Appendix B
Paul Erdős

Working with Paul Erdős was like taking a walk in the hills. Every time when I thought that we had achieved our goal and deserved a rest, Paul pointed to the top of another hill and off we would go.

– Fan Chung

B.1 PAPERS

Paul Erdős was the most prolific mathematician of the twentieth century, with over 1500 written papers and more than 490 collaborators. This highly subjective list gives only some of the papers that created and shaped the subject matter of this volume. **MR** and **Zbl.** refer to reviews in *Math Reviews* and *Zentralblatt*, respectively. Chapter and section reference are to pertinent areas of this volume.

- A combinatorial problem in geometry, *Compositio Math* **2** (1935), 463–470 (with George Szekeres); **Zbl.** 12, 270.
  Written when Erdős was still a teenager this gem contains a rediscovery of Ramsey's Theorem and the Monotone Subsequence Theorem. Many authors have written that this paper played a key role in moving Erdős toward a more combinatorial view of mathematics.

- Some remarks on the theory of graphs, *Bull. Am. Math. Soc.* **53** (1947), 292–294, **MR** 8# 479d; **Zbl** 32, 192.
  The three-page paper that "started" the probabilistic method, giving an exponential lower bound on the Ramsey number $R(k,k)$. Section 1.1.

- The Gaussian law of errors in the theory of additive number theoretic functions, *Am. J. Math.* **62** (1940), 738–742 (with Mark Kac); **MR** 2# 42c; **Zbl.** 24, 102.
  Showing that the number of prime factors of x chosen uniformly from 1 to n has an asymptotically normal distribution. A connection between probability and number theory that was extraordinary for its time. Section 4.2.

- Problems and results in additive number theory, *Colloque sur la Théorie des Nombres, Bruxelles, 1955*, 127–137, George Thone, Liège; Masson and Cie, Paris, 1956; **MR** 18# 18a; **Zbl.** 73, 31.
  Using random subsets to prove the existence of a set of integers such that every n is represented $n = x + y$ at least once but at most $c \ln n$ times. Resolving a problem Sidon posed to Erdős in the 1930s. This problem continued to fascinate Erdős: see, e.g., Erdős and Tetali (1990). Section 8.6.

- On a combinatorial problem, *Nordisk Mat. Tidsskr.* **11** (1963), 220–223; **MR** 28# 4068; **Zbl.** 122, 248.
  On a combinatorial problem II, *Acta Math. Acad. Sci. Hung.* **15** (1964), 445–447; **MR** 29# 4700; **Zbl.** 201, 337.
  Property B. Probabilistic proofs that any $m < 2^{n-1}$ n-sets can be two-colored with no set monochromatic yet there exist $cn^2 2^n$ n-sets that cannot be so colored. Section 1.3.

- On the evolution of random graphs, *Magyar. Tud. Akad. Mat. Kutató Int. Közl.* **5** (1960), 17–61 (with Alfred Rényi); **MR** 23# A2338; **Zbl.** 103, 163.
  Rarely in mathematics can an entire subject be traced to one paper. For random graphs this is the paper. Chapter 10.

- Graph theory and probability, *Can. J. Math.* **11** (1959), 34–38; **MR** 21# 876; **Zbl.** 84, 396.
  Proving by probabilistic methods the existence of graphs with arbitrarily high girth and chromatic number. This paper convinced many of the power of the methodology, as the problem had received much attention but no construction had been found. The Probabilistic Lens: High Girth and High Chromatic Number, following Chapter 3.

- Graph theory and probability II, *Can. J. Math.* **13** (1961), 346–352; **MR** 22# 10925; **Zbl.** 97, 391.
  Showing the existence of a triangle-free graph on n vertices with no independent set of size $cn^{1/2} \ln n$ vertices, and hence that the Ramsey number $R(3,k) = \Omega(k^2 \ln^{-2} k)$. A technical *tour de force* that uses probabilistic methods in a very subtle way, particularly considering the early date of publication.

- On circuits and subgraphs of chromatic graphs, *Mathematika* **9** (1962), 170–175; **MR** 25 # 3035; **Zbl.** 109, 165.
  Destroying the notion that chromatic number is necessarily a local property, Erdős proves the existence of a graph on n vertices that cannot be k-colored but for which every εn vertices can be three-colored. The Probabilistic Lens: Local Coloring, following Chapter 8.

- On a combinatorial game, *J. Combin. Theory, Ser. A* **14** (1973), 298–301 (with John Selfridge); **MR** 48# 5655; **Zbl.** 293, 05004.
  Players alternate turns selecting vertices and the second player tries to stop the first from getting a winning set. The weight function method used was basically probabilistic and was an early use of derandomization. Section 16.1.

B.2 CONJECTURES

Conjectures were always an essential part of the mathematical life of Paul Erdős. Here are some of our favorites.

- Do sets of integers of positive density necessarily contain arithmetic progressions of arbitrary length? In finite form, is there for all k and all $\varepsilon > 0$, an n_0 so that if $n \geq n_0$ and S is a subset of the first n integers of size at least εn then S necessarily contains an arithmetic progression of length k? This conjecture was first made by Paul Erdős and Paul Turán in the 1930s. It was solved (positively) by Szemerédi in the 1970s. Let $F(k,\varepsilon)$ denote the minimal n_0 that suffices above. The growth rate of F remains an intriguing question with very recent results due to Gowers.

- Call distinct S, T, U a Δ-system if $S \cap T = S \cap U = T \cap U$. Let $F(n)$ be the minimal m such that given any m n-sets some three form a Δ-system. Erdős and Rado showed that $F(n)$ exists and gave the upper bound $F(n) \leq 2^n n!$. Erdős conjectured that $F(n) < C^n$ for some constant C.

- What are the asymptotics of the Ramsey function $R(k,k)$? In particular, what is the value c (if it exists) of $\lim_k R(k,k)^{1/k}$? The classic 1947 paper of Erdős gives $c \geq \sqrt{2}$ and $c \leq 4$ follows from the proof of Ramsey's Theorem but a half-century has seen no further improvements in c, though there have been some results on lower order terms.

- Write $r_S(n)$ for the number of solutions to the equation $n = x + y$ with $x, y \in S$. Does there exist a set S of positive integers such that $r_S(n) > 0$ for all but finitely many n yet $r_S(n)$ is bounded by some constant K? The 1955 paper of Erdős referenced above gives S with $r_S(n) = \Theta(\ln n)$.

- Let $m(n)$, as defined in Section 1.3, denote the minimal size of a family of n-sets that cannot be two-colored without forming a monochromatic set. What are the asymptotics of $m(n)$? In 1963 and 1964 Erdős found the bounds $\Omega(2^n) \leq$

$m(n) = O(2^n n^2)$ and the lower bound of Radhakrishnan and Srinivasan, shown in Section 3.5, is now $\Omega(2^n (n/\ln n)^{1/2})$.

- Given $2^{n-2}+1$ points in the plane, no three on a line, must some n of them form a convex set? This conjecture dates back to the 1935 paper of Erdős and Szekeres referenced above.

- Let $m(n,k,l)$ denote the size of the largest family of k-element subsets of an n-set such that no l-set is contained in more than one of them. Simple counting gives $m(n,k,l) \le \binom{n}{l}/\binom{k}{l}$. Erdős and Hanani conjectured in 1963 that for fixed $l < k$ this bound is asymptotically correct; that is, the ratio of $m(n,k,l)$ to $\binom{n}{l}/\binom{k}{l}$ goes to one as $n \to \infty$. Erdős had a remarkable ability to select problems that were very difficult but not impossible. This conjecture was settled affirmatively by Vojtech Rödl in 1985, as discussed in Section 4.7. The asymptotics of the difference $\binom{n}{l}/\binom{k}{l} - m(n,k,l)$ remains open.

B.3 ON ERDŐS

There have been numerous books and papers written about the life and mathematics of Paul Erdős. Three deserving particular mention are:

- *The Mathematics of Paul Erdős* (Ron Graham and Jarik Nešetřil, eds.), Springer-Verlag, Berlin, 1996 (Vols. I and II).

- *Combinatorics, Paul Erdős Is Eighty* (D. Miklós, V. T. Sós, T. Szönyi, eds.), Bolyai Soc. Math. Studies, Vol. I (1990) and Vol. II (1993).

- *Erdős on Graphs — His Legacy of Unsolved Problems*, Fan Chung and Ron Graham, A.K. Peters, 1998.

Of the many papers by mathematicians we note the following:

- László Babai, In and out of Hungary: Paul Erdős, his friends, and times. In *Combinatorics, Paul Erdős Is Eighty* (listed above), Vol. II, 7–93.

- Béla Bollobás, Paul Erdős — Life and work, in *The Mathematics of Paul Erdős* (listed above), Vol. II, 1–42.

- A. Hajnal, Paul Erdős' Set theory, in *The Mathematics of Paul Erdős* (listed above), Vol. II, 352–393.

- János Pach, Two places at once: a remembrance of Paul Erdős, *Math Intelligencer*, Vol. 19 (1997), no. 2, 38–48.

Two popular biographies of Erdős have appeared:

- *The Man Who Loved Only Numbers*, Paul Hoffman, Hyperion, New York, 1998.

- *My Brain Is Open — The Mathematical Journies of Paul Erdős*, Bruce Schechter, Simon & Schuster, New York, 1998.

Finally, George Csicsery has made a documentary film, *N Is a Number, A Portrait of Paul Erdős*, available from the publishers A. K. Peters, which allows one to see and hear Erdős in lecture and among friends, proving and conjecturing.

B.4 UNCLE PAUL

Paul Erdős died in September 1996 at the age of 83. His theorems and conjectures permeate this volume. This tribute,[1] given by Joel Spencer at the National Meeting of the American Mathematical Society in January 1997, attempts to convey some of the special spirit that we and countless others took from this extraordinary man.

Paul Erdős was a searcher, a searcher for mathematical truth.

Paul's place in the mathematical pantheon will be a matter of strong debate for in that rarefied atmosphere he had a unique style. The late Ernst Straus said it best, in a commemoration of Erdős' seventieth birthday.

> In our century, in which mathematics is so strongly dominated by "theory constructors" he has remained the prince of problem solvers and the absolute monarch of problem posers. One of my friends — a great mathematician in his own right — complained to me that "Erdős only gives us corollaries of the great metatheorems which remain unformulated in the back of his mind." I think there is much truth to that observation but I don't agree that it would have been either feasible or desirable for Erdős to stop producing corollaries and concentrate on the formulation of his metatheorems. In many ways Paul Erdős is the Euler of our times. Just as the "special" problems that Euler solved pointed the way to analytic and algebraic number theory, topology, combinatorics, function spaces, etc.; so the methods and results of Erdős' work already let us see the outline of great new disciplines, such as combinatorial and probabilistic number theory, combinatorial geometry, probabilistic and transfinite combinatorics and graph theory, as well as many more yet to arise from his ideas.

Straus, who worked as an assistant to Albert Einstein, noted that Einstein chose physics over mathematics because he feared that one would waste one's powers in pursuing the many beautiful and attractive questions of mathematics without finding the central questions. Straus goes on,

> Erdős has consistently and successfully violated every one of Einstein's prescriptions. He has succumbed to the seduction of every beautiful problem he has encountered — and a great many have succumbed to him. This just proves to me that in the search for truth there is room for Don Juans like Erdős and Sir Galahads like Einstein.

[1] Reprinted with permission from the *Bulletin of the American Mathematical Society*.

I believe, and I'm certainly most prejudiced on this score, that Paul's legacy will be strongest in Discrete Math. Paul's interest in this area dates back to a marvellous paper with George Szekeres in 1935 but it was after World War II that it really flourished. The rise of the Discrete over the past half century has, I feel, two main causes. The first was The Computer, how wonderful that this physical object has led to such intriguing mathematical questions. The second, with due respect to the many others, was the constant attention of Paul Erdős with his famous admonition "Prove and Conjecture!" Ramsey Theory, Extremal Graph Theory, Random Graphs, how many turrets in our mathematical castle were built one brick at a time with Paul's theorems and, equally important, his frequent and always penetrating conjectures.

My own research specialty, The Probabilistic Method, could surely be called The Erdős Method. It was begun in 1947 with a three page paper in the *Bulletin of the American Math Society*. Paul proved the existence of a graph having certain Ramsey property without actually constructing it. In modern language he showed that an appropriately defined random graph would have the property with positive probability and hence there must exist a graph with the property. For the next twenty years Paul was a "voice in the wilderness," his colleagues admired his amazing results but adaption of the methodology was slow. But Paul persevered — he was always driven by his personal sense of mathematical aesthetics in which he had supreme confidence — and today the method is widely used in both Discrete Math and in Theoretical Computer Science.

There is no dispute over Paul's contribution to the spirit of mathematics. Paul Erdős was the most inspirational man I have ever met. I began working with Paul in the late 1960s, a tumultuous time when "do your own thing" was the admonition that resonated so powerfully. But while others spoke of it, this was Paul's modus operandi. He had no job; he worked constantly. He had no home; the world was his home. Possessions were a nuisance, money a bore. He lived on a web of trust, travelling ceaselessly from Center to Center, spreading his mathematical pollen.

What drew so many of us into his circle? What explains the joy we have in speaking of this gentle man? Why do we love to tell Erdős stories? I've thought a great deal about this and I think it comes down to a matter of belief, or faith. We mathematicians know the beauties of our subject and we hold a belief in its transcendent quality. God created the integers, the rest is the work of Man. Mathematical truth is immutable, it lies outside physical reality. When we show, for example, that two nth powers never add to an nth power for $n \geq 3$ we have discovered a Truth. This is our belief, this is our core motivating force. Yet our attempts to describe this belief to our nonmathematical friends are akin to describing the Almighty to an atheist. Paul embodied this belief in mathematical truth. His enormous talents and energies were given entirely to the Temple of Mathematics. He harbored no doubts about the importance, the absoluteness, of his quest. To see his faith was to be given faith. The religious world might better have understood Paul's special personal qualities. We knew him as Uncle Paul.

I do hope that one cornerstone of Paul's, if you will, theology will long survive. I refer to The Book. The Book consists of all the theorems of mathematics. For each theorem there is in The Book just one proof. It is the most aesthetic proof, the most insightful proof, what Paul called The Book Proof. And when one of Paul's myriad conjectures was resolved in an "ugly" way Paul would be very happy in congratulating the prover but would add, "Now, let's look for The Book Proof." This platonic ideal spoke strongly to those of us in his circle. The mathematics was there, we had only to discover it.

The intensity and the selflessness of the search for truth were described by the writer Jorge Luis Borges in his story "The Library of Babel." The narrator is a worker in this library which contains on its infinite shelves all wisdom. He wanders its infinite corridors in search of what Paul Erdős might have called The Book. He cries out,

> To me, it does not seem unlikely that on some shelf of the universe there lies a total book. I pray the unknown gods that some man — even if only one man, and though it have been thousands of years ago! — may have examined and read it. If honor and wisdom and happiness are not for me, let them be for others. May heaven exist though my place be in hell. Let me be outraged and annihilated but may Thy enormous Library be justified, for one instant, in one being.

In the summer of 1985 I drove Paul to what many of us fondly remember as Yellow Pig Camp — a mathematics camp for talented high school students at Hampshire College. It was a beautiful day — the students loved Uncle Paul and Paul enjoyed nothing more than the company of eager young minds. In my introduction to his lecture I discussed The Book but I made the mistake of describing it as being "held by God." Paul began his lecture with a gentle correction that I shall never forget. "You don't have to believe in God," he said, "but you should believe in The Book."