The Great Trinomial Hunt

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Trinomials and Polynomials over a Field

A *trinomial* is a polynomial in one variable with three nonzero terms, for example

$$P = 6x^7 + 3x^3 - 5.$$

If the coefficients of a polynomial P (in this case $6, 3, -5$) are in some ring or field F, we say that P is a polynomial over F, and write $P \in F[x]$.

The operations of addition and multiplication of polynomials in $F[x]$ are defined in the usual way, with the operations on coefficients performed in F.

The Ground Field

Classically the most common cases are $F = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ or $\mathbf{C}$, respectively the integers, rationals, reals or complex numbers. However, polynomials over finite fields are important in applications.

We restrict our attention to polynomials over the simplest finite field: the field $\mathrm{GF}(2)$ of two elements, usually written as 0 and 1.

Other notations for $\mathrm{GF}(2)$ are $\mathbf{Z}_2$ and $\mathbf{Z}/2\mathbf{Z}$. The field operations of addition and multiplication are defined as for integers modulo 2, so $0 + 1 = 1$, $1 + 1 = 0$, $0 \times 1 = 0$, $1 \times 1 = 1$, etc.

Squaring in $\mathrm{GF}(2)[x]$

An important consequence of the definitions is that, for polynomials $P, Q \in GF(2)[x]$, we have

$$(P + Q)^2 = P^2 + Q^2$$

because the “cross term” $2PQ$ vanishes.

High school algebra would have been much easier if we had used polynomials over $\mathrm{GF}(2)$ instead of over **R**.

Fibonacci-like Recurrences

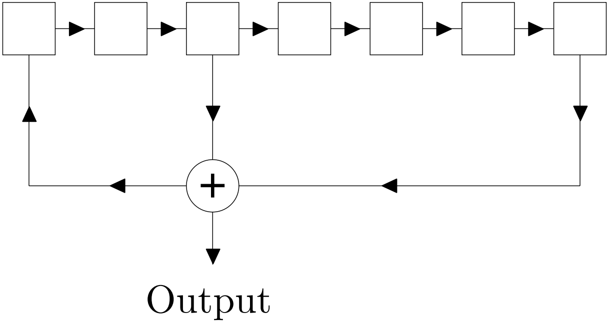
Trinomials over $\mathrm{GF}(2)$ are important in cryptography and random number generation. To illustrate why this might be true, consider a sequence $(z_0, z_1, z_2, \ldots)$ satisfying the recurrence

$$z_n = z_{n-s} + z_{n-r} \bmod 2,$$

where r and s are given positive integers, $r > s > 0$, and the initial values $z_0, z_1, \ldots, z_{r-1}$ are also given. The recurrence then defines all the remaining terms $z_r, z_{r+1}, \ldots$ in the sequence.

Hardware Implementation

It is easy to build hardware to implement the recurrence $z_n = z_{n-s} + z_{n-r} \bmod 2$. All we need is a shift register capable of storing r bits, and a circuit capable of computing the addition mod 2 (equivalently, the "exclusive or") of two bits separated by $r - s$ positions in the shift register and feeding the output back into the shift register. For example, with $r = 7$, $s = 3$:



Pseudo-Random Sequences

The recurrence $z_n = z_{n-s} + z_{n-r} \bmod 2$ looks similar to the well-known Fibonacci recurrence

$$F_n = F_{n-1} + F_{n-2};$$

indeed the Fibonacci numbers mod 2 satisfy our recurrence with $r = 2$, $s = 1$. This gives a sequence $(0, 1, 1, 0, 1, 1, \ldots)$ with period 3: not very interesting. However, if we take r larger we can get much longer periods.

Getting a Long Period

The period can be as large as $2^r - 1$, which makes such sequences interesting as components in pseudo-random number generators or stream ciphers. In fact, the period is $2^r - 1$ if the initial values are not all zero and the associated trinomial

$$x^r + x^s + 1,$$

regarded as a polynomial over $\mathrm{GF}(2)$, is *primitive*.

A primitive polynomial is one that is irreducible (it has no nontrivial factors), and satisfies an additional condition (to be described soon).

Mersenne Primes

A *Mersenne prime* is a prime of the form $2^r - 1$.

They are named after Marin Mersenne (1588–1648), who corresponded with many of the scholars of his day, and in 1644 gave a list (containing several errors) of the Mersenne primes with $r \le 257$.

A *Mersenne exponent* is the exponent r of a Mersenne prime $2^r - 1$. A Mersenne exponent is necessarily prime, but not conversely.

For example, 11 is not a Mersenne exponent because

$$2^{11} - 1 = 23 \cdot 89$$

is not prime.

The Trinomial Search

This talk is about a search for primitive trinomials of large degree r, and its interplay with a search for large Mersenne primes.

First, we need to explain the connection between these two topics, and briefly describe the GIMPS project.

Next we describe the algorithms used in our search, which can be split into two distinct periods, “classical” and “modern”.

Finally, we describe the results obtained in the modern period.

Mathematical Foundations

We consider polynomials over the finite field $\mathrm{GF}(2)$. An *irreducible polynomial* is a polynomial that is not divisible by any non-trivial polynomial other than itself.

For example, $x^5 + x^2 + 1$ is irreducible, but $x^5 + x + 1$ is not, since $x^5 + x + 1 = (x^2 + x + 1)(x^3 + x^2 + 1)$ in $\mathrm{GF}(2)[x]$.

We do not consider binomials $x^r + 1$, because they are divisible by $x + 1$, and thus reducible for $r > 1$.

Representing Large Finite Fields

An irreducible polynomial P of degree $r > 1$ yields a representation of the finite field $\mathrm{GF}(2^r)$ of 2^r elements: any polynomial of degree less than r represents an element.

Addition is just polynomial addition.

Multiplication is defined modulo P: one first multiplies both inputs, and then reduces their product modulo P.

Thus

$$\mathrm{GF}(2^r) \simeq \mathrm{GF}(2)[x]/P(x) \; .$$

Primitive polynomials

An irreducible polynomial P of degree $r > 0$ over $\mathrm{GF}(2)$ is said to be *primitive* iff $P(x) \neq x$ and the residue classes $x^k \bmod P$, $0 \leq k < 2^r - 1$, are distinct.

In other words, P is primitive if x is a generator of the multiplicative group of $\mathrm{GF}(2^r) \simeq \mathrm{GF}(2)[x]/P(x)$.

In order to check primitivity of an irreducible polynomial P, it is only necessary to check that $x^k \neq 1 \bmod P$ for those k that are maximal non-trivial divisors of $2^r - 1$.

For example, $x^5 + x^2 + 1$ is primitive, as $2^5 - 1 = 31$ is prime.

$x^6 + x^3 + 1$ is irreducible but not primitive, since $x^9 = 1 \bmod (x^6 + x^3 + 1)$. Here 9 divides $2^6 - 1 = 63$ and is a maximal divisor as $63/9 = 7$ is prime.

Testing Primitivity for Large Degrees

If r is large and $2^r - 1$ is not prime, it can be difficult to test primitivity of a polynomial of degree r, because we need to know the prime factors of $2^r - 1$. Thanks to the Cunningham project, these are known for all $r < 863$, but not in general for larger r.

On the other hand, if $2^r - 1$ is prime, then all irreducible polynomials of degree r are primitive. This is the reason why we consider degrees r that are Mersenne exponents.

Starting the Search

In the year 2000 Paul Zimmermann, Samuli Larvala and I were in contact via email (mainly because of our common interest in factoring large integers) when the topic of efficient algorithms for testing irreducibility or primitivity of trinomials over $\mathrm{GF}(2)$ arose.

Publication of a paper by Kumada *et al.*, describing a search for primitive trinomials of degree 859 433 (a Mersenne exponent), prompted the three of us to embark on a search for primitive trinomials of degree r, for r ranging over all known Mersenne exponents.

The computational challenge

At that time, the largest known Mersenne exponents were 3 021 377 and 6 972 593. The existing programs took time proportional to r^3.

Since $(6972593/859433)^3 \approx 534$, and the computation by Kumada *et al.* had taken three months on 19 processors, it was quite a challenge.

Using Kumada's program and computational resources, the task would take about 133 years!

The GIMPS project

GIMPS stands for *Great Internet Mersenne Prime Search*. It is a distributed computing project started by George Woltman, with home page `www.mersenne.org`. The goal of GIMPS is to find new Mersenne primes.

As of October 2009, GIMPS has found 13 new Mersenne primes in 13 years, and has held the record of the largest known prime since the discovery of M_{35} in 1996.

Mersenne primes are usually numbered in increasing order of size: $M_1 = 2^2 - 1 = 3$, $M_2 = 2^3 - 1 = 7$, $M_3 = 2^5 - 1 = 31$, $M_4 = 2^7 - 1 = 127, \ldots, M_{38} = 2^{6972593} - 1$, etc.

The ordering problem

Since GIMPS does not always find Mersenne primes in order, there can be some uncertainty in numbering the largest known Mersenne primes. We write M'_n for the n-th Mersenne prime in order of discovery.

There are gaps in the search above $M_{39} = 2^{13466917} - 1$. Thus we can have $M'_n > M'_{n+1}$ for $n > 39$.

For example, $M'_{45} = 2^{43112609} - 1$ was found before $M'_{46} = 2^{37156667} - 1$ and $M'_{47} = 2^{42643801} - 1$.

At the present time, 47 Mersenne primes are known, and the largest is $M'_{45} = 2^{43112609} - 1$.

To avoid ambiguity, we'll write r_n for the exponent of M_n, and r'_n for the exponent of M'_n. For example, $r'_{45} = 43\,112\,609$.

Swan's Theorem

We state a useful theorem, known as Swan's theorem, since it appeared in a paper by Swan (1962), although the result was actually found much earlier by Pellet (1878) and Stickelberger (1897).

Swan's theorem gives information on the *parity* of the number of irreducible factors of a trinomial.

Theorem [Swan]

Let $r > s > 0$, and assume $r + s$ is odd. Then $T_{r,s}(x) = x^r + x^s + 1$ has an even number of irreducible factors over $\mathrm{GF}(2)$ in the following cases:
a) r even, $r \neq 2s$, $rs/2 = 0$ or $1 \bmod 4$.
b) r odd, s not a divisor of $2r$, $r = \pm 3 \bmod 8$.
c) r odd, s a divisor of $2r$, $r = \pm 1 \bmod 8$.
In all other cases $x^r + x^s + 1$ has an odd number of irreducible factors.

Applying the theorem in other cases

If both r and s are even, then $T_{r,s}$ is a square. If both r and s are odd, we can apply the theorem to $T_{r,r-s}$. Thus, Theorem 1 is applicable to any trinomial over $\mathrm{GF}(2)$.

Application of Swan's theorem

For r an odd prime, and excluding the easily-checked cases $s = 2$ or $r - 2$, case (b) says that the trinomial has an even number of irreducible factors, and hence must be reducible, if $r = \pm 3 \bmod 8$.

Thus, we only need to consider those Mersenne exponents with $r = \pm 1 \bmod 8$.

Of the 14 known Mersenne exponents $r > 10^6$, only 8 satisfy this condition.

Cost of the Basic Operations

The basic operations that we need are squarings modulo the trinomial $T = x^r + x^s + 1$, multiplications modulo T, and greatest common divisors (GCDs) between T and a polynomial of degree less than r.

We measure the cost of these operations in terms of the number of bit or word-operations required to implement them.

In $\mathrm{GF}(2)[x]$, squarings cost $O(r)$, due to the fact that the square of $x^i + x^j$ is $x^{2i} + x^{2j}$.

The reduction modulo T of a polynomial of degree less than $2r$ costs $O(r)$, due to the sparsity of T; thus modular squarings cost $O(r)$.

Cost of modular multiplication

Modular multiplications cost $O(M(r))$, where $M(r)$ is the cost of multiplication of two polynomials of degree less than r over $\mathrm{GF}(2)$.

The reduction modulo T costs $O(r)$, so the multiplication cost dominates the reduction cost.

The "classical" polynomial multiplication algorithm has $M(r) = O(r^2)$, but an algorithm due to Schönhage has $M(r) = O(r \log r \log \log r)$.

Note: this algorithm is not the Schönhage-Strassen integer-multiplication algorithm, although both are based on the idea of using a fast Fourier transform.

Cost of GCD

A GCD computation for polynomials of degree bounded by r costs $O(M(r)\log r)$ using a "divide and conquer" approach combined with Schönhage's fast polynomial multiplication. This follows from the recurrence

$$G(2r) = 2G(r) + O(M(r)) \ .$$

Summary of costs

The costs are summarized in the table.

modular squaring	$O(r)$
modular product	$O(M(r))$
GCD	$O(M(r)\log r)$

Table: Cost of the basic operations.

Testing Irreducibility

Let $\mathbf{P}_r(x) = x^{2^r} + x$. (We could equally well write $x^{2^r} - x$.) Then $\mathbf{P}_r(x)$ is the product of all irreducible polynomials of degree d, where d runs over the divisors of r.
For example,

$$\mathbf{P}_3(x) = x(x+1)(x^3+x+1)(x^3+x^2+1)$$

in $\mathrm{GF}(2)[x]$. Here x and $x+1$ are the irreducible polynomials of degree 1, and the other factors are the irreducible polynomials of degree 3.

If r is an odd prime, then a polynomial $P(x) \in \mathrm{GF}(2)[x]$ with degree r is irreducible iff

$$x^{2^r} = x \bmod P(x)\ . \tag{1}$$

If r is not prime, we have to check a further condition to guarantee irreducibility.

Testing irreducibility for odd prime r

When r is an odd prime, equation (1) gives a simple test for irreducibility (or primitivity, in the case that r is a Mersenne exponent): just perform r modular squarings, starting from x, and check if the result is x. Since the cost of each squaring is $O(r)$, the cost of the irreducibility test is $O(r^2)$.

"Faster" irreducibility tests

There are more sophisticated algorithms for testing irreducibility, based on modular composition (Kedlaya and Umans, 2008) and fast matrix multiplication (Brent and Kung, 1978). However, these algorithms are actually slower than the classical algorithm when applied to trinomials of degree less than about 10^7.

The reciprocal trinomial

When searching for irreducible trinomials of degree r, we can assume that $s \leq r/2$, since

$$T(x) = x^r + x^s + 1$$

is irreducible iff the "reciprocal polynomial"

$$x^r T(1/x) = x^r + x^{r-s} + 1$$

is irreducible.

This simple observation saves a factor of 2.

In the following, we always assume that $s \leq r/2$.

Degrees of Factors

In order to predict the expected behaviour of our algorithm, we need to know the expected distribution of degrees of irreducible factors.

Our complexity estimates are based on the assumption that trinomials of degree r behave like the set of all polynomials of the same degree, up to a constant factor:

Assumption 1

Over all trinomials $x^r + x^s + 1$ of degree r over $\mathrm{GF}(2)$, the probability π_d that a trinomial has no non-trivial factor of degree $\le d$ is at most c/d, where c is a constant and $1 < d \le r/\ln r$.

This assumption is plausible and in agreement with experiments, though not proven. It is not critical, because the correctness of our algorithms does not depend on the assumption – only the predicted running time depends on it.

Evidence for the Assumption

Some evidence for the assumption, in the case $r = r_{38}$, is presented in the Table. The maximum value of $d\pi_d$ is 2.08, occurring at $d = 226\,887$.

d	$d\pi_d$	d	$d\pi_d$
1	1.00	2	1.33
3	1.43	4	1.52
5	1.54	6	1.60
7	1.60	8	1.67
9	1.64	10	1.65
100	1.77	1000	1.76
10000	1.88	226887	2.08

It would be interesting to try to explain the exact values of $d\pi_d$ for small d, but this would lead us too far afield.

Sieving

When testing a large integer N for primality, it is sensible to check if it has any small factors before applying a primality test such as the AKS, ECPP, or (if we are willing to accept a small probability of error) Rabin-Miller test.

Similarly, when testing a high-degree polynomial for irreducibility, it is wise to check if it has any small factors before applying the $O(r^2)$ test.

Since the irreducible polynomials of degree d divide $\mathbf{P}_d(x)$, we can check if a trinomial T has a factor of degree d (or some divisor of d) by computing

$$\gcd(T, \mathbf{P}_d).$$

If $T = x^r + x^s + 1$ and $2^d < r$, we can reduce this to the computation of a GCD of polynomials of degree less than 2^d.

How to Sieve Efficiently

Let $d' = 2^d - 1$, $r' = r \bmod d'$, $s' = s \bmod d'$. Then $\mathbf{P}_d = x(x^{d'} - 1)$,

$$T = x^{r'} + x^{s'} + 1 \bmod (x^{d'} - 1),$$

so we only need to compute

$$\gcd(x^{r'} + x^{s'} + 1, x^{d'} - 1).$$

The process is called “sieving” by analogy with the process of sieving out small prime factors of integers, even though it is performed using GCD computations.
If the trinomials that have factors of degree less than $\log_2(r)$ are excluded by sieving, then by Assumption 1 we are left with $O(r/\log r)$ trinomials to test.

Thus the overall search has cost $O(r^3/\log r)$.

The "Missing" Trinomial

Primitive trinomials of degree $r < r_{32} = 756\,839$ are listed in Heringa *et al* (1992). Kumada *et al* (2000) reported a search for primitive trinomials of degree $r_{33} = 859\,433$.

They found one primitive trinomial; however they missed the trinomial

$$x^{859433} + x^{170340} + 1$$

because of a bug in their sieving routine.

We discovered the missing trinomial in June 2000 while testing our program on the known cases.

Certificates

This motivated us to produce *certificates* of reducibility for all the trinomials that we tested (excluding, of course, the small number that turned out to be irreducible).

A certificate of reducibility is, ideally, a non-trivial factor. If a trinomial T is found by sieving to have a small factor, then it is easy to keep a record of this factor.

If we do not know a factor, but the trinomial fails an irreducibility test, then we can record the residue $R(x) = x^{2^r} - x \bmod T$.

Because the residue can be large, we might choose to record only part of it, e.g., $R(x) \bmod x^{32}$.

The Classical Period

The period 2000–2003 could be called the *classical* period. In this period we used efficient implementations of the classical algorithms.

Since different trinomials could be tested on different computers, it was easy to conduct a search in parallel, using as many processors as were available. We often made use of PCs in an undergraduate teaching laboratory during the vacation, when the students were away.

In this way, we found three primitive trinomials of degree $r_{32} = 756\,839$ (in June 2000), two of degree $r_{37} = 3\,021\,377$ (August and December 2000), and one of degree $r_{38} = 6\,972\,593$ (in August 2002).

Primitive trinomials of degree r_{34}, r_{35} and r_{36} were ruled out by Swan's theorem, as were r_{39} and r'_{40}..

Degree 6 972 593

The computation for degree $r_{38} = 6\,972\,593$ was completed and double-checked by July 2003.

There turned out to be only one primitive trinomial $x^r + x^s + 1$:

$$x^{6972593} + x^{3037958} + 1 \ .$$

The trinomial was named *Bibury* after the village that the three authors BLT were visiting on the day that it was discovered.

Bibury

"The most beautiful village in England" (William Morris)

Unreliability of Long Computations

How can we be sure that we did not miss any primitive trinomials?

For each non-primitive trinomial we had a certificate, and these certificates were checked in an independent computation.

We found a small number of discrepancies, possibly due to memory parity errors in some of the older PCs that were used. This is a risk in any long computation – we should not assume that computers are infallible.

The same phenomenon was observed by Nicely in his computation of Brun's constant (which also uncovered the infamous "Pentium bug").

What Next?

Since we had caught up with the GIMPS project, we thought (not for the last time) that this game had finished, and published our results in *Math. Comp.*

However, GIMPS soon overtook us by finding several larger Mersenne primes with exponents $\pm 1 \bmod 8$:
$r'_{41} = 24\,036\,583, \ldots, r'_{44} = 32\,582\,657$.
The search for degree $r_{38} = 6\,972\,593$ had taken more than two years (February 2001 to July 2003), so it did not seem feasible to tackle the new Mersenne exponents $r'_{41}, \ldots, r'_{44}$.

The Modern Period

We realised that, in order to extend the computation, we had to find more efficient algorithms. The expensive part of the computation was testing irreducibility using repeated squaring.

If we could sieve much further, we could avoid most of the irreducibility tests. From Assumption 1, if we could sieve to degree $r/\ln r$, then we would expect only $O(\log r)$ irreducibility tests.

What we needed was an algorithm that would find the smallest factor of a sparse polynomial (specifically, a trinomial) in a time that was fast *on average*.

Factoring polynomials

There are many algorithms for factoring polynomials over finite fields. The cost of most of them is dominated by GCD computations.

It is possible to replace most GCD computations by modular multiplications, using a process called *blocking*, introduced by Pollard (1975) in the context of integer factorization, and by von zur Gathen and Shoup (1992) for polynomial factorization.

The idea of blocking is simple: instead of computing $\gcd(T, P_1), \ldots, \gcd(T, P_k)$ in the hope of finding a non-trivial GCD (and hence a factor of T), we compute $\gcd(T, P_1 P_2 \cdots P_k \bmod T)$, and backtrack if necessary to split factors if they are not irreducible.

Since a GCD typically takes about 40 times as long as a modular multiplication for $r \approx r'_{41}$, blocking can give a large speedup.

A New Idea – More Blocking

- In February 2007, we realised that a second level of blocking could be used to replace most modular multiplications by squarings.
- Since a modular multiplication might take 400 times as long as a squaring (for $r \approx r'_{41}$), this second level of blocking can provide another large speedup.
- We won't describe the details here, since they are rather technical, but note that m multiplications and m squarings can be replaced by one multiplication and m^2 squarings.
- The optimal value of m is $m_0 \approx \sqrt{M(r)/S(r)}$, where $M(r)$ is the cost of a modular multiplication and $S(r)$ is the cost of a modular squaring, and the resulting speedup is about $m_0/2$.
- If $M(r)/S(r) = 400$, then $m_0 \approx 20$ and the speedup over single-level blocking is roughly a factor of ten.

Consequences of the New Idea

Using two levels of blocking, combined with a fast implementation of polynomial multiplication and a subquadratic GCD algorithm, we were able to find ten primitive trinomials of degrees $r'_{41}, \ldots, r'_{44}$ by January 2008.

Once again, we thought we were finished and published our results, only to have GIMPS leap ahead again by discovering M'_{45} in August 2008, and M'_{46} and M'_{47} shortly afterwards.

The Last Two Exponents(?)

The exponent r'_{46} was ruled out by Swan's theorem, but we had to set to work on degrees $r'_{45} = 43\,112\,609$ and (later) the slightly smaller $r'_{47} = 42\,643\,801$.

The search for degree r'_{45} ran from September 2008 to May 2009, with assistance from Dan Bernstein and Tanja Lange who kindly allowed us to use their computing resources in Eindhoven, and resulted in four primitive trinomials of record degree.

The search for degree r'_{47} ran from June 2009 to August 2009, and found five primitive trinomials. In this case we were lucky to have access to a new computing cluster with 224 processors at the Australian National University, so the computation took less time than the earlier searches.

Computational Results

The results of our computations in the "Modern Period" are given in the table. There does not seem to be any predictable pattern in the s values.

r	s
24 036 583	8 412 642, 8 785 528
25 964 951	880 890, 4 627 670, 4 830 131, 6 383 880
30 402 457	2 162 059
32 582 657	5 110 722, 5 552 421, 7 545 455
42 643 801	55 981, 3 706 066, 3 896 488, 12 899 278, 20 150 445
43 112 609	3 569 337, 4 463 337, 17 212 521, 21 078 848

The number of primitive trinomials for a given Mersenne exponent $r = \pm 1 \bmod 8$ appears to follow a Poisson distribution with mean about 3.2.

The Modern Algorithm – Summary

To summarize the "modern" algorithm for finding primitive trinomials, we improve on the classical algorithm by sieving much further to find a factor of smallest degree, using a factoring algorithm based on fast multiplication and two levels of blocking.

Given a trinomial $T = x^r + x^s + 1$, we search for a factor of smallest degree $d \le r/3$ (using Swan's theorem).

If such a factor is found, we know that T is reducible, so the program outputs "reducible" and saves the factor for a certificate of reducibility. The factor can be found by taking the GCD of T and $x^{2^d} + x$; if this GCD is non-trivial, then T has at least one factor of degree dividing d. If factors of degree smaller than d have already been ruled out, then the GCD only contains factors of degree d (possibly a product of several such factors). This is known as *distinct degree factorization* (DDF).

Equal Degree Factorization and Unique Certificates

If the GCD has degree λd for $\lambda > 1$, and one wants to split the product into λ factors of degree d, then an *equal degree factorization* algorithm (EDF) is used.

If the EDF is necessary it is usually cheap, since the total degree λd is usually small if $\lambda > 1$.

In this way we produce certificates of reducibility that consist just of a non-trivial factor of smallest possible degree, and the lexicographically least such factor if there are several.

It is worth going to the trouble to find the lexicographically least factor, since this makes the certificate unique and allows us to compare different versions of the program and locate bugs more easily than would otherwise be the case.

The certificates can be checked, for example with an independent program using NTL, much faster than the original computation (typically in less than one hour for any of the degrees considered).

Large Factors

The critical fact is that most trinomials have a small factor, so the search runs fast on average. However, occasionally we encounter a trinomial with no small factor.

After searching unsuccessfully for factors of degree $d < 10^6$ say, we could switch to the classical irreducibility test, which is faster than factoring if the factor has degree greater than about 10^6.

However, in that case our list of certificates would be incomplete. Since it is rare to find a factor of degree greater than 10^6, we let the program run until it finds a factor or outputs "irreducible".

Of course, we verify irreducibility using the classical test, just in case there is a bug in the factoring program.
So far no discrepancies have been found.

The Largest Smallest Factor

Of the smallest irreducible factors found during our searches (apart from irreducible trinomials themselves), the largest is a factor

$$P(x) = x^{10199457} + x^{10199450} + \cdots + x^4 + x + 1$$

of the trinomial

$$x^{42643801} + x^{3562191} + 1 \; .$$

Although the trinomial is sparse and has a compact representation, the factor is dense and hence too large to present here in full!

Comparison: Classical versus Modern

For simplicity we'll use the $\widetilde{O}$ notation which ignores log factors. For example, $27r^{3/2}\log r/\log\log r$ is $\widetilde{O}(r^{3/2})$.

The "classical" algorithm takes an expected time $\widetilde{O}(r^2)$ per trinomial, or $\widetilde{O}(r^3)$ to cover all trinomials of degree r.

The "modern" algorithm takes expected time $\widetilde{O}(r)$ per trinomial, or $\widetilde{O}(r^2)$ to cover all trinomials of degree r.

In practice, the new algorithm is faster by a factor of about 160 for $r = r_{38} = 6\,972\,593$, and by a factor of about 1000 for $r = r'_{45} = 43\,112\,609$.

Comparing the computation for $r = r'_{45}$ with that for $r = r_{38}$: using the classical algorithm would take about 240 times longer (impractical), but using the modern algorithm saves a factor of 1000.

How to Speed up the Search – Classical and Modern

- Since the computations for each trinomial can be performed independently, it is easy to conduct a search in parallel, using as many computers as are available.
- There is a one-one correspondence between polynomials of degree $< d$ and binary numbers with d bits. Thus, on a 64-bit computer we can encode a polynomial of degree d in $\lceil (d+1)/64 \rceil$ computer words and do 64 operations in parallel (word-parallelism).
- Squaring of polynomials over $\mathrm{GF}(2)$ can be done in *linear time* (linear in the degree of the polynomial), because the cross terms in the square vanish.
- Reduction of a polynomial of degree $2(r-1)$ modulo a trinomial $T = x^r + x^s + 1$ of degree r can also be done in linear time.

How to Speed up the Search – Modern

- Most GCD computations involving polynomials can be replaced by multiplication of polynomials, using the “blocking” technique described above.
- Most multiplications of polynomials can be replaced by squarings, using another level of blocking.
- Asymptotically fast algorithms can be used for the polynomial multiplications and GCDs that are unavoidable.

Conclusion

The combination of these seven ideas makes it feasible to find primitive trinomials of very large degree.

In fact, the current record degree is the same as the largest known Mersenne exponent, $r = r'_{45} = 43\,112\,609$.

We are ready to find more primitive trinomials as soon as GIMPS finds another Mersenne prime that is not ruled out by Swan's Theorem.

Our task is easier than that of GIMPS, because finding a primitive trinomial of degree r, and verifying that a single value of r is a Mersenne exponent, both cost about the same: $\widetilde{O}(r^2)$.

Byproducts

The trinomial hunt has resulted in improved software for operations on polynomials over $\mathrm{GF}(2)$, and has shown that the best algorithms in theory are not always the best in practice.

It has also provided a large database of factors of trinomials over $\mathrm{GF}(2)$, leading to several interesting conjectures which are a topic for future research.

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