FAST SOLUTION OF TOEPLITZ SYSTEMS OF EQUATIONS AND COMPUTATION OF PADÉ APPROXIMANTS

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Abstract

We present two new algorithms, ADT and MDT, for solving order-n Toeplitz systems of linear equations $Tz = b$ in time $O(n \log^2 n)$ and space $O(n)$. The fastest algorithms previously known, such as Trench's algorithm, require time $\Omega(n^2)$ and require that all principal submatrices of T be nonsingular. Our algorithm ADT requires only that T be nonsingular.

Both our algorithms for Toeplitz systems are derived from algorithms for computing entries in the Padé table for a given power series. We prove that entries in the Padé table can be computed by the Extended Euclidean Algorithm. We describe an algorithm EMGCD (Extended Middle Greatest Common Divisor) which is faster than the algorithm HGCD of Aho, Hopcroft and Ullman, although both require time $O(n \log^2 n)$, and we generalize EMGCD to produce PRSDC (Polynomial Remainder Sequence Divide and Conquer) which produces any iterate in the PRS, not just the middle term, in time $O(n \log^2 n)$. Applying PRSDC to the polynomials $U_0(x) = x^{2n+1}$ and $U_1(x) = a_0 + a_1 x + \cdots + a_{2n} x^{2n}$ gives algorithm AD (Anti-Diagonal), which computes any (m, p) entry along the antidiagonal $m + p = 2n$ of the Padé table for U_1 in time $O(n \log^2 n)$.

Our other algorithm, MD (Main-Diagonal), computes any diagonal entry (n, n) in the Padé table for a normal power series, also in time $O(n \log^2 n)$. MD is related to Schönhage's fast continued fraction algorithm. A Toeplitz matrix T is naturally associated with U_1, and the (n, n) Padé approximation to U_1 gives the first column of T^{-1}. Thus, the Padé table algorithms AD and MD give $O(n \log^2 n)$ Toeplitz algorithms ADT and MDT. Trench's formula breaks down in certain degenerate cases, but in such cases a companion formula, the discrete analog of the Christoffel-Darboux formula, is valid and may be used to compute z in time $O(n \log^2 n)$ via the fast computation (by algorithm AD) of at most four Padé approximants.

We also apply our results to obtain new complexity bounds for the solution of banded Toeplitz systems and for BCH decoding via Berlekamp's algorithm.

Comments

Only the Abstract is given here. The full paper appeared as [1]. For related work and more recent references, see [2, 3, 4].

1991 *Mathematics Subject Classification.* Primary 65Y20; Secondary 41A21, 47B35, 68Q25, 94B35.

Key words and phrases. Toeplitz system, Padé approximant, Padé table, Euclidean algorithm, polynomial remainder sequence, continued fraction, Trench's algorithm, BCH decoding, Berlekamp's algorithm, fast algorithm, complexity.

Received December 22, 1979; revised April 20, 1980.

rpb059a typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

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