evaluations as much as possible, we first observe that after having determined a Gram block B_j of length $L \geq 2$, we already have implicitly detected $L-2$ sign changes of $Z(t)$. Hence, the problem reduces to finding the "missing two" sign changes. Next we observe that these missing two (if existing) must both lie in one and the same Gram interval of the block B_j. Some preliminary experiments with the LR & W-program revealed that in the majority of cases the missing two are situated in one of the *outer* Gram intervals of B_j. Therefore, we first search in (g_j, g_{j+1})

or (g_{j+L-1}, g_{j+L}) according to which of $\text{abs}(Z(g_j) + Z(g_{j+1}))$ and $\text{abs}(Z(g_{j+L-1}) + Z(g_{j+L}))$ is the smallest. In the selected interval an efficient parabolic interpolation search routine is invoked. (Here is the main improvement over Brent's method, which used random search rather than parabolic interpolation.) If this routine terminates without having found the missing two sign changes, the other outer Gram interval of the block is treated in the same manner. In case the missing two are still not found, another search routine is called, depending on the length L of the block $B_j = [g_j, g_{j+L})$.

If $L = 2$, the interval (g_j, g_{j+2}) is scanned again, and if $L > 2$ we continue to search in the interval (g_{j+1}, g_{j+L-1}). In both cases, the search is performed by means of a refinement of a search routine described by LEHMAN [9]. For more details we refer the reader to the source text of the LR & W - program in [12].

If at some instant one of the search routines has detected the missing two, a new Gram block is set up and we continue as described above. In the opposite case (which occurs very rarely) the program prints a message and a "plot" of Z(t) corresponding to the whole Gram block under investigation and proceeds by pretending (!) that the missing two were found indeed. These plots of Z(t) were inspected afterwards (if necessary) "by hand". So far, the missing two were always easily found either in the Gram block under consideration or in an adjacent Gram block (compare BRENT [2, Section 4]).

After having covered the range $[g_{156,800,000}, g_{200,000,000})$ we ran the computation a little further, and found 4 Gram blocks in $[g_{200,000,000}, g_{200,000,004})$, all of them satisfying Rosser's rule. By applying Theorem 3.2 of BRENT [2] we completed the proof of our claim that the first $n = 200,000,001$ zeros of $\zeta(s)$ are simple and lie on $\sigma = \frac{1}{2}$.

3. COMPUTATION OF Z(t) AND ERROR ANALYSIS

3.0. Introduction

In principle, Brent and LR & W's methods of computing Z(t) and error analysis are exactly as described in Section 5 of BRENT [2]. We will only mention here some details of LR & W's computations and error analysis. The full details are given in [12].

The unambiguous determination of the sign of Z(t) requires a rigorous bound for the error, committed when one actually computes Z(t) on a computer.

In our program we actually used two methods (A and B) for evaluating $Z(t)$.

Method A is a very fast and efficient method which usually gives the correct sign of $Z(t)$.

Method B is a comparitively slow, but very accurate method which is invoked when $|Z(t)|$ is too small for method A.

We used the well-known Riemann-Siegel formula (with two correction terms in either case):

$$(3.1) \qquad Z(t) = 2\sum_{k=1}^{m} k^{-\frac{1}{2}}\cos[t.\ell n(k) - \theta(t)] + (-1)^{m-1}\tau^{-\frac{1}{4}}\sum_{j=0}^{1}\Phi_j(z)\tau^{-j/2} + R_1(t),$$

where $m = \lfloor\tau^{\frac{1}{2}}\rfloor$, $\tau = t/(2\pi)$, $z = 1 - 2(\tau^{\frac{1}{2}} - m)$,

$$(3.2) \qquad \theta(t) = \arg[\pi^{-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\,it)], \quad \theta(t) \text{ continuous and } \theta(0) = 0,$$

$$(3.3) \qquad \Phi_0(z) = \cos[\pi(4z^2+3)/8]/\cos(\pi z) =: \sum_{k=0}^{\infty} c_{2k}^{(0)} z^{2k}, \quad -1 < z \le 1,$$

$$(3.4) \qquad \Phi_1(z) = \frac{d^3}{dz^3}\,\Phi_0(z)/(12\pi^2) =: \sum_{k=0}^{\infty} c_{2k+1}^{(1)} z^{2k+1}.$$

The last term $R_1(t)$ will be dropped in our actual computation. GABCKE [5] and BRENT & SCHOENFELD [4] have given bounds on $R_n(t)$ (here, $n+1$ denotes the number of terms in the second sum in (3.1), hence $n = 1$ in our situation). We used the bound (GABCKE [5])

$$(3.5) \qquad |R_1(t)| < 0.053t^{-5/4} < 0.0054\tau^{-5/4}, \quad \text{for } t \ge 200.$$

The floating point machine approximations of Z by means of methods A and B will be denoted by $\tilde{Z}_A$ and $\tilde{Z}_B$, respectively. Throughout this section, the result of the floating point machine approximation of some quantity q will be denoted by $\tilde{q}$.

We present an error analysis which accounts for *all* possible errors in $\tilde{Z}$, for any t (resp. τ) in the range,

$$(3.6) \qquad 3.5\times 10^7 < t < 3.72\times 10^8 \quad (\text{resp. } 5.5\times 10^6 < \tau < 5.92\times 10^7).$$

This covers the range of zero #81,000,001 till zero #1,000,000,000 of $\zeta(s)$ in the critical strip, which we had originally planned to investigate ($\gamma_{81,000,001} \approx 35,018,261.166$, $\gamma_{1,000,000,000} \approx 371,870,203.837$).

The computations were carried out on a CDC CYBER 175 computer having a 60-bit word, and single-precision (SP) and double-precision (DP) floating-point arithmetic using 48- and 96- bit binary fractions, respectively. In the sequel we will frequently work with the unit roundoffs $\varepsilon_s := 2^{-47}$ and $\varepsilon_d := 2^{-95}$.

3.1. Computation of Z(t)

At the start of the program four tables are precomputed:

(i) $\ln(k)$ for $1 \le k \le m_0$ in DP, where m_0 is large enough to cover the range of the current job;

(ii) $k^{-\frac{1}{2}}$ for $1 \le k \le m_0$ in DP, truncated to SP;

(iii) $\cos(2\pi k.2^{-13})$ for $0 \le k \le 2^{13}+1$ in DP, truncated to SP;

(iv) $\cos(2\pi(k+1)2^{-13}) - \cos(2\pi k 2^{-13})$ for $0 \le k \le 2^{13}$ in DP, truncated to SP.

Methods A and B run essentially as follows.

Method A. Given a τ as a DP floating point number, $t = 2\pi\tau$ and $\theta(t)$ are computed in DP; $f^{(1)} := \mathrm{frac}\{\theta(t)(2\pi)^{-1}\}$ is computed in DP, and truncated to SP. Next, the main loop (corresponding to the first sum in (3.1)) is executed. This loop has been written in COMPASS (machine language of the CYBER) and optimized using the specific properties of the CYBER's central processing units. One cycle of the loop executes in about 2.1 μ sec. $f^{(2)} := \mathrm{frac}\{\tau \ln(k)\}$ (where $\ln(k)$ is looked up in the precomputed table) is computed as follows: the DP product of τ and $\ln(k)$ is decreased with the integer part of the SP product of τ and $\ln(k)$ and the result is truncated to SP. This programming "trick" saves a considerable amount of time in the main loop. $x = \mathrm{abs}(f^{(1)}-f^{(2)})$ is computed in SP, and $\cos(2\pi x)$ is approximated by linear interpolation in the precomputed cosine-table, using the precomputed cosine-difference table. The result is multiplied by the precomputed $k^{-\frac{1}{2}}$ and the product is accumulated in an SP sum. End of the main loop. Next, the two terms in the asymptotic expansion of the Riemann-Siegel formula (3.1) are approximated using the truncated Taylor series expansions

$$(3.7)\qquad \Phi_0(z) \cong \sum_{k=0}^{N_0} c_{2k}^{(0)} z^{2k} \quad \text{and} \quad \Phi_1(z) \cong \sum_{k=0}^{N_1} c_{2k+1}^{(1)} z^{2k+1}.$$

For the values of N_0 and N_1 actually used, see Section 3.4. The total correction is computed and added to 2 times the sum obtained in the main loop. The computations after the main loop are carried out in SP.

Method B. The same as Method A, with *all* computations in DP. The value of $\cos(2\pi x)$ is computed using the available standard FORTRAN DP library function DCOS.

3.2. Error analysis

In our error analysis we assume that τ is exactly representable as a floating point number. The positive integer m $(= \lfloor \tau^{\frac{1}{2}} \rfloor)$ is *exactly* computed from τ by testing the inequalities $m^2 \le \tau < (m+1)^2$ and by correcting the machine-computed value, if necessary. Now let

$$(3.8) \qquad s(t) := 2 \sum_{k=1}^{m} k^{-\frac{1}{2}} \cos[t.\ell n(k) - \theta(t)], \quad (t = 2\pi\tau).$$

By $\tilde{s}(\tilde{t})$ we denote the computed value of $s(t)$, where errors may be made in the computation of t, $\ell n(k)$, $\theta(t)$, $t.\ell n(k) - \theta(t)$, $\cos(.)$, $k^{-\frac{1}{2}}$ and the final inner product. The following lemma accounts for *all* these errors.

LEMMA 3.1. *Suppose that* $|t-\tilde{t}| \le \delta_0 t$, $|\ell n(k) - \widetilde{\ell n}(k)| \le \delta_1 \ell n(k)$ *for* $k = 1,2,\ldots,m$, *and* $|\theta(u) - \tilde{\theta}(u)| \le \delta_2 \theta(u)$; *let* $f_k := \mathrm{frac}\{\tau\widetilde{\ell n}(k) - \tilde{\theta}(\tilde{t})(2\pi)^{-1}\}$ *and suppose that* $|f_k - \tilde{f}_k| \le \delta_3$ *for* $k = 1,2,\ldots,m$. *Moreover, suppose that* $|\cos(x) - \widetilde{\cos}(x)| \le \delta_4$ *for* $0 \le x \le 2\pi + h$, *where* h *is fixed**), $|k^{-\frac{1}{2}} - \widetilde{k^{-\frac{1}{2}}}| \le \delta_5 k^{-\frac{1}{2}}$ *for* $k = 1,2,\ldots,m$, *and that the inner product of the two vectors with components* $\widetilde{k^{-\frac{1}{2}}}$ *and* $\widetilde{\cos}(2\pi\tilde{f}_k)$, *respectively,* $(1 \le k \le m)$ *is computed with a relative error in the basic arithmetic operations (+,−,* and /) bounded by* ε. *Then we have*

$$(3.9) \qquad \begin{aligned} |s(t) - \tilde{s}(\tilde{t})| \le\ & 4\pi\tau^{5/4}\ell n(\tau)[2\delta_0 + \delta_1(1+\delta_0) + \delta_2] + \\ & + 4\tau^{\frac{1}{4}}[2\pi\delta_3 + \delta_4 + (1+\delta_4)\{\delta_5 + (1+\delta_5)((1+\varepsilon)^m - 1)\}]. \end{aligned}$$

This lemma is similar to Lemma 5.3 of BRENT [2], the difference being that we explicitly account for *all* possible errors in the computation of s(t). The proof is routine and uses the technique of backward error analysis (cf. WILKINSON [18]) for the inner product computation (cf. PARLETT [14, pp. 30-32]) and for the other basic arithmetic operations.

Let

$$(3.10) \qquad \chi(\tau) := (-1)^{m-1}\tau^{-\frac{1}{4}}[\Phi_0(z) + \tau^{-\frac{1}{2}}\Phi_1(z)].$$

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*) The reason for the occurrence of this (small) number h in this lemma will be clarified in Section 3.3.

By $\tilde{\chi}(\tau)$ we denote the computed value of $\chi(\tau)$ where errors may be made in the computation of $\tau^{-\frac{1}{2}}$, $\tau^{-\frac{1}{4}}$, z, $\Phi_0(z)$, $\Phi_1(z)$, and in the other arithmetic operations. The following lemma accounts for *all* these errors.

<u>LEMMA 3.2</u>. *Let* ε *be as in Lemma* 3.1 *and let the relative error in the square root computation be bounded by* $a\varepsilon$. *Moreover, suppose that* $|z-\tilde{z}| \le \delta_6$ *and that* $\Phi_0(z)$ *and* $\Phi_1(z)$ *are approximated by* $\tilde{\Phi}_0(z) := \Sigma_{k=0}^{N_0} \widetilde{c_{2k}^{(0)}} z^{2k}$ *and* $\tilde{\Phi}_1(z) := \Sigma_{k=0}^{N_1} \widetilde{c_{2k+1}^{(1)}} z^{2k+1}$, *respectively, where* $|c_{2k}^{(0)} - \widetilde{c_{2k}^{(0)}}| \le \delta_7$ *and* $|c_{2k+1}^{(1)} - \widetilde{c_{2k+1}^{(1)}}| \le \delta_7$. *Then*

$$
\begin{aligned}
|\chi(\tau) - \tilde{\chi}(\tau)| \le\ & \tau^{-\frac{1}{4}}[2\delta_6 + 2\delta_7(N_0+1) + \frac{1}{(N_0+1)!}(\tfrac{\pi}{2})^{N_0+1} + (5N_0+2a+4)\varepsilon] + \\
& + \tau^{-3/4}[\tfrac{1}{4}\delta_6 + 2\delta_7(N_1+1) + \frac{1}{6}\,\frac{N_1+5/2}{(N_1+1)!}(\tfrac{\pi}{2})^{N_1+1} + (5N_1+3a+7)\varepsilon].
\end{aligned}
\tag{3.11}
$$

In the proof of this lemma, which we omit, use is made of the inequalities $|\Phi_0(z)| \le 1$, $|\Phi_1(z)| \le 1$, $|\Phi_0'(z)| \le 1$ and $|\Phi_1'(z)| \le \frac{1}{4}$ for $|z| \le 1$ (see GABCKE [5, Theorem 1, p. 60]) and of the bounds given in GABCKE [5, Theorem 2, p. 62] on the error induced by truncating the infinite series in (3.3) and (3.4).

3.3. <u>Estimates for $\delta_0,\ldots,\delta_7$ for methods A and B</u>

Because of the programming "trick" mentioned in 3.1 we must take into account the possibility that the *computed* value of $f^{(2)}$ may be (slightly) larger than 1 by an amount which is bounded by $2.5\varepsilon_s \tau \ell n(k)$. In the t-range (3.6) this excess is bounded by 10^{-5}. Instead of correcting $f^{(2)}$ by subtracting 1, which is needed only very rarely, we use *one* extra element in the cosine interpolation table beyond $\cos(2\pi)$, viz. $\cos(2\pi+h)$, where $h = 2\pi.2^{-13} \approx$ $\approx 7.7 \times 10^{-4}$ $(> 10^{-5})$.

In [12] we have given an account of our computation of the values of $\delta_0,\ldots,\delta_7$. The results are summarized in Table 3.1.

<u>Table 3.1.</u>

Values of $\delta_0,\ldots,\delta_7$ for methods A and B

method	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
A	1.01×10^{-28}	5.1×10^{-29}	3.6×10^{-27}	5×10^{-14}	7.36×10^{-8}	7.2×10^{-15}	7.2×10^{-15}	5×10^{-14}
B	1.01×10^{-28}	5.1×10^{-29}	3.6×10^{-27}	1.2×10^{-17}	1.5×10^{-27}	1.01×10^{-28}	2.0×10^{-24}	5×10^{-28}

3.4. The error bounds on $\tilde{Z}$ for methods A and B

To complete the error analysis we apply Lemmas 3.1 and 3.2 with $\delta_0, \ldots, \delta_7$ as given in Table 3.1, $\varepsilon = \varepsilon_s = 2^{-47}$, $a = 10$, $N_0 = 16$ and $N_1 = 17$ for method A, and $\varepsilon = \varepsilon_d = 2^{-95}$, $a = 10$, $N_0 = N_1 = 29$ for method B; including the inherent error (3.5) this yields

$$(3.13) \qquad |Z(t) - \tilde{Z}_A(\tilde{t})| \le 3 \times 10^{-7} \tau^{1/4}$$

and

$$(3.14) \qquad |Z(t) - \tilde{Z}_B(\tilde{t})| \le 5.4 \times 10^{-3} \tau^{-5/4} + 3.1 \times 10^{-16} \tau^{1/4} + 4.1 \times 10^{-24} \tau^{-1/4} + 5 \times 10^{-26} \tau^{5/4} \ell n(\tau),$$

for any t ($= 2\pi\tau$) in the interval $(3.5 \times 10^7, 3.72 \times 10^8)$. In this interval, safe upper bounds for the errors are 2.7×10^{-5} and 2.0×10^{-11}, respectively. In the LR & W-program (see [12]) the extremely conservative *fixed* bounds $\varepsilon_1 = 10^{-4}$ and $\varepsilon_2 = 2.5 \times 10^{-6}$ were used, respectively. In case $|\tilde{Z}_A(\tilde{t})|$ was less than ε_1, a few rather small shifts with $\tilde{t}$ were tried. If still no "clear" value was found with method A, method B was invoked. Until now not a single t occurred for which method B could not determine the sign of $Z(t)$ rigorously.

4. STATISTICS

The LR & W-program was organized in such a way that in case the value of $Z(t)$, obtained with method A, was too small for a rigorous sign determination, a few small shifts of the argument were tried before method B was invoked. Therefore, the LR & W-program uses, in relatively few cases, an approximation of the Gram point g_j instead of g_j itself. (In a run of 2,500,000 zeros, with error bound 10^{-4} for method A, the *total* number of shifts was always less than 370. Most of them were made when separating the zeros *inside* the Gram blocks. Only a few of them were made in Gram points. Also see the text introducing Table 4.3.) Consequently, the statistics found by LR & W cannot, strictly speaking, be accumulated to those, found by Brent. Nevertheless, just for convenience, we have put together all results. This should be kept in mind when reading the tables.

In Table 4.1 we present a list of 104 exceptions to Rosser's rule up to $g_{200,000,000}$ found by Brent and LR & W, including the 15 exceptions up to $g_{75,000,000}$ from [2], for completeness. Moreover, the types (see Table 4.2) are given in parentheses, followed by the local extreme values of S(t) (see BRENT [2]) near B_n. It is possible that for n ≥ 156,800,000 the LR & W - program has not detected *all* exceptions to Rosser's rule, due to

Table 4.1

(extension of Table 3 of BRENT [2])

104 exceptions to Rosser's rule up to $g_{200,000,000}$

Notation: n (type) extreme S(t)

where n is the index of the Gram block B_n containing no zeros.

13,999,525(1)	-2.0041	100,788,444(1)	-2.0230	146,130,246(2)	2.0005	173,737,614(2)	2.0221
30,783,329(1)	-2.0026	106,236,172(1)	-2.0184	147,059,770(1)	-2.0498	174,102,513(1)	-2.0180
30,930,927(2)	2.0506	106,941,328(2)	2.1559	147,896,100(2)	2.0391	174,284,990(1)	-2.0181
37,592,215(1)	-2.0764	107,287,955(1)	-2.0786	151,097,113(1)	-2.0043	174,500,513(1)	-2.0125
40,870,156(1)	-2.0038	107,532,017(2)	2.0728	152,539,438(1)	-2.0026	175,710,609(1)	-2.0193
43,628,107(1)	-2.0242	110,571,044(1)	-2.0458	152,863,169(2)	2.0459	176,870,844(2)	2.0125
46,082,042(1)	-2.0311	111,885,254(2)	2.0247	153,522,727(2)	2.0027	177,332,733(2)	2.0146
46,875,667(1)	-2.0046	113,239,783(1)	-2.0306	155,171,525(2)	2.0437	177,902,862(2)	2.0223
49,624,541(2)	2.0018	120,159,903(1)	-2.0589	155,366,607(1)	-2.0277	179,979,095(1)	-2.0182
50,799,238(1)	-2.0288	121,424,392(2)	2.0515	157,260,687(2)	2.0363	181,233,727(2)	2.1018
55,221,454(2)	2.0242	121,692,932(2)	2.0616	157,269,224(1)	-2.0329	181,625,435(1)	-2.0401
56,948,780(2)	2.0177	121,934,171(2)	2.1719	157,755,123(1)	-2.0205	182,105,257(6)	2.0084
60,515,663(1)	-2.0081	122,612,849(2)	2.0072	158,298,485(2)	2.0273	182,223,560(2)	2.0156
61,331,766(3)	-2.0543	126,116,567(1)	-2.0106	160,369,051(2)	2.0071	191,116,405(2)	2.0195
69,784,844(2)	2.0637	127,936,513(1)	-2.1105	162,962,787(1)	-2.0115	191,165,600(2)	2.0283
75,052,114(1)	-2.0045	128,710,278(2)	2.0444	163,724,709(1)	-2.0163	191,297,535(5)	-2.1490
79,545,241(2)	2.0113	129,398,903(2)	2.0431	164,198,114(2)	2.0235	192,485,616(1)	-2.0416
79,652,248(2)	2.0066	130,461,097(2)	2.0963	164,689,301(1)	-2.1579	193,264,636(6)	2.0055
83,088,043(1)	-2.1328	131,331,948(2)	2.0047	164,880,229(2)	2.0308	194,696,968(1)	-2.0664
83,689,523(2)	2.0775	137,334,072(2)	2.0239	166,201,932(1)	-2.0024	195,876,805(1)	-2.0143
85,348,958(1)	-2.0095	137,832,603(1)	-2.0134	168,573,836(1)	-2.0159	195,916,549(2)	2.0546
86,513,820(1)	-2.0154	138,799,472(2)	2.0135	169,750,763(1)	-2.1036	196,395,161(2)	2.0326
87,947,597(2)	2.0523	139,027,791(1)	-2.0031	170,375,507(1)	-2.0009	196,676,303(1)	-2.0135
88,600,095(1)	-2.1394	141,617,806(1)	-2.1253	170,704,880(2)	2.0249	197,889,883(2)	2.0034
93,681,183(1)	-2.0165	144,454,931(1)	-2.0380	172,000,993(2)	2.0608	198,014,122(1)	-2.0333
100,316,552(2)	2.0233	145,402,380(2)	2.0012	173,289,941(1)	-2.0378	199,235,289(1)	-2.0205

possible shifts in Gram points. For instance, an exception of type 2 (see Table 4.2) may have been detected as a Gram block of length 3 with "2 1 0" zero-pattern. It may be noted, however, that in the range $[g_{81,000,000}, g_{120,000,000})$ LR & W have found exactly the same exceptions to Rosser's rule as Brent.

In addition to the types 1, 2 and 3 introduced by BRENT [2] we have defined the types 4, 5 and 6, the meaning of which should be clear from Table 4.2. This table also gives the frequencies of the occurrences of the various types in $[g_{-1}, g_{200,000,000})$. Note that an exception of type 4 has not yet been found, so that at the time of writing we still know only one Gram interval with four zeros, viz. $G_{61,331,768}$, found by BRENT [2].

Table 4.2

Various types of exceptions to Rosser's rule and their frequencies in $[g_{-1}, g_{200,000,000})$.

Gram block of length 2 without any zeros: $[g_n, g_{n+2})$

$[g_{n-2}, g_{n-1})$	$[g_{n-1}, g_n)$	$[g_n, g_{n+1})$	$[g_{n+1}, g_{n+2})$	$[g_{n+2}, g_{n+3})$	$[g_{n+3}, g_{n+4})$	type	frequency
		0	0	3		1	53
	3	0	0			2	47
		0	0	4	0	3	1
0	4	0	0			4	0
		0	0	2	2	5	1
2	2	0	0			6	2

Very recently, KARKOSCHKA and WERNER [8] have developed a method for detecting exceptions to Rosser's rule with relatively small computational effort, i.e., by searching in certain selected small ranges of a given t-interval. A comparison of their results with Table 4.1 shows the power of their method: in $[g_{3,500,000}, g_{50,000,000})$ they found all 9 exceptions to Rosser's rule, and in $[g_{100,000,000}, g_{120,000,000})$ they found 6 of the 9 exceptions.

Table 4.3 is a continuation of Table 1 of BRENT [2]. Six Gram blocks of length 8 were found. The average block length up to n = 200,000,000 is 1.1951. We have compared the results of LR & W with those of Brent in the range $[g_{110,000,000}, g_{120,000,002})$. Brent's program counted 7,011,482 Gram blocks of length 1, 1,055,511 of length 2 and 230,234 of length 3. The corresponding figures obtained by LR & W were 7,011,494, 1,055,508 and 230,232, respectively. The numbers of Gram blocks of length ≥ 4 were the same for both programs.

Table 4.3

(continuation of Table 1 of BRENT [2])

Number of Gram blocks of given length

n	J(1,n)	J(2,n)	J(3,n)	J(4,n)	J(5,n)	J(6,n)	J(7,n)	J(8,n)
80,000,000	56,942,025	8,386,072	1,714,271	260,637	18,807	1,033	34	
90,000,000	63,977,026	9,439,917	1,941,455	299,932	22,257	1,240	46	
100,000,000	71,004,697	10,493,487	2,169,610	340,360	25,813	1,436	54	
110,000,000	78,023,506	11,547,936	2,399,154	381,216	29,601	1,644	61	
120,000,000	85,034,988	12,603,447	2,629,388	422,721	33,500	1,841	74	1
130,000,000	92,041,326	13,659,032	2,860,087	464,955	37,495	2,070	92	1
140,000,000	99,041,526	14,713,754	3,092,451	507,686	41,631	2,332	102	1
150,000,000	106,038,874	15,768,532	3,325,400	550,630	45,795	2,591	114	3
156,800,000	110,793,769	16,486,479	3,484,026	579,999	48,731	2,780	120	3
200,000,000	140,956,084	21,047,520	4,497,856	771,607	68,631	4,031	213	6

Table 4.4 is a continuation of Table 2 of BRENT [2]. The percentages of the numbers of Gram intervals up to n = 200,000,000 containing exactly m zeros are 13.4, 73.4, 13.0 and 0.2 for m = 0, 1, 2 and 3, respectively.

Table 4.4

(continuation of Table 2 of BRENT [2])

Number of Gram intervals containing exactly m zeros

n	m = 0	m = 1	m = 2	m = 3	m = 4
80,000,000	10,513,316	59,105,832	10,248,390	132,461	1
90,000,000	11,854,362	66,440,792	11,555,331	149,514	1
100,000,000	13,197,331	73,771,910	12,864,188	166,570	1
110,000,000	14,543,760	81,096,629	14,175,463	184,147	1
120,000,000	15,892,224	88,416,806	15,489,718	201,251	1
130,000,000	17,242,449	95,733,829	16,804,996	218,725	1
140,000,000	18,594,089	103,047,955	18,121,824	236,131	1
150,000,000	19,946,624	110,360,313	19,439,504	253,558	1
156,800,000	20,867,682	115,330,181	20,336,593	265,543	1
200,000,000	26,731,720	146,878,417	26,048,007	341,855	1

Table 4.5 continues Table 4 of BRENT [2]. As yet, no Gram block of type (7,1) was found. Due to the shifts, we may have missed earlier occurrences of blocks of types (7,7), (8,3) and (8,7), although we consider this unlikely.

Table 4.5

(continuation of Table 4 of BRENT [2])

First occurrences of Gram blocks of various types

j	k	n
7	7	195,610,937 (LR & W)
8	2	112,154,948 (BRENT)
8	3	175,330,804 (LR & W)
8	6	145,659,810 (BRENT)
8	7	165,152,519 (LR & W)

In Table 4.6 we list the number of Gram blocks of type (j,k), $1 \le j \le 8$, $1 \le k \le j$, in the interval $[g_{156,800,000}, g_{200,000,000})$, as they were actually counted by the LR & W-program. On the line with $j = 2$ we also mention the numbers of Gram blocks of length 2 with zero-pattern "0 0" and those with pattern "2 2" which could, of course, neither be classified as type (2,1) nor as (2,2). The 43 blocks with "0 0"-pattern correspond to the exceptions to Rosser's rule in $[g_{156,800,000}, g_{200,000,000})$ and the 3 blocks with "2 2"-pattern correspond to the exceptions of types 5 and 6 (cf. Table 4.2). The entries in parentheses give the approximate percentages with respect to the total number of blocks of length j, given in the final column.

Our main purpose of presenting this table is to render support to the LR & W-strategy of dealing with Gram blocks of length $j \ge 2$. The table shows that this strategy is successful for $2 \le j \le 5$. However, for $j \ge 6$ the missing two zeros in B_n show an increasing tendency to lie either in (g_{n+1}, g_{n+2}) or in (g_{n+j-2}, g_{n+j-1}). Only one of the 93 blocks of length $j = 7$ has its missing two zeros in one of the outer Gram intervals!

Table 4.6

Number of Gram blocks of type (j,k), $1 \le j \le 8$, $1 \le k \le j$, *in the interval* $[g_{156,800,000}, g_{200,000,000})$

↓j	k→ 1	2	3	4	5	6	7	8	total
1	30,162,315								30,162,315
2	2,279,942 (50)	2,281,053 (50)	43 blocks with 0 0 zero-pattern 3 blocks with 2 2 zero-pattern						4,561,041
3	479,720 (47)	53,497 (5)	480,613 (47)						1,013,830
4	87,367 (46)	8,592 (4)	8,499 (4)	87,150 (45)					191,608
5	7,581 (38)	1,811 (9)	948 (5)	1,882 (9)	7,678 (39)				19,900
6	156 (12)	337 (27)	119 (10)	126 (10)	366 (29)	147 (12)			1,251
7	0	29	17	3	17	26	1		93
8	0	0	1*)	0	0	1*)	1*)	0	3

*) viz. B_n, for n = 175,330,804, 181,390,731 and 165,152,519.

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Added in Proof. In the meanwhile VAN DE LUNE & TE RIELE have extended the computations so far that we can now (December 1982) say that the first 307 000 000 non-trivial zeros of $\zeta(s)$ are all simple and lie on $\sigma = \frac{1}{2}$.