Lecture 11. Geodesics and completeness

In this lecture we will investigate further the metric properties of geodesics of the Levi-Civita connection, and use this to characterise completeness of a Riemannian manifold in terms of the exponential map.

11.1 Geodesic polar coordinates and the Gauss Lemma

Let (M, g) be a Riemannian manifold, and $x \in M$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_xM, and induce an isomorphism from $\mathbb{R}^n$ to T_xM. This in turn induces a map from $S^{n-1} \times (0, \infty) \to M$, by sending (ω, r) to $\exp_x(r\omega)$. This is called *geodesic polar coordinates from x*.

Proposition 11.1.1 *In geodesic polar coordinates the metric takes the form*

$$g(\partial_r, \partial_r) = 1;$$
$$g(\partial_r, u) = 0,$$

for any $u \in TS^{n-1}$.

In other words, the image under the exponential map of the unit radial vector in T_xM is always a unit vector, and the image of a vector tangent to a sphere about the origin is always orthogonal to the image of the radial vector. Think of a polar 'grid' of radial lines and spheres in T_xM, mapped onto M by the exponential map T_xM. Then this map says that the images of the spheres are everywhere orthogonal to the images of the radial lines (which are of course geodesics).

Proposition 11.1.1 is often called the Gauss Lemma.

Proof. The first part of the claim follows from the fact that ∂_r is the tangent vector to a geodesic, so $\nabla_r \partial_r = 0$. Hence by compatibility,

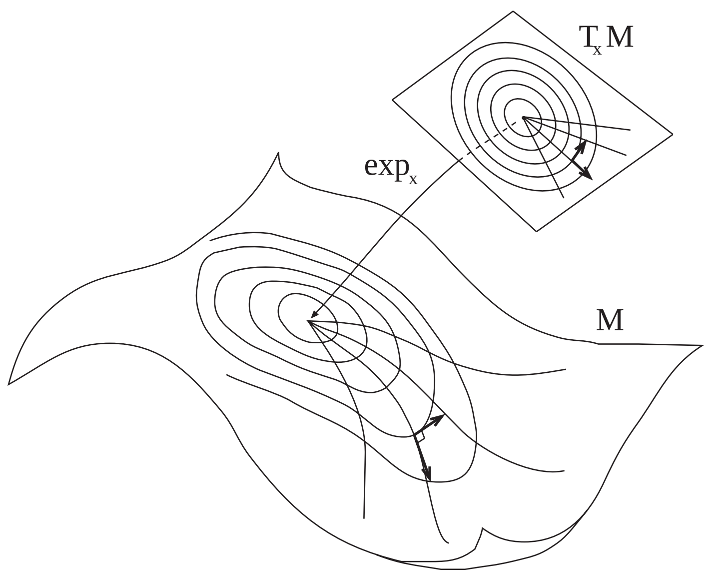
$$\partial_r g(\partial_r, \partial_r) = 2g(\partial_r, \nabla_r \partial_r) = 0,$$

and the length of ∂_r is constant in the r direction. But when $r = 0$ we have $|\partial_r| = 1$.

To prove the second part of the proposition, let $\gamma : [0, r] \times (-\varepsilon, \varepsilon) \to M$ be given by $\gamma(r, s) = \exp_x(r\omega(s))$, where $\omega(s) \in S^{n-1}(0) \subset T_xM$. Then in T_xM, $\omega'(s)$ is tangent to a sphere about the origin, and we need to show that the image ∂_s of this vector is orthogonal to the radial vector ∂_r:

$$\begin{aligned}\partial_r g(\partial_r, \partial_s) &= g(\nabla_r \partial_r, \partial_s) + g(\partial_r, \nabla_r \partial_s)\\ &= g(\partial_r, \nabla_s \partial_r)\\ &= \frac{1}{2}\partial_s g(\partial_r, \partial_r)\\ &= 0,\end{aligned}$$

since we know $g(\partial_r, \partial_r) = 1$ everywhere. But as before, we have $g(\partial_r, \partial_s) = 0$ when $r = 0$. □

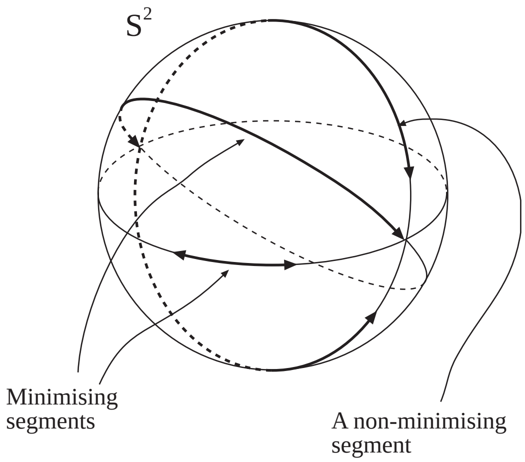


11.2 Minimising properties of geodesics

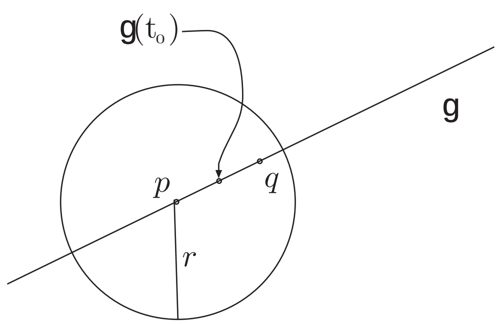
An important consequence of the Gauss Lemma is the fact that geodesics of the Levi-Civita connection, restricted to sufficiently short intervals, have smaller length than any other path between their endpoints.

Proposition 11.2.1 *Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection of g. Let $\gamma : I \to M$ be a ∇-geodesic. Then for any $t \in I$ there exists $\delta > 0$ such that* $L\left[\gamma\big|_{[t_0-\delta, t_0+\delta]}\right] = d(\gamma(t_0 - \delta), \gamma(t_0 + \delta))$.

Note that we cannot expect that geodesics minimise length on long intervals – consider the example of the sphere S^2: Geodesics are great circles, and these achieve the distance between their endpoints on intervals of length no greater then π, but not on longer intervals.



Proof. Choose ε sufficiently small so that $\exp_{\gamma(t_0-\varepsilon)}$ is a diffeomorphism on a ball of radius $r > 2\varepsilon$ about the origin. For convenience we denote by p the point $\gamma(t_0 - \varepsilon)$, and by q the point $\gamma(t_0 + \varepsilon)$.



Now let σ be any other curve joining the points p and q. Suppose first that σ remains in the set $B_r(p) = \exp_p(B_r(0))$. Then we can write $\sigma(t) = r(t)\omega(t)$ where $r > 0$ and $|\omega(t)| = 1$. The squared length of the tangent vector σ' is then

$$\begin{aligned} |\sigma'(t)|^2 &= (r')^2 g(\partial_r, \partial_r) + 2rr' g(\partial_r, \omega') + r^2 g(\omega', \omega') \\ &= (r')^2 + r^2|\omega'|^2 \\ &\geq (r')^2, \end{aligned}$$

with equality if and only if ω' vanishes. Therefore we have

$$L[\sigma] \geq \int |r'|dt \geq 2\varepsilon,$$

with equality if and only if ω is constant and r is monotone, in which case σ is simply a reparametrisation on γ.

The other possibility is that σ leaves the ball. But then the same argument applies on the portion of sigma joining p to the boundary of the ball, giving $L[\sigma] \geq r > 2\varepsilon$.

Therefore $L[\sigma] \geq L[\gamma]$, with equality if and only if σ is a reparametrisation of σ. □

Proposition 11.2.2 *Suppose $\gamma : [0,1] \to M$ is a piecewise smooth path for which $L[\gamma] = d(\gamma(0), \gamma(1))$. Then $\gamma = \sigma \circ f$, where $f : [0,1] \to [0,1]$ is a piecewise smooth monotone increasing function, and σ is a geodesic.*

Proof. First observe that γ achieves the distance between any pair of its points: If there were some subinterval on which this were not true, then replacing γ by a shorter path on that subinterval would also yield a shorter path from $\gamma(0)$ to $\gamma(1)$.

For any $t \in (0,1)$, there is a sufficiently small neighbourhood J of t in $[0,1]$ such that $\gamma\big|_J$ is contained in a diffeomorphic image of the exponential map from one of its endpoints, and so by the Gauss Lemma $\gamma\big|_J$ is a reparametrised geodesic. □

This is a nice feature of the Levi-Civita connection: The geodesics of ∇ are precisely the length-minimising paths, re-parametrised to have constant speed.

11.3 Convex neighbourhoods

We know that points which are sufficiently close to each other can be connected by a unique 'short' geodesic. This result can be stated somewhat more cleany in the special case of the Levi-Civita connection than it can in the general case:

Proposition 11.3.1 *Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection of g. Then for every $p \in M$ there exist constants $0 < \varepsilon \leq \eta$ such that for every pair of points q and r in $B_\varepsilon(p)$ there exists a unique geodesic γ_{qr} of length $L[\gamma_{qr}] < \eta$ such that $\gamma_{qr}(0) = q$ and $\gamma_{qr}(1) = r$. Furthermore, $L[\gamma_{qr}] = d(q,r)$.*

Proof. The idea is exactly the same as the proof of Proposition 8.11.5: The map $\widetilde{\exp}$ is a local diffeomorphism from a neighbourhood of $(p, 0)$ in TM to a neighbourhood of (p, p) in $M \times M$. Choose η sufficiently small to ensure that

$\tilde{\exp}$ is a diffeomorphism on $\mathcal{O} = \{(q, v)|\ q \in B_\eta,\ |v| \le \eta\}$, and then choose ε sufficiently small to ensure that $B_\varepsilon(p) \times B_\varepsilon(p) \subset \tilde{\exp}\mathcal{O}$.

This gives the existence of a geodesic γ_{qr} of length less than η joining q to r. Also γ_{qr} achieves the distance between its endpoints, by the Gauss Lemma. □

We will improve the result slightly:

Proposition 11.3.2 *For any $p \in M$ there exists a constant $\varepsilon > 0$ such that for every pair of points q and r in $B_\varepsilon(p)$, there exists a unique geodesic γ_{qr} for which $\gamma(0) = q$, $\gamma(1) = r$, and $d(p, \gamma(t)) \le \max\{d(p, q), d(p, r)\}$ for all $t \in [0, 1]$. Furthermore $L[\gamma_{qr}] = d(q, r)$.*

Proof. By Proposition 11.5.1, ${\Gamma_{ij}}^k(0) = 0$ in exponential normal coordinates. Therefore there is $\eta > 0$ such that $\left|\sum_{k,i,j} x^k {\Gamma_{ij}}^k \xi^i \xi^j\right| < \frac{1}{2}\sum_k \left(\xi^k\right)^2$ provided $\sum_k \left(x^k\right)^2 < 2\eta$, and such that for some $\varepsilon \in (0, \eta)$ the conditions of Proposition 11.3.1 hold.

Then by Proposition 11.3.1 there is a geodesic γ_{qr} of length less than η joining q to r; since p and q are within distance ε of p, the entire geodesic γ_{qr} stays within distance 2η of p.

Now along γ_{qr} we compute:

$$\begin{aligned}
\frac{d^2}{dt^2} d(p, \gamma(t))^2 &= \frac{d^2}{dt^2} \sum_k \left(x^k\right)^2 \\
&= 2\frac{d}{dt} \sum_k \left(x^k \dot{x}^k\right) \\
&= 2\sum_k \left(\dot{x}^k\right)^2 + 2\sum_k x^k \frac{d^2}{dt^2} x^k \\
&= 2\sum_k \left(\dot{x}^k\right)^2 - 2\sum_k x^k {\Gamma_{ij}}^k \dot{x}^i \dot{x}^j \\
&\ge 2\left(\sum_k \left(\dot{x}^k\right)^2 - \frac{1}{2}\sum_{[} \left(\dot{x}^k\right)^2\right) \\
&\ge 0,
\end{aligned}$$

since we are within the ball of radius 2η about p. Therefore $d(p, \gamma(t))^2$ is a convex function along γ_{qr}, so the maximum value is attained at the endpoints. □

11.5 Completeness and the Hopf-Rinow Theorem

Now we are ready to prove the main result of this section:

Theorem 11.5.1 The Hopf-Rinow Theorem. *Let (M, g) be a connected Riemannian manifold. The following are equivalent:*

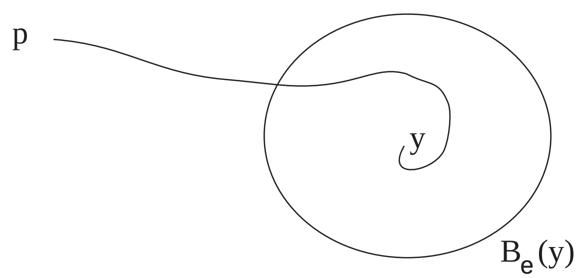
(1). M is a complete metric space with the distance function d;
(2). exp *is defined on all of TM (i.e. all geodesics can be extended indefinitely);*
(3). There exists $p \in M$ for which $\exp_p$ *is definied on all of T_pM.*

Furthermore, each of these conditions implies

(*). For every p and q in M, there exists a geodesic $\gamma : [0, 1] \to M$ for which $\gamma(0) = p$, $\gamma(1) = q$, and $L[\gamma] = d(p, q)$.

Proof. (2) $\implies$ (3) trivially. We prove (1) $\implies$ (2), and then (3) $\implies$ $(*_p)$ and ((3) and $(*_p)$) $\implies$ (1), where $(*_p)$ is the statement that every point $q \in M$ can be connected to p by a length-minimising geodesic.

Suppose (M, d) is a complete metric space. If (2) does not hold, then there is $p \in M$, $v \in T_pM$ with $|v| = 1$, and $T < \infty$ such that $\gamma(t) = \exp_p(tv)$ exists for $t < T$ but not for $t = T$.



But $d(\gamma(s), \gamma(t)) \leq |s - t|$, so $\gamma(t)$ is Cauchy as t approaches T. By completeness, $\gamma(t)$ converges to a limit $y \in M$ as $t \to T$. Choose $\varepsilon > 0$ such that the ball $B_\varepsilon(y)$ is convex in the sense of Proposition 10.3.2. Then for $s, t > T - \varepsilon$ we have $\gamma(s)$ and $\gamma(t)$ contained in this geodesically convex set, and γ a geodesic joining them; therefore γ achieves the distance between $\gamma(s)$ and $\gamma(t)$:

$$d(\gamma(s), \gamma(t)) = |t - s|, \qquad \text{for } t, s > T - \varepsilon.$$

Then we have by continuity of the distance function,

$$d(\gamma(t), y) = \lim_{s \to T} d(\gamma(t), \gamma(s)) = \lim_{s \to T} (s - t) = T - t = L[\gamma\big|_{(t,T)}].$$

Therefore $\gamma|_{(t,T)}$ is a minimising path, and must be a radial geodesic from y:

$$\gamma(t) = \exp_y(w(T-t)).$$

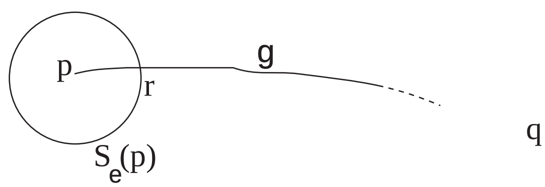
But then γ can be extended beyond y by taking $t > T$ in this formula. This is a contradiction, so condition (2) must hold.

Next we prove (3) $\Longrightarrow$ $(*_p)$. Let $p \in M$ be such that (3) holds, and $q \in M$ any other point. Choose $\varepsilon > 0$ such that $\exp_p$ is a diffeomorphism on a set containing the closure of $B_\varepsilon(0)$. If $q \in B_\varepsilon(p)$ then we are done, so assume not. Then $S_\varepsilon(p) = \exp_p(S_\varepsilon(0))$ is the image of a compact set under a continuous map, and so the function $d(.,q)$ attains its minimum on this set. In other words, there exists $r = \exp_p(\varepsilon v)$ such that $d(r,q) = d(S_\varepsilon(p),q) = \inf\{d(r',q) :\ r' \in S_\varepsilon(p)\}$. Then we must have

$$d(p,q) = \varepsilon + d(r,q).$$

The inequality $d(p,q) \leq \varepsilon + d(r,q)$ follows by the triangle inequality since $d(p,r) = \varepsilon$, and the other inequality follows because any path from p to q must pass through $S_\varepsilon(p)$.

Define $\gamma(t) = \exp_p(tv)$.



Let $A \subset [0, d(p,q)]$ be the set of t for which

$$d(p,q) = t + d(\gamma(t),q).$$

A is non-empty, as we have just shown; it is closed, by continuity of the distance function and because γ can be extended indefinitely. We also have that $t \in A$ implies $[0,t] \subset A$: If $s < t$ we have by the triangle inequality and the fact that $\gamma|_{[0,t]}$ is a minimising path

$$\begin{aligned} d(p,q) &= t + d(\gamma(t),q) \\ &= s + (t-s) + d(\gamma(t),q) \\ &\geq s + d(\gamma(s),q) \\ &\geq d(p,q), \end{aligned}$$

so equality must hold throughout, and $s \in A$.

Finally, we will prove that A is open in $[0, d(p,q)]$. This will imply $A = [0, d(p,q)]$, so in particular $d(\gamma(d(p,q)),q) = 0$, and $\gamma(d(p,q)) = q$.

Suppose $T \in A$, and write $p' = \gamma(T)$. Choose $\delta > 0$ such that $\exp_{p'}$ is a diffeomorphism on $B_\delta(0)$. If $q \in B_\delta(p')$, then write $q = \exp_{p'}(d(p',q)w)$, and define

$$\sigma(t) = \begin{cases} \gamma(t), & \text{for } t \le T; \\ \exp_{p'}((t-T)w) & \text{for } t \ge T. \end{cases}$$

Then σ is a curve joining p to q, and by assumption

$$d(p,q) = T + d(p',q) = L[\sigma].$$

Therefore σ is a minimising curve, hence a geodesic, so $\sigma(t) = \exp_p(t)$ for all $t \in [0, d(p,q)]$ and we are done.

Otherwise, $q \notin B_\delta(p')$, and we can choose $r' \in S_\delta(p')$ such that $d(r',q) = d(B_\delta(p'), q)$. Then as before, we have

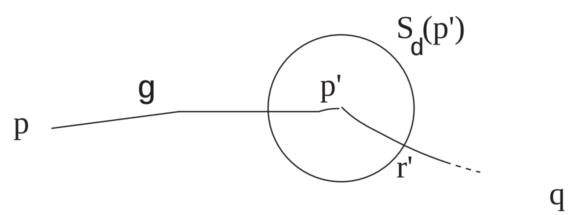
$$d(p',q) = \delta + d(r',q),$$

and so by assumption

$$d(p,q) = d(p,p') + d(p',q) = T + \delta + d(r',q).$$

Let σ be the unit speed curve given by following γ from p to p', then following the radial geodesic from p' to r'. Then $L[\sigma] = d(p,p') + \delta$. Therefore,

$$d(p,q) = L[\sigma] + d(r',q).$$



It follows that $L[\sigma] = d(p,r')$, since if there were any shorter path σ' from p to r', we would have

$$d(p,q) \le L[\sigma'] + d(r',q) < L[\sigma'] + d(r',q) = d(p,q)$$

which is a contradiction. Therefore σ is a geodesic, and $\sigma(t) = \gamma(t)$; and $[0, T+\delta] \subset A$. Therefore A is open as claimed, and we have proved $(*_p)$.

Now we complete the proof by showing that (3) and $(*_p)$ together imply condition (1). For p satisfying (3) and $(*_p)$, let $M_k = \exp_p\left(\overline{B_k(0)}\right)$. This is the image of a compact set under a continuous map, and so is compact. By $(*_p)$ we have $\cup_{k=1}^{\infty} M_k = \infty$.

Suppose $\{z_i\}$ is a Cauchy sequence in M. Then in particular $\{z_i\}$ is contained in some M_k, and hence converges by the compactness of M_k.

This completes the proof of the Theorem. $\square$