Lecture 13. Differential forms

In the last few lectures we have seen how a connection can be used to differentiate tensors, and how the introduction of a Riemannian metric gives a canonical choice of connection. Before exploring the properties of Riemannian spaces more thoroughly, we will first look at a special class of tensors for which there is a notion of differentiation that makes sense even without a connection or a metric. These are called differential forms, and they play an extremely important role in differential geometry.

13.1 Alternating tensors

We will first look a little more at the linear algebra of tensors at a point. We will consider a natural subspace of the space of k-tensors, namely the **alternating tensors**.

Definition 13.1.1 A k-tensor $\omega \in \otimes^k T_x^*M$ is **alternating** if it is antisymmetric under interchange of any two of its arguments. Equivalently, for any k vectors $v_1, \dots, v_k \in T_xM$, and any permutation $\sigma \in S_k$,

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}\sigma\omega(v_1, \dots, v_k),$$

where $\mathrm{sgn}\sigma = 1$ if σ is an even permutation, $\mathrm{sgn}\sigma = -1$ is σ is an odd permutation.

The space of alternating k-tensors at x is denoted $\Lambda^k T^*M$. Note that $\Lambda^1 T_x^*M = T_x^*M$, so alternating 1-tensors are just covectors.

There is a natural projection $\mathcal{A} : \otimes^k T_x^*M \to \Lambda^k T_x^*M$ defined as follows:

$$\mathcal{A}T(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\sigma T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Then T is alternating if and only if $\mathcal{A}T = T$.

Example 13.1.1 The geometric meaning of this definition is probably not clear at this stage. An illustrative example is the following: Choose an orthonormal basis $\{\phi^1, \ldots, \phi^n\}$ for T_x^*M. Then we can define an alternating n-tensor A by taking

$$A(v_1, \ldots, v_n) = \det[\phi^i(v_j)].$$

This is antisymmetric since the determinant is antisymmetric under interchange of any pair of columns. Geometrically, the result $A(v_1, \ldots, v_n)$ is the (signed) n-dimensional volume of the parallellopiped generated by $v_1, \ldots, v_n$.

Given a basis $\{\partial_1, \ldots, \partial_n\}$ for T_xM, we can define a natural basis for $\Lambda^k T_x^*M$: For each k-tuple $i_1, \ldots, i_k$, we define

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k} = k!\mathcal{A}\left(dx^{i_1} \otimes \ldots \otimes dx^{i_k}\right).$$

Note that this is zero if the k-tuple is not distinct, and that if we change the order of the k-tuple then the result merely changes sign (depending whether the k-tuple is rearranged by an even or an odd permutation).

The factor $k!$ is included in our definition for the following reason: If we apply the alternating k-tensor $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (where $i_1, \ldots, i_k$ are distinct) to the k vectors $\partial_{i_1}, \ldots, \partial_{i_k}$, the result is 1. If we apply it to the same k vectors in a different order (say, rearranged by some permutation σ), then the result is just the sign of σ. Any other k vectors yield zero.

These 'elementary alternating k-tensors' have a geometric interpretation similar to that in Example 13.1.1: The value of $dx^I(v_1, \ldots, v_k)$ is the determinant of the matrix with (m, n) coefficient $dx^{i_m}(v_n)$, and this gives the signed k-dimensional volume of the projection of the parallelopiped generated by $v_1, \ldots, v_k$ onto the subspace generated by $\partial_{i_1}, \ldots, \partial_{i_k}$. This relationship between alternating forms and volumes will be central in the next lecture when we define integration of differential forms and prove Stokes' theorem.

Proposition 13.1.1
(1). $dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathrm{sgn}\sigma \, dx^{i_{\sigma(1)}} \otimes \ldots \otimes dx^{i_{\sigma(k)}} = k!\mathcal{A}(dx^{i_1} \otimes \ldots \otimes dx^{i_k})$.
(2). For each k, $\{dx^{i_1} \wedge \ldots \wedge dx^{i_k} : \ 1 \le i_1 < \ldots < i_k \le n\}$ *is a basis for* $\Lambda^k T_x^*M$. *In particular the space of alternating* k*-tensors at* x *has dimension* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. (1) is immediate, since the value on any k-tuple of coordinate basis vectors agrees. To prove (2), we note that by Proposition 12.2.2, any alternating tensor can be written as a linear combination of the basis elements $dx^{i_1} \otimes \ldots \otimes dx^{i_k}$. Invariance under $\mathcal{A}$ shows that this is the same as a linear combination of k-forms of the form $\mathcal{A}(dx^{i_1} \otimes \ldots \otimes dx^{i_k})$, and these are all of the form given. It remains to show the supposed basis is linearly independent, but this is also immediate since if $I = (i_1, \ldots, i_k)$ then $dx^I(\partial_{i_1}, \ldots, \partial i_k) = 1$, but $dx^J(\partial_{i_1}, \ldots, \partial i_k) = 0$ for any increasing k-tuple $J \ne I$. $\square$

It follows that any alternating k-tensor T can be written in the form

$$T = \sum_{1 \leq i_1 < \ldots < i_k \leq n} T_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

for some coefficients $T_{i_1 \ldots i_k}$. Some caution is required here, because T can also be written in the form

$$T = \frac{1}{k!} \sum_{i_1, \ldots, i_k} T_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where the coefficients are the same as above for increasing k-tuples, and to be given from these by antisymmetry in other cases. Thus the coefficients in this expression differ by a factor $k!$ from those given in the expression after Proposition 12.2.2.

13.2 The wedge product

The projection $\mathcal{A}$ immediately gives us a notion of product on alternating tensors, which we have already implicity built into our notation for the basis elements for $\Lambda^k T_x^* M$:

Definition 13.2.1 Let $S \in \Lambda^k T_x^* M$ and $T \in \Lambda^l T_x^* M$ be alternating tensors. Then the **wedge product** $S \wedge T$ of S and T is the alternating $k + l$-tensor given by

$$S \wedge T = \frac{(k+l)!}{k!l!} \mathcal{A}(S \otimes T).$$

This may not seem the obvious definition, because of the factor on the right. This is chosen to make our notation consistent with that in our definition of the basis elements: Take an increasing k-tuple $i_1, \ldots, i_k$ and an increasing l-tuple $j_1, \ldots, j_l$, and assume for simplicity that $i_k < j_1$. Then we can form the alternating tensors $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, $dx^{j_1} \wedge \ldots \wedge dx^{j_l}$ and $dx^{j_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}$, and we would like to know that the third of these is the wedge product of the first two.

Proposition 13.2.1
The wedge product is characterized by the following properties:
(i). *Associativity:* $f \wedge (g \wedge h) = (f \wedge g) \wedge h$;
(ii). *Homogeneity:* $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$;
(iii). *Distributivity: If* f *and* g *are in* $\Lambda^k T_x^* M$ *then*

$$(f + g) \wedge h = (f \wedge h) + (g \wedge h);$$

(iv). *Anticommutativity: If* $f \in \Lambda^k T_x^* M$ *and* $g \in \Lambda^l T_x^* M$, *then*

$$g \wedge f = (-1)^{kl} f \wedge g;$$

(v). *In any chart,*

$$(dx^{i_1}\wedge\ldots\wedge dx^{i_k})\wedge(dx^{j_1}\wedge\ldots\wedge dx^{j_l}) = dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}.$$

Proof. We start by proving (v). Choose a chart about x. Then

$$\begin{aligned}
&(dx^{i_1}\wedge\ldots\wedge dx^{i_k})\wedge(dx^{j_1}\wedge\ldots\wedge dx^{j_l})\\
&\quad= \frac{(k+l)!}{k!l!}\mathcal{A}\left((dx^{i_1}\wedge\ldots\wedge dx^{i_k})\otimes(dx^{j_1}\wedge\ldots\wedge dx^{j_l})\right)\\
&\quad= \frac{(k+l)!}{k!l!}\mathcal{A}\left(\sum_{\sigma\in S_k,\tau\in S_l}\mathrm{sgn}\sigma\,\mathrm{sgn}\tau\, dx^{i_{\sigma(1)}}\otimes\ldots\otimes dx^{i_{\sigma(k)}}\otimes dx^{j_{\tau(1)}}\otimes\ldots\otimes dx^{j_{\tau(l)}}\right)\\
&\quad= \frac{1}{k!l!}\sum_{\sigma\in S_k,\tau\in S_l}\mathrm{sgn}\sigma\,\mathrm{sgn}\tau\, dx^{i_{\sigma(1)}}\wedge\ldots\wedge dx^{i_{\sigma(k)}}\wedge dx^{j_{\tau(1)}}\wedge\ldots\wedge dx^{j_{\tau(l)}}\\
&\quad= dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}.
\end{aligned}$$

The homogeneity and distributivity properties of the wedge product are immediate from the definition. From this we can deduce the following expression for the wedge product in local coordinates: For $S = S_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}$ and $T = T_{j_1\ldots j_l}dx^{j_1}\wedge\ldots\wedge dx^{j_l}$ (summing over increasing k-tuples and l-tuples respectively)

$$S\wedge T = \frac{1}{k!l!}\sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} S_{i_1\ldots i_k}T_{j_1\ldots j_l}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}.$$

The associativity property can now be checked straightforwardly. Finally, we derive the anticommutativity property (iv): If $g = g_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}$ and $f = f_{j_1\ldots j_l}dx^{j_1}\wedge\ldots\wedge dx^{j_l}$, then

$$\begin{aligned}
g\wedge f &= \frac{1}{k!l!}\sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} g_{i_1\ldots i_k}f_{j_1\ldots j_l}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}\\
&= \frac{(-1)^k}{k!l!}\sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} g_{i_1\ldots i_k}f_{j_1\ldots j_l}dx^{j_1}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_2}\wedge\ldots\wedge dx^{j_l}\\
&\ldots\\
&= \frac{(-1)^{kl}}{k!l!}\sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} g_{i_1\ldots i_k}f_{j_1\ldots j_l}dx^{j_1}\wedge\ldots\wedge dx^{j_l}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\\
&= (-1)^{kl}f\wedge g.
\end{aligned}$$

□

13.3 Differential forms

Definition 13.3.1 A k-form ω on a differentiable manifold M is a smooth section of the bundle of alternating k-tensors on M. Equivalently, ω associates to each $x \in M$ an alternating k-tensor ω_x, in such a way that in any chart for M, the coefficients $\omega_{i_1 \dots i_k}$ are smooth functions. The space of k-forms on M is denoted $\Omega^k(M)$.

In particular, a 1-form is a covector field. We will also interpret a 0-form as being a smooth function on M, so $\Omega^0(M) = C^\infty(M)$.

By using the local definition in section 13.2, we can make sense of the wedge product as an operator which takes a k-form and an l-form to a $k+l$-form, which is associative, C^∞-linear in each argument, distributive and anticommutative.

13.4 The exterior derivative

Now we will define a differential operator on differential k-forms.

Proposition 13.4.1 *There exists a unique linear operator* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ *such that*
(i). *If $f \in \Omega^0(M) = C^\infty(M)$, then $\mathrm{d}f$ agrees with the differential of f (Definition 4.2.1);*
(ii). *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^k \omega \wedge (\mathrm{d}\eta);$$

(iii). $\mathrm{d}^2 = 0$.

Proof. Choose a chart $\varphi : U \to V$ with coordinate tangent vector fields $\partial_1, \dots, \partial_n$.

We will first produce the operator d acting on differential forms on $U \subseteq M$. On this region we have the smooth functions $x^1, \dots, x^n$ given by the components of the map φ. The differentials of these are the one-forms $dx^1, \dots, dx^n$, and in agreement with condition (iii) we assume that $d(dx^i) = 0$ for each i.

By induction and condition (ii), we deduce that $\mathrm{d}(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = 0$ for any k-tuple $i_1, \dots, i_k$.

Now let $f = \frac{1}{k!} \sum_{i_1,\dots,i_k} f_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$. The linearity of d, together with condition (ii) and condition (i), imply

$$\mathrm{d}f = \frac{1}{k!} \sum_{i_0,i_1,\dots,i_k} \partial_{i_0} f_{i_1 \dots i_k} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

One can easily check that this formula defines an operator which satisfies the required conditions. In particular we can compute d^2 to check that it vanishes:

$$\begin{aligned}
&\mathrm{d}^2\left(\omega_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right)\\
&\quad=\mathrm{d}\left(\partial_i\omega_{i_1\ldots i_k}dx^i\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right)\\
&\quad=\partial_j\partial_i\omega_{i_1\ldots i_k}dx^j\wedge dx^i\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\\
&\quad=\frac{1}{2}\left(\frac{\partial^2\omega_{i_1\ldots i_k}}{\partial x^j\partial x^i}-\frac{\partial^2\omega_{i_1\ldots i_k}}{\partial x^i\partial x^j}\right)dx^j\wedge dx^i\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\\
&\quad=0.
\end{aligned}$$

To extend this definition to all of M we need to check that it does not depend on the choice of coordinate chart. Let η be any other chart, with components $y^1,\ldots,y^n$. On the common domain of η and φ, we have $x^i = F^i(y)$, where $F=\varphi\circ\eta^{-1}$, and

$$dx^i=\frac{\partial F^i}{\partial y^j}dy^j.$$

This implies that

$$dx^{i_1}\wedge\ldots\wedge dx^{i_k}=\sum_{i_1,\ldots,i_k,j_1,\ldots,j_k}\frac{\partial F^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial F^{i_k}}{\partial y^{j_k}}dy^{j_1}\wedge\ldots\wedge dy^{j_k}.$$

Now we can check that if we define the operator d in the y coordinates, then $\mathrm{d}(dx^{i_1}\wedge\ldots\wedge dx^{i_k})=0$:

$$\begin{aligned}
\mathrm{d}\left(dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right)&=\sum_{I,J}\mathrm{d}\left(\frac{\partial F^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial F^{i_k}}{\partial y^{j_k}}dy^{j_1}\wedge\ldots\wedge dy^{j_k}\right)\\
&=\sum_{I,J,j_0}\sum_{m=1}^{k}\frac{\partial^2F^{i_m}}{\partial y^{j_m}\partial y^{j_0}}\prod_{p\neq m}\frac{\partial F^{i_p}}{\partial y^{j_p}}dy^{j_0}\wedge\ldots\wedge dy^{j_k}\\
&=\frac{1}{2}\sum_{I,J,j_0}\sum_{m=1}^{k}\left(\frac{\partial^2F^{i_m}}{\partial y^{j_m}\partial y^{j_0}}-\frac{\partial^2F^{i_m}}{\partial y^{j_0}\partial y^{j_m}}\right)\\
&\qquad\qquad\times\left(\prod_{p\neq m}\frac{\partial F^{i_p}}{\partial y^{j_p}}\right)dy^{j_0}\wedge\ldots\wedge dy^{j_k}\\
&=0
\end{aligned}$$

It follows (by linearity and distributivity) that the differential operators defined in the two charts agree. □

The differential operator may seem somewhat mysterious. The following example may help:

Example 13.4.1 (The exterior derivative on $\mathbb{R}^3$) The exterior derivative in $\mathbb{R}^3$ captures the differential operators which are normally defined as part of vector calculus: First, the differential operator of a 0-form (i.e. a function f) is just the differential of the function, which we can identify with the gradient vector field ∇f.

Next, consider d applied to a 1-form: For purposes of visualisation, we can identify a 1-form with a vector field by duality: The 1-form $\omega = \omega_1 dx^1 \wedge \omega_2 dx^2 + \omega_3 dx^3$ is identified with the vector field $(\omega_1, \omega_2, \omega_3)$. Applying d to ω, we obtain

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}(\omega_i dx^i) \\ &= \partial_j \omega_i dx^j \wedge dx^i \\ &= (\partial_1\omega_2 - \partial_2\omega_1) dx_1 \wedge dx^2 + (\partial_2\omega_3 - \partial_3 dx^2) dx^2 \wedge dx^3 \\ &\quad + (\partial_3\omega_1 - \partial_1\omega_3) dx^3 \wedge dx^1. \end{aligned}$$

The result is a 2-form. We identify 2-forms with vector fields again, by sending $adx^1 \wedge dx^2 + bdx^2 \wedge dx^3 + cdx^3 \wedge dx^1$ to the vector field (b, c, a). With this identification, the exterior derivative on 1-forms is equivalent to the curl operator on vector fields.

Finally, consider d applied to a 2-form (which we again associate to a vector field $V = (V_1, V_2, V_3)$). We find

$$\begin{aligned} &\mathrm{d}\left(V_3 dx^1 \wedge dx^2 + V_1 dx^2 \wedge dx^3 + V_2 dx^3 \wedge dx^1\right) \\ &\qquad = \left(\partial_3 V_3 + \partial_1 V_1 + \partial_2 V_2) dx^1 \wedge dx^2 \wedge dx^3\right) \\ &\qquad = (\mathrm{div} V)\ dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

Thus the exterior derivative acting on 2-forms is equivalent to the divergence operator acting on vector fields. The familiar identities from vector calculus that the curl of a gradient is zero and that the divergence of a curl is zero are therefore special cases of the identity $\mathrm{d}^2 = 0$.

13.5 Pull-back invariance

Now we will prove a remarkable result which really makes the theory of differential forms work:

Proposition 13.5.1 *Suppose M and N are differentiable manifolds, and $F : M \to N$ is a smooth map. Then for any $\omega \in \Omega^k(N)$ and $\eta \in \Omega^l(N)$,*

$$F_*(\omega \wedge \eta) = F_*(\omega) \wedge F_*\eta$$

and

$$\mathrm{d}(F_*\omega) = F_*(\mathrm{d}\omega).$$

Proof. The proof of the first statement is immediate from the definition. The second statement is proved by an argument identical to that used to prove that the definition of the exterior derivative does not depend on the chart in Proposition 13.4.1, except that the map F may be a smooth map between Euclidean spaces of different dimension. $\square$

13.6 Differential forms and orientability

There is a useful relationship between orientability of a differentiable manifold M^n and the space of n-forms $\Omega^n(M)$:

Proposition 13.6.1 *A differentiable manifold M is orientable if and only if there exists an n-form $\omega \in \Omega^n(M)$ which is nowhere vanishing on M.*

Proof. Suppose there exists such an n-form ω. Let $\mathcal{A}$ be the set of charts $\varphi : U \to V$ for M for which $\omega(\partial_1, \ldots, \partial_n) > 0$. Then $\mathcal{A}$ is an atlas for M, since any chart for M is either in $\mathcal{A}$ or has its composition with a reflection in $\mathcal{A}$ — in particular charts in $\mathcal{A}$ cover M. Furthermore $\mathcal{A}$ is an oriented atlas: For any pair of charts φ and η in $\mathcal{A}$ with non-trivial common domain of definition in M, we have

$$\partial_i^{(\eta)} = \left(D_i \eta \circ \varphi^{-1}\right)_i^j \partial_j^{(\varphi)},$$

and therefore by linearity and antisymmetry of ω,

$$\omega(\partial_1^{(\eta)}, \ldots, \partial_n^{(\eta)}) = \det D(\eta \circ \varphi^{-1}) \omega(\partial_1^{(\varphi)}, \ldots, \partial_n^{(\varphi)}).$$

By assumption, $\omega(\partial_1^{(\eta)}, \ldots, \partial_n^{(\eta)})$ and $\omega(\partial_1^{(\varphi)}, \ldots, \partial_n^{(\varphi)})$ are positive and non-zero. It follows that $\det D(\eta \circ \varphi^{-1}) > 0$.

Conversely, suppose M has an oriented atlas $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$. Let $\{\rho_\beta\}_{\beta \in \mathcal{J}}$ be a partition of unity subordinate to the cover $\{U_\alpha : \alpha \in \mathcal{I}\}$, so that for each $\beta \in \mathcal{J}$ there exists $\alpha(\beta) \in \mathcal{I}$ such that $\text{supp}\rho_\beta \subseteq U_{\alpha(\beta)}$. Define

$$\omega = \sum_{\beta \in \mathcal{J}} \rho_\beta \, dx^1_{\varphi_{\alpha(\beta)}} \wedge \ldots \wedge dx^n_{\varphi_{\alpha(\beta)}}.$$

Then ω is everywhere non-zero, since $dx^1_{\varphi_{\alpha(\beta_1)}} \wedge \ldots \wedge dx^n_{\varphi_{\alpha(\beta_1)}}$ is a positive multiple of $dx^1_{\varphi_{\alpha(\beta_2)}} \wedge \ldots \wedge dx^n_{\varphi_{\alpha(\beta_2)}}$ for $\beta_1 \neq \beta_2$. $\square$

We can interpret this in a slightly different way: For each $x \in M$, let $\text{Or}_x M$ be the set of equivalence classes of non-zero alternating n-tensors at x, where $\omega \sim \eta$ if ω is a positive multiple of η. $\text{Or}_x M$ has exactly two elements for each $x \in M$. Then we take $\text{Or}M = \cup_{x \in M} \text{Or}_x M$, which is the **orientation bundle** of M. On any chart $\varphi : U \to V$ for M, the restriction of this bundle to U is diffeomorphic to $U \times \mathbb{Z}_2$, but this is not necessarily true globally.

A slight modification of the proof of Proposition 13.6.1 gives the result that M is orientable if and only if the orientation bundle of M is trivial (that is, diffeomorphic to $M \times \mathbb{Z}_2$).

13.7 Frobenius' Theorem revisited

Differential forms allow an alternative formulation of the Theorem of Frobenius that we proved in Lecture 7 (Proposition 7.3.5). In order to formulate this, let $\mathcal{D}$ be a k-dimensional distribution on M. We relate this distribution to differential forms by considering the subspace $\Omega_0(\mathcal{D})$ of $\Omega(M)$ consisting of differential forms which yield zero when applied to vectors in the distribution $\mathcal{D}$. This subspace is closed under C^∞ scalar multiplication and under wedge products.

Proposition 13.7.1 *The distribution $\mathcal{D}$ is integrable if and only if the subspace $\Omega_0(\mathcal{D})$ is closed under exterior differentiation.*

Proof. First suppose $\mathcal{D}$ is integrable. Then locally we can choose charts $\varphi : U \to V \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ where the first k directions are tangent to $\mathcal{D}$.

In such a chart, forms in $\Omega_0^l(\mathcal{D})$ have the form

$$\omega_{i_1 \dots i_l} dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

where $\omega_{i_1 \dots i_l} = 0$ whenever $i_1, \dots, i_l \leq k$. This implies that $\partial_m \omega_{i_1 \dots i_l} = 0$ for $i_1, \dots, i_l \leq k$ and arbitrary m. Applying the exterior derivative, we find

$$\mathrm{d}\omega = \partial_m \omega_{i_1 \dots i_k} dx^m \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

which is clearly in $\Omega_0^{l+1}(\mathcal{D})$.

Next suppose $\mathcal{D}$ is not integrable. Then by Frobenius's theorem we can find vector fields X and Y in $\mathcal{X}(\mathcal{D})$ such that $[X, Y] \notin \mathcal{X}(\mathcal{D})$, say in particular $[X, Y]_x \notin \mathcal{D}_x$ for some $x \in M$. Choose a 1-form $\omega \in \Omega_0^1(\mathcal{D})$ such that $\omega_x([X, Y]_x) \neq 0$ (How would you construct such a 1-form?)

Then we compute:

$$\begin{aligned}
\mathrm{d}\omega(X, Y) &= X^i Y^j (\partial_i \omega_j - \partial_j \omega_i) \\
&= X^i \partial_i (Y^j \omega_j) - Y^j \partial_j (X^i \omega_i) - X^i (\partial_i Y_j) \omega_j + Y^j (\partial_j X_i) \omega_i \\
&= X\omega(Y) - Y\omega(X) - \omega([X, Y]) \\
&\neq 0 \quad \text{at } x,
\end{aligned}$$

since $\omega(Y) = 0$ and $\omega(X) = 0$ everywhere, and $\omega([X, Y]) \neq 0$ at x by assumption. Therefore $\Omega_0(\mathcal{D})$ is not closed under exterior differentiation. □

Exercise 13.7.1 The identity $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$ for the exterior derivative of a one-form generalises to an expression for exterior derivatives of k-forms: If $\omega \in \Omega^k(M)$, then

$$\begin{aligned} \mathrm{d}\omega(X_0,\dots,X_k) = &\sum_{i=0}^{k}(-1)^i X_i\omega(X_0,\dots,\hat{X}_i,\dots,X_k) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k). \end{aligned}$$

Prove this identity.