Linear Algebra and its Applications 466 (2015) 21–26

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Note on best possible bounds for determinants of matrices close to the identity matrix

Richard P. Brent [a,*], Judy-anne H. Osborn [b], Warren D. Smith [c]

[a] Australian National University, Canberra, ACT 0200, Australia
[b] The University of Newcastle, Callaghan, NSW 2308, Australia
[c] Center for Range Voting, 21 Shore Oaks Drive, Stony Brook, NY 11790, USA

ARTICLE INFO

Article history:
Received 20 March 2014
Accepted 27 September 2014
Available online 16 October 2014
Submitted by R. Brualdi

MSC:
65F40
05B20
15A15
15A42
15B34

Keywords:
Determinant
Perturbation bound
Diagonally dominant matrix
Skew-Hadamard matrix
Fredholm determinant
Maximal determinant

ABSTRACT

We give upper and lower bounds on the determinant of a small perturbation of the identity matrix. The lower bounds are best possible, and in most cases they are stronger than well-known bounds due to Ostrowski and other authors. The upper bounds are best possible if a skew-Hadamard matrix of the same order exists.

* Corresponding author at: PO Box 501, Cardiff, NSW 2285, Australia.
E-mail address: pert@rpbrent.com (R.P. Brent).

http://dx.doi.org/10.1016/j.laa.2014.09.041

Table 1
Summary of lower bound results.

Case	Lower bound	Condition
(A) general	$(1-\delta-(n-1)\varepsilon)(1-\delta+\varepsilon)^{n-1}$	$\delta+(n-1)\varepsilon \le 1$
(B) $\delta=\varepsilon$	$1-n\varepsilon$	$n\varepsilon \le 1$
(C) $\delta=0$	$(1-(n-1)\varepsilon)(1+\varepsilon)^{n-1}$	$(n-1)\varepsilon \le 1$

Table 2
Summary of upper bound results.

Case	Upper bound
(A) general	$((1+\delta)^2+(n-1)\varepsilon^2)^{n/2}$
(B) $\delta=\varepsilon$	$(1+2\varepsilon+n\varepsilon^2)^{n/2}$
(C) $\delta=0$	$(1+(n-1)\varepsilon^2)^{n/2}$

1. Introduction

Many bounds on determinants of diagonally dominant matrices have been given in the literature. See, for example, Bhatia and Jain [1], Elsner [5], Horn and Johnson [8], Ipsen and Rehman [9], Ostrowski [11–13], and Price [14]. We consider the case of a matrix $A = I - E$, where I is the $n \times n$ identity matrix and the elements e_{ij} of E are small. Thus, A is "close" to the identity matrix. A more general case, where A is close to a nonsingular diagonal matrix, can be reduced to this case by row and/or column scaling.

To make precise the sense in which E is small, we introduce two non-negative parameters δ and ε, and require

$$|e_{ij}| \le \begin{cases} \delta & \text{if } i = j; \\ \varepsilon & \text{otherwise.} \end{cases}$$

We consider three cases: (A) is the general case, (B) is when $\delta = \varepsilon$, and (C) is when $\delta = 0$. These cases are all of interest. Case (B) is the simplest, and was considered by Ostrowski [13] and others. Case (C) arises naturally if scaling is used to reduce the diagonal elements to 1. Case (A) is an obvious generalization which unifies the cases (B)–(C), and is required to obtain sharp results in some applications where δ and ε have different orders of magnitude.[1]

For the reader's convenience, the lower and upper bound results are summarized in Tables 1–2. A comparison with previously-published bounds is given in Section 2. Our lower bounds are given in Section 3, and the upper bounds in Section 4.

2. Comparison with previous bounds

It is perhaps surprising that we have only found one of the six bounds (cases (A)–(C), lower and upper) in the literature, although their proofs use standard techniques and are not difficult.

[1] For example, in [2, Corollary 5], an optimization problem leads to the choice $\delta \asymp \varepsilon^2$.

In case (B), Ostrowski [13, Eq. (5,5)] gives the lower bound

$$\det(I - E) \geq 1 - n\varepsilon, \tag{1}$$

subject to $n\varepsilon \leq 1$. This is best-possible, as it is attained if $E = \varepsilon J$, where J is the $n \times n$ matrix of all ones. Ostrowski [13] also gives an upper bound

$$\det(I - E) \leq \frac{1}{1 - n\varepsilon}, \tag{2}$$

subject to $n\varepsilon < 1$, but this is not best-possible. Our upper bound

$$\det(I - E) \leq \left(1 + 2\varepsilon + n\varepsilon^2\right)^{n/2}, \tag{3}$$

which is easily proved using Hadamard's inequality [7], and holds for arbitrary $\varepsilon \geq 0$, is smaller than Ostrowski's bound (2) for all $\varepsilon \in (0, 1/n)$. The difference between the bounds is $n\varepsilon^2 + O(\varepsilon^3)$. The bound (3) is best-possible if a skew-Hadamard matrix of order n exists.[2]

In case (C), Ostrowski [11] gives a more general result that implies the lower bound

$$\det(I - E) \geq \left(1 - (n-1)\varepsilon\right)^n, \tag{4}$$

subject to $(n-1)\varepsilon \leq 1$, and the same bound follows from Gerschgorin's theorem [6]. However, this bound is $1 - n(n-1)\varepsilon + O(\varepsilon^2)$, whereas our bound is $1 - n(n-1)\varepsilon^2/2 + O(\varepsilon^3)$, which shows that the perturbation in the determinant is of order ε^2, not of order ε. Ostrowski [12, Satz VI] gives a lower bound that reduces (under the same assumption on ε) to

$$\det(I - E) \geq \left(1 - (n-1)^2\varepsilon^2\right)^{\lfloor n/2 \rfloor}. \tag{5}$$

This is better than (4) as it shows that the perturbation is of order ε^2. For small ε, the bound (5) is $1 - (n-1)^2\lfloor n/2 \rfloor\varepsilon^2 + O(\varepsilon^4)$, which is worse than our (best possible) bound if $n \geq 3$.

A different lower bound, due to von Koch [10] (see Ostrowski [11, §2]), reduces (under the same assumption on ε) to

$$\det(I - E) \geq e^{n(n-1)\varepsilon}\left(1 - (n-1)\varepsilon\right)^n. \tag{6}$$

For small ε von Koch's bound is $1 - n(n-1)^2\varepsilon^2/2 + O(\varepsilon^3)$, which is worse than our bound if $n \geq 2$.

[2] This is true for $n = 1, 2$, all multiples of four up to and including 4×68, as well as infinitely many larger n, such as all powers of two, see [4]. Sharp bounds for small orders for which a skew-Hadamard matrix does not exist (e.g. $n = 3$) are considered in [3, §4.1].

In case (C), Ostrowski [12, Satz VI] gives an upper bound, which reduces to

$$\det(I - E) \le \bigl(1 + (n-1)^2\varepsilon^2\bigr)^{\lfloor n/2 \rfloor}, \tag{7}$$

assuming that $(n-1)\varepsilon \le 1$. Our upper bound is better if $n \ge 3$, and does not require the condition on ε.

To illustrate the lower bounds in case (C) with some numerical values, suppose that $n = 5$ and $\varepsilon = 1/8$. Then Gerschgorin/Ostrowski (4) gives the bound $2^{-5} = 0.03125$, von Koch (6) gives $e^{5/2}/2^5 \approx 0.3807$, Ostrowski (5) gives $9/16 = 0.5625$, while our Corollary 2 gives $3^8/2^{13} \approx 0.8009$, which is best-possible.

3. Lower bounds

We start with a general result, then deduce Theorem 1, which gives our lower bound in case (A). Our lower bounds in cases (B)–(C) are special cases of Theorem 1.

Proposition 1. *Let $F \in \mathbb{R}^{n\times n}$, $f_{ij} \ge 0$, $\rho(F) \le 1$. If $A = I - E \in \mathbb{R}^{n\times n}$, where $|e_{ij}| \le f_{ij}$, then*

$$\det(A) \ge \det(I - F).$$

Proof. A proof using the Fredholm determinant formula is given in [3, §3]. □

Theorem 1. *Let $A = I - E \in \mathbb{R}^{n\times n}$, where $|e_{ij}| \le \varepsilon$ for $i \ne j$, $|e_{ii}| \le \delta$ for $1 \le i \le n$, and $\delta + (n-1)\varepsilon \le 1$. Then*

$$\det(A) \ge \bigl(1 - \delta - (n-1)\varepsilon\bigr)(1 - \delta + \varepsilon)^{n-1},$$

and the inequality is sharp.

Proof. The result is immediate if $n = 1$, so suppose that $n \ge 2$. Define $F := (\delta - \varepsilon)I + \varepsilon J$, so F is a Toeplitz matrix with diagonal entries δ and off-diagonal entries ε.

Observe that $Je = ne$, so J has an eigenvalue $\lambda_1(J) = n$; the other $n-1$ eigenvalues are zero since J has rank 1.

Since εJ has one eigenvalue equal to $n\varepsilon$ and $n-1$ eigenvalues equal to zero, it is immediate that F has eigenvalues $\delta - \varepsilon + n\varepsilon = \delta + (n-1)\varepsilon$ and $\delta - \varepsilon$. Thus

$$\rho(F) = \max\bigl(\delta + (n-1)\varepsilon, |\delta - \varepsilon|\bigr) = \delta + (n-1)\varepsilon \le 1.$$

Also, the eigenvalues of $I - F$ are $1 - \delta - (n-1)\varepsilon$ with multiplicity 1, and $1 - \delta + \varepsilon$ with multiplicity $n-1$, so

$$\det(I - F) = \bigl(1 - \delta - (n-1)\varepsilon\bigr)(1 - \delta + \varepsilon)^{n-1}.$$

Thus, the inequality follows from Proposition 1. It is sharp because equality holds for $A = I - F$. $\square$

A generalization of Theorem 1 is given in [3, Corollary 1]. The generalization, which follows from Theorem 1 by a scaling argument, shows that the condition $|e_{ii}| \le \delta$ may be replaced by the one-sided condition $e_{ii} \le \delta$.

Corollaries 1–2 are simple consequences of Theorem 1. Corollary 1 is essentially Ostrowski's lower bound (1), although Ostrowski did not explicitly state that the bound is sharp.

Corollary 1 *(Ostrowski). If $A = I - E \in \mathbb{R}^{n\times n}$, $|e_{ij}| \le \varepsilon$ for $1 \le i, j \le n$, and $n\varepsilon \le 1$, then*

$$\det(A) \ge 1 - n\varepsilon,$$

and the inequality is sharp.

Proof. This is the case $\delta = \varepsilon$ of Theorem 1. The result is sharp as there is equality for $E = \varepsilon J$. $\square$

Corollary 2. *If $A = I - E \in \mathbb{R}^{n\times n}$, $|e_{ij}| \le \varepsilon$ for $1 \le i, j \le n$, $e_{ii} = 0$ for $1 \le i \le n$, and $(n-1)\varepsilon \le 1$, then*

$$\det(A) \ge \bigl(1 - (n-1)\varepsilon\bigr)(1+\varepsilon)^{n-1},$$

and the inequality is sharp.

Proof. This is the case $\delta = 0$ of Theorem 1. The result is sharp as there is equality for $E = \varepsilon(J - I)$. $\square$

4. Upper bounds

The following theorem gives upper bounds in cases (B) and (C). The more general case (A) given in Table 2 is similar.

Theorem 2. *If $A = I - E \in \mathbb{R}^{n\times n}$, $|e_{ij}| \le \varepsilon$ for $1 \le i, j \le n$, then*

$$\det(A) \le \bigl(1 + 2\varepsilon + n\varepsilon^2\bigr)^{n/2}. \tag{8}$$

If, in addition, $e_{ii} = 0$ for $1 \le i \le n$, then

$$\det(A) \le \bigl(1 + (n-1)\varepsilon^2\bigr)^{n/2}. \tag{9}$$

Proof. Let the columns of A be $u_1, u_2, \dots, u_n$. From Hadamard's inequality,

$$\det(A) \le \prod_{i=1}^{n} \|u_i\|_2.$$

However, the condition $|e_{ij}| \le \varepsilon$ implies that

$$\|u_i\|_2^2 \le (1+\varepsilon)^2 + (n-1)\varepsilon^2 = 1 + 2\varepsilon + n\varepsilon^2.$$

Hence, the result (8) follows. The proof of (9) is similar. □

The upper bounds (A)–(C) are attained if a skew-Hadamard matrix H of order n exists. To see this, consider $A = (1+\delta)I + \varepsilon(H - I)$. Conversely, if the upper bound (C) is best-possible for all sufficiently small ε, then a skew-Hadamard matrix of order n exists – see [3, Theorem 4] for a proof.

Acknowledgements

We thank L.N. Trefethen and the anonymous referees for their comments on an earlier version of this note. The first author was supported in part by Australian Research Council grant DP140101417.

References

[1] R. Bhatia, T. Jain, Higher order derivatives and perturbation bounds for determinants, Linear Algebra Appl. 431 (2009) 2102–2108.
[2] R.P. Brent, J.H. Osborn, W.D. Smith, Lower bounds on maximal determinants of binary matrices via the probabilistic method, arXiv:1402.6817v2, 14 Mar. 2014, 37 pp.
[3] R.P. Brent, J.H. Osborn, W.D. Smith, Bounds on determinants of perturbed diagonal matrices, arXiv:1401.7048v7, 2014, 18 pp.
[4] C.J. Colbourn, J.H. Dinitz, Handbook of Combinatorial Designs, 2nd edition, CRC Press, New York, 2006.
[5] L. Elsner, Bounds for determinants of perturbed M-matrices, Linear Algebra Appl. 257 (1997) 283–288.
[6] S. Gerschgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk 6 (1931) 749–754.
[7] J. Hadamard, Résolution d'une question relative aux déterminants, Bull. Sci. Math. 17 (1893) 240–246.
[8] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.
[9] I.C.F. Ipsen, R. Rehman, Perturbation bounds for determinants and characteristic polynomials, SIAM J. Matrix Anal. Appl. 30 (2008) 762–776.
[10] H. von Koch, Über das Nichtverschwinden einer Determinante nebst Bemerkungen über Systeme unendlich vieler linearer Gleichungen, Jber. Deutschen Math. Verein. 22 (1913) 285–291.
[11] A.M. Ostrowski, Sur la détermination des bornes inférieures pour une classe des déterminants, Bull. Sci. Math. (2) 61 (1937) 19–32.
[12] A.M. Ostrowski, Über die Determinanten mit überwiegender Hauptdiagonale, Comment. Math. Helv. 10 (1937) 69–96.
[13] A.M. Ostrowski, Sur l'approximation du déterminant de Fredholm par les déterminants des systèmes d'equations linéaires, Ark. Math. Stockholm Ser. A 26 (1938) 1–15.
[14] G.B. Price, Bounds for determinants with dominant principal diagonal, Proc. Amer. Math. Soc. 2 (1951) 497–502.