11 [7.15, 9.10].—F. D. CRARY & J. B. ROSSER, *High Precision Coefficients Related to the Zeta Function,* MRC Technical Summary Report #1344, Univ. of Wisconsin, Madison, May 1975, 64 pp. text and 107 pp. tables.

The Riemann-Siegel formula [4], [5], [7] is the most efficient procedure known for the computation of the Riemann zeta function on the critical line $R(s) = ½$. This report gives accurate coefficients for the first seven terms in the Riemann-Siegel formula.

Let $s = ½ + it$, $t = 2\pi\tau > 0$, $m = \lfloor\tau^{½}\rfloor$, $z = 2(\tau^{½} - m) - 1$, $n \geq -1$, $\theta(t) = \arg[\pi^{-½it}\Gamma(¼ + ½it)]$, and $Z(t) = \exp(i\theta(t))\zeta(s)$. It is known that $Z(t)$ is real, so a search for zeros of $\zeta(s)$ on the critical line reduces to a search for changes of sign of $Z(t)$. The Riemann-Siegel asymptotic expansion for $Z(t)$ is

$$Z(t) = \sum_{k=1}^{m} 2k^{-½}\cos(t \cdot \ln(k) - \theta(t)) - R(t),$$

where

$$R(t) = (-1)^m\tau^{-¼} \sum_{j=0}^{n} \Phi_j(z)(-1)^j\tau^{-½j} + O(\tau^{-(2n+3)/4}) \tag{1}$$

as $\tau \longrightarrow \infty$, and the $\Phi_j(z)$ are certain entire functions which may be expressed in terms of the derivatives of

$$\Phi_0(z) = \Phi(z) = \cos[\pi(4z^2 + 3)/8]/\cos(\pi z).$$

Expressions for Φ_1 to Φ_4 were given in [3]. The present report gives the derivation of (1) and expressions for Φ_1 to Φ_8. These have been verified by the reviewer, using the representation

$$\Phi_j(z) = \sum_{0 \leq k \leq 3j/4} \frac{r_{k,j}\Phi^{(3j-4k)}(z)}{4^j\pi^{2(j-k)}(3j-4k)!},$$

where

$$r_{k,j} = \begin{cases} \rho_{j/4} & \text{if } 3j = 4k \geq 0, \\ 0 & \text{if } k < 0 \text{ or } 3j < 4k, \\ r_{k-1,j-1} + (3j-4k-1)(3j-4k-2)r_{k,j-1} & \text{otherwise,} \end{cases}$$

and ρ_0 to ρ_{12} are given in [1] ($\rho_0 = 1$, $\rho_1 = 2$, $\rho_2 = 82$, $\rho_3 = 10572$, $\rho_4 = 2860662$, etc.).

For computational purposes the explicit expressions for $\Phi_j(z)$ are unwieldy. Much more convenient are the Taylor series coefficients $c_{j,k}$ defined by $\Phi_j(z) = \sum_{k=0}^{\infty} c_{j,k}z^k$. (Note that $c_{j,k} = 0$ if $j + k$ is odd.) Since $\Phi_j(z)$ is entire, $|c_{j,k}|$ decreases rapidly as k increases. Haselgrove and Miller [3] gave $c_{j,k}$ for $j = 0(1)4$ with an accuracy ranging from almost 20D (for $c_{0,k}$) to almost 11D (for $c_{4,k}$). The present table gives $c_{j,k}$ to 70D for $j = 0(1)6$ and $k = 0(1)(100 - 3j)$. The entries are correctly rounded to 70D (verified by the reviewer by comparison with [1]). The range includes all $c_{j,k}$ with $j \leq 6$ and $|c_{j,k}| > 10^{-41}$: the largest $|c_{j,k}|$ omitted is $c_{6,84} = 1.155 \times 10^{-42}$. Some auxiliary quantities (not checked by the reviewer) are also tabulated to 70D. To compensate for cancellation the computation was carried out, using the equivalent of 157 decimal digits, with Crary's multiple-precision package [2].

The report under review does not give any rigorous bounds for the error in (1).

(A subsequent report is promised.) However, we can estimate the accuracy obtainable by using (1) with $n = 6$ and the 70D tables of $c_{j,k}$. Let $M_j = \max_{-1 \leq z \leq 1} |\Phi_j(z)|$. From [1] (which gives M_j for $j = 0(1)64$) we have $M_7 < 2 \times 10^{-5}$ and $M_8 < 3 \times 10^{-6}$. Thus, so long as τ is large enough for exponentially decreasing terms to be negilgible, the error in the computed $R(t)$ should be bounded by $A\tau^{-15/4} + B$, where A is of order 3×10^{-5} and B of order 10^{-40}. For $\tau \simeq 1000$ this gives an accuracy of about 16D. To obtain greater accuracy, more terms (given in [1]) could be used in (1), or the Euler-Maclaurin formula could be used [4], [5]. For $\tau \simeq 3 \times 10^5$ (at the limit of the computation of [6]) an accuracy of about 25D is obtainable, and this should be more than enough for most applications.

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1. R. P. BRENT, *Numerical Investigation of the Riemann-Siegel Approximation*, manuscript, 1976.

2. F. D. CRARY, *Multiple-Precision Arithmetic Design With an Implementation on the Univac* 1108, MRC Technical Summary Report #1123, Univ. of Wisconsin, Madison, May 1971.

3. C. B. HASELGROVE in collaboration with J. C. P. MILLER, *Tables of the Riemann Zeta Function*, Roy. Soc. Math. Tables No. 6, Cambridge Univ. Press, New York, 1960; RMT **6**, *Math. Comp.*, v. 15, 1961, pp. 84–86. MR **22** #8679.

4. D. H. LEHMER, "Extended computation of the Riemann zeta-function," *Mathematika*, v. 3, 1956, pp. 102–108; RMT **108**, *MTAC*, v. 11, 1957, p. 273. MR 19, p. 121.

5. D. H. LEHMER, "On the roots of the Riemann zeta-function," *Acta Math.*, v. 95, 1956, pp. 291–298; RMT **52**, *MTAC*, v. 11, 1957, pp. 107–108. MR **19**, p. 121.

6. J. B. ROSSER, J. M. YOHE & L. SCHOENFELD, *Rigourous Computation and the Zeros of the Riemann Zeta-Function*, Information Processing 68 (Proc. IFIP Congress, Edinburgh, 1968), v. 1: Mathematics, Software; North-Holland, Amsterdam, 1969, pp. 70–76. MR **41** #2892.

7. C. L. SIEGEL, *Über Riemanns Nachlass zur analytischen Zahlentheorie*, Quellen Studien zur Geschichte der Math. Astron. und Phys. Abt. B: Studien 2, 1932, pp. 45–80. (Also in "Gesammelte Abhandlungen", v. 1, Springer-Verlag, New York, 1966.)

12 [13.20].—OVE SKOVGAARD, IB A. SVENDSEN, IVAR G. JONSSON & OLE BRINK-KJAER, *Sinusoidal and Cnoidal Gravity Waves—Formulae and Tables*, Institute of Hydrodynamics and Hydraulic Engineering, Technical University of Denmark, Lyngby, Denmark, 1974, 8 pp., 21 cm. Price Dkr. 5.

This is an eight-page fold-out booklet made of plastic-covered cardboard. It contains basic formulas derived from linear theory (sinusoidal) and from nonlinear theory (cnoidal) pertaining to progressive surface water waves. The formulas provide expressions to calculate various water wave properties such as phase velocity, group velocity, mean energy density, and pressure. Evaluation of the formulas requires the use of tables of complete elliptic integrals of the first and second kind. In particular, the formula for the wave profile of a cnoidal wave is expressed in terms of the Jacobian elliptic function $\mathrm{cn}(\theta, m)$, hence the term cnoidal, analogous to sinusoidal.

For the case of sinusoidal waves basic formulas are given together with deep-water and shallow-water approximations. For the case of cnoidal waves only the basic formulas are given, as cnoidal wave theory is applicable only for shallow water (water depth small compared to wave length). In addition to the formulas there are tables (to 3 and 4S) of functions that are used to evaluate the formulas for various parameters such as wave period and wave length. Furthermore, directions are provided for using the formulas and tables to determine wave properties such as length and celerity, given other properties, as water depth, wave height, and wave period. The directions apply to the use of the formulas for waves progressing over water of constant depth (referred to as