Estimation of sums over zeros of the Riemann zeta-function

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Abstract

Consider sums of the form $\sum \phi(\gamma)$ where ϕ is a given function and γ ranges over the ordinates of nontrivial zeros of the Riemann zeta-function in a given interval. We show how the numerical estimation of such sums can be accelerated, improving in many cases on a well-known lemma of Lehman (1966), and give an example involving an analogue of the harmonic series.

This is joint work with Dave Platt and Tim Trudgian.

For a preprint, see `https://arxiv.org/abs/2009.13791`.

Motivation

- In analytic number theory we often encounter sums of the form $\sum \phi(\gamma)$ where the sum is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, perhaps restricted to some interval $[T_1, T_2]$ or $[T_1, \infty)$.
- For example, consider $\sum_{0<\gamma\leqslant T} 1/\gamma^2$. In some applications it is sufficient to know that the sum converges as $T \to \infty$. In other applications, especially when obtaining "explicit" bounds, we may need numerical upper and lower bounds on the sum (for specific values of T, or as $T \to \infty$).
- Similarly for $\sum_{0<\gamma\leqslant T} 1/\gamma$, except that here the sum diverges as $T \to \infty$, and we may want bounds on its rate of growth.

Some notation

$\mathcal{F}$ is the set of positive γ, where $\rho = \beta + i\gamma$ is a non-trivial zero of $\zeta(s)$.

If $0 < T \notin \mathcal{F}$, then $N(T)$ is #zeros with $0 < \gamma \leqslant T$ and $S(T) = \pi^{-1} \arg \zeta(\frac{1}{2} + iT)$ defined in the usual way.

If $T \in \mathcal{F}$ then $N(T) = \lim_{\varepsilon \to 0} \frac{N(T+\varepsilon)+N(T-\varepsilon)}{2}$, and similarly for $S(T)$.

$\sum'_{T_1 \leqslant \gamma \leqslant T_2} \phi(\gamma)$ indicates that if $\gamma = T_1$ or $\gamma = T_2$ then the term $\phi(\gamma)$ is given weight $\frac{1}{2}$.

Some useful results

In Titchmarsh, Ch. 9, we find $N(T) = L(T) + Q(T)$, where

$$L(T) = \frac{T}{2\pi}\left(\log\left(\frac{T}{2\pi}\right) - 1\right) + \frac{7}{8}$$

and the "remainder term" $Q(T) = S(T) + O(1/T)$.

More precisely, we can prove that, for all $t \geqslant 2\pi$,

$$|Q(t) - S(t)| \leqslant \frac{1}{150t}\,.$$

It is known that $S(T) \ll \log T$, so $Q(T) \ll \log T$.

Also, if $S_1(T) := \int_0^T S(t)\,dt$, then $S_1(T) \ll \log T$.

Explicit bounds on $S(T)$ and $S_1(T)$ are known.

Lehman's Lemma – motivation

We want to estimate sums of the form

$$\sum_{T_1 \leqslant \gamma \leqslant T_2}{}' \phi(\gamma) \, .$$

We can think of this as a Riemann sum approximating

$$\int_{T_1}^{T_2} \phi(t) w(t) \, dt,$$

where $w(t)$ is a weight function that takes into account the non-uniform spacing of the γs. The natural weight function is

$$w(t) := L'(t) = \frac{1}{2\pi} \log(t/2\pi) \, .$$

Lehman's lemma bounds the error (the difference between the sum and integral) if we use this weight function.

Lehman's Lemma

Lemma (Lehman, 1966)

If $2\pi e \leqslant T_1 \leqslant T_2$ and $\phi : [T_1, T_2] \mapsto [0, \infty)$ is monotone decreasing on $[T_1, T_2]$, and

$$E(T_1, T_2) := \sideset{}{'}\sum_{T_1 \leqslant \gamma \leqslant T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi)\, dt,$$

then

$$|E(T_1, T_2)| \leqslant A \left(2\phi(T_1) \log T_1 + \int_{T_1}^{T_2} \frac{\phi(t)}{t}\, dt \right),$$

where A is an absolute constant.

Remark 1: We may take $A = 0.28$.

Remark 2: If $\int_{T_1}^{\infty} \phi(t)/t\, dt < \infty$, we can allow $T_2 \to \infty$.

An assumption

Assumption: From now on we assume that $\phi(t)$ is in $C^2[T_0, \infty)$ and satisfies $\phi'(t) \leqslant 0$ and $\phi''(t) \geqslant 0$ on $[T_0, \infty)$.

These conditions are stronger than those assumed in Lehman's Lemma.

In most applications $\phi(t)$ is in $C^\infty[T_0, \infty)$. Thus, essentially the only new condition is that $\phi''(t) \geqslant 0$.

Our lemma

Lemma (BPT, 2020)

If $2\pi \leqslant T_0 \leqslant T_1 \leqslant T_2$ *and*

$$E(T_1, T_2) := \sideset{}{'}\sum_{T_1 \leqslant \gamma \leqslant T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi)\, dt\,,$$

(as in Lehman's lemma), then

$$E(T_1, T_2) = \phi(T_2)Q(T_2) - \phi(T_1)Q(T_1) + E_2(T_1, T_2), \textit{ where}$$

$$E_2(T_1, T_2) = -\int_{T_1}^{T_2} \phi'(t)Q(t)\, dt, \textit{ and}$$

$$|E_2(T_1, T_2)| \leqslant 2(A_0 + A_1 \log T_1)\, |\phi'(T_1)| + (A_1 + A_2)\phi(T_1)/T_1\,.$$

Remark 3: We may take $A_0 = 2.067$, $A_1 = 0.059$, $A_2 = 0.007$.

Idea of the proof

Write the sum as a Stieltjes integral involving $dN(t) = dL(t) + dQ(t)$, then use integration by parts to obtain the first expression for E_2 as an integral involving $\phi'(t)Q(t)$ (so far this is as in the proof of Lehman's Lemma).

Replace $Q(t)$ by $S(t)$ in the integral, and bound the error introduced, using $Q(t) - S(t) \ll 1/t$.

Use integration by parts again to obtain an integral involving $\phi''(t)S_1(t)$, and bound this integral using an explicit bound on $S_1(t)$. This gives a bound involving integrals of $\phi''(t)$ and $\phi''(t)\log t$.

Finally, use integration by parts once again, to avoid expressions involving $\phi''(t)$, and simplify.

It is interesting to note that some terms involving T_2 cancel. This also occurs in the proof of Lehman's lemma.

Corollary – convergent sums

Theorem (BPT, 2020)

Suppose that $2\pi \leqslant T_0 \leqslant T$ *and* $\int_T^\infty \phi(t)\log(t/2\pi)\,dt < \infty$. *Let*

$$E(T) := {\sum_{T \leqslant \gamma}}' \phi(\gamma) - \frac{1}{2\pi}\int_T^\infty \phi(t)\log(t/2\pi)\,dt\,.$$

Then $E(T) = -\phi(T)Q(T) + E_2(T)$, *where*

$$E_2(T) = -\int_T^\infty \phi'(t)Q(t)\,dt$$

and

$$|E_2(T)| \leqslant 2(A_0 + A_1\log T)\,|\phi'(T)| + (A_1 + A_2)\phi(T)/T.$$

Proof.

Let $T_2 \to \infty$ in our lemma, and replace T_1 by T. □

Example

Consider the convergent sum $c_1 := \sum_{\gamma>0} 1/\gamma^2$.
A first approximation is the sum over $0 < \gamma \leqslant T$ (a finite sum involving $\ll T \log T$ terms).
The error is $\sum_{\gamma>T} 1/\gamma^2 \sim \log(T)/2\pi T$.

We can do better by using Lehman's lemma with $(T_1, T_2) \to (T, \infty)$. This gives

$$\sum_{\gamma \geqslant T}{}' 1/\gamma^2 = \frac{1}{2\pi} \int_T^\infty \frac{\log t/2\pi}{t^2}\, dt + E(T),$$

where $|E(T)| \leqslant 0.28(0.5 + 2\log T)/T^2$.

Using integration by parts, the integral here is $\dfrac{1 + \log(T/2\pi)}{2\pi T}$.

Thus, using Lehman's lemma decreases the error bound by a factor of order T, from $O(\log(T)/T)$ to $O(\log(T)/T^2)$.

Example continued

If we use our theorem instead of Lehman's lemma, the error term becomes $E_2(T)$, where

$$|E_2(T)| \leqslant \frac{8.334 + 0.236 \log T}{T^3} .$$

Thus, we get another factor of order T, from $O(\log(T)/T^2)$ to $O(\log(T)/T^3)$.

For example, taking $T = 1000$ (corresponding to the first 649 nontrivial zeros), we have the following error bounds.

- Naive truncation of series: 9.7×10^{-4}.
- Using Lehman's lemma: 4.009×10^{-6}.
- Using our theorem: 9.965×10^{-9}.

The improvement over Lehman's lemma is a factor of 400.

Example continued

Corollary (BPT, 2020)

$c_1 = 0.023104993115418970788933810 4 + \vartheta(5 \times 10^{-28})$, *where* $|\vartheta| \leqslant 1$.

Proof.

This follows from our theorem by an interval-arithmetic computation using the first $n = 10^{10}$ zeros, with $T = 3293531632.542\cdots \in (\gamma_n, \gamma_{n+1})$. $\square$

Divergent sums – existence of a limit $F(T_0)$

We can handle certain divergent sums in much the same way as convergent sums.

Theorem (BPT, 2020)

Suppose that $T_0 \geqslant 2\pi$, *and*

$$\int_{T_0}^{\infty} \frac{\phi(t)}{t}\,dt < \infty.$$

Then there exists

$$F(T_0) := \lim_{T\to\infty}\left(\sum_{T_0\leqslant\gamma\leqslant T}{}' \phi(\gamma) - \frac{1}{2\pi}\int_{T_0}^{T} \phi(t)\log(t/2\pi)\,dt\right),$$

and

$$F(T_0) = -\phi(T_0)Q(T_0) - \int_{T_0}^{\infty} \phi'(t)Q(t)\,dt.$$

Example – a "harmonic" series

Corollary (BPT, 2020)

Let $G(T) := \sum'_{0<\gamma\leqslant T} 1/\gamma$. *Then there exists*

$$H := \lim_{T\to\infty}\left(G(T) - \frac{\log^2(T/2\pi)}{4\pi}\right)$$

and

$$H = \int_{2\pi}^{\infty} \frac{Q(t)}{t^2}\,dt - \frac{1}{16\pi}\,.$$

Proof.

Take $\phi(t) = 1/t$ and $T_0 = 2\pi$ in our Theorem, and observe that $Q(2\pi) = 1/8$. □

Example continued

Remark 1. The definition of H is analogous to the usual definition of Euler's constant C:

$$C := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right),$$

and the expression for H is analogous to the result

$$C = 1 - \int_1^\infty \frac{x - \lfloor x \rfloor}{x^2}\,dx\,.$$

Remark 2. Hassani (2016) asked about the existence of the limit H. Hassani and several other authors gave bounds on $G(T)$, but did not prove the existence of H.

Remark 3. An independent proof of the existence of H uses only integration by parts (as in the proof of Lehman's lemma) and the fact that $Q(t) \ll \log t$. For details, see Theorem 1 of https://arxiv.org/abs/2009.05251.

Divergent sums – approximation of $F(T_0)$

Theorem (BPT, 2020)

Suppose that $2\pi \leqslant T_0 \leqslant T_1$ *and that* $\phi(t)$ *and* $F(T_0)$ *are as in the previous theorem. Then*

$$F(T_0) = {\sum_{T_0 \leqslant \gamma \leqslant T_1}}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T_1} \phi(t) \log(t/2\pi)\, dt - \phi(T_1) Q(T_1) + E_2(T_1),$$

where

$$|E_2(T_1)| \leqslant 2(A_0 + A_1 \log T_1)\, |\phi'(T_1)| + (A_1 + A_2)\phi(T_1)/T_1.$$

Remark. The bound on $|E_2|$ is the same as in the convergent case. This is not surprising, since both results depend on our Lemma.

Example – approximation of H

Suppose we wish to approximate the constant H by summing over $\gamma \in (0, T])$. Lehman's lemma gives

$$H = G(T) - \frac{\log^2(T/2\pi)}{4\pi} + A\vartheta\left(\frac{2\log T + 1}{T}\right),$$

where $|\vartheta| \leqslant 1$, and we can take $A = 0.28$.

However, we can do better by a factor of order T. Using the theorem above, we obtain

$$H = G(T) - \frac{\log^2(T/2\pi)}{4\pi} - \frac{Q(T)}{T} + E_2(T),$$

where

$$|E_2(T)| \leqslant \frac{4.2 + 0.12\log T}{T^2}.$$

Numerical approximation of H

Corollary

Let H be defined as above. Then

$$H = -0.017159404307098149 5 + \vartheta(10^{-18}),$$

where $|\vartheta| \leqslant 1$.

Proof.

This follows from the method on the previous slide, using an interval-arithmetic computation using the first $n = 10^{10}$ zeros, with $T = \gamma_n \approx 3293531632.4$. $\square$

References

R. P. Brent, D. J. Platt, and T. S. Trudgian,
On a harmonic sum over non-trivial zeros of the Riemann zeta-function, *Bull. Austral. Math. Soc.*, online 20 Nov. 2020. Also `https://arxiv.org/abs/2009.05251`.

R. P. Brent, D. J. Platt, and T. S. Trudgian,
Accurate estimation of sums over zeros of the Riemann zeta-function, `https://arxiv.org/abs/2009.13791`.

M. Hassani, Explicit approximation of the sums over the imaginary part of the non-trivial zeros of the Riemann zeta function, *Appl. Math. E-Notes* **16**:109–116, 2016.

R. S. Lehman, On the difference $\pi(x) - \mathrm{li}(x)$, *Acta Arith.* **11**:397–410, 1966.

E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed. (edited by D. R. Heath-Brown), Oxford, 1986.