# Lecture 19. The Cartan Moving Frame Method

In this lecture we will explore the use of differential forms to compute connection coefficients and curvatures for given Riemannian metrics.

We start with a technical Lemma (Cartan's Lemma):

**Lemma 19.1** *Let $\omega_i$, $i = 1, \dots, n$ be a collection of 1-forms on a region $U$ which form a basis for the space of covectors at each point. Suppose $\eta_{ij}$, $1 \leq i, j \leq n$ be a collection of 1-forms which satisfy*

$$\sum_{j=1}^{n} \eta_{ij} \wedge \omega_j = 0$$

*for each $i$, and*

$$\eta_{ij} + \eta_{ji} = 0$$

*for each $i$ and $j$. Then $\eta_{ij} = 0$ for all $i$ and $j$ at every point of $U$.*

*Proof.* We can write uniquely

$$\eta_{ij} = \sum_k a_{ijk} \omega_k.$$

The identity $\eta_{ij} \wedge \omega_j$ then becomes

$$a_{ijk} - a_{ikj} = 0$$

for each $i$, $j$ and $k$. The second identity gives

$$a_{ijk} + a_{jik} = 0.$$

Thus we have

$$a_{ijk} = -a_{jik} = -a_{jki} = a_{kji} = a_{kij} = -a_{ikj} = -a_{ijk},$$

so that $a_{ijk} = 0$ and $\eta_{ij} = 0$. □

To proceed, suppose we have a manifold $M$ equipped with a Riemannian metric $g$, such that there is a smoothly defined orthonormal collection of vector fields $e_1, \dots, e_n$. Let $\omega_1, \dots, \omega_n$ be the dual basis of 1-forms.

Then we have the following result:

**Proposition 19.2** *There exists a unique collection of 1-forms* $\omega_{ij}$ *for* $1 \leq i, j \leq n$ *such that*

$$d\omega_i = \omega_{ij} \wedge \omega_j$$

*and*

$$\omega_{ij} + \omega_{ji} = 0.$$

*Proof.* We start by proving uniqueness: Suppose that $\bar{\omega}_{ij}$ is any other collection of one-forms satisfying the same conditions. Then let $\eta_{ij} = \bar{\omega}_{ij} - \omega_{ij}$. Then we have

$$\eta_{ij} \wedge \omega_j = \bar{\omega}_{ij} \wedge \omega_j - \omega_{ij} \wedge \omega_j = d\omega_i - d\omega_i = 0$$

and

$$\eta_{ij} + \eta_{ji} = 0.$$

By Lemma 19.1, we have $\eta_{ij} = 0$, and therefore $\omega_{ij} = \bar{\omega}_{ij}$.

Now we prove existence. We set $\omega_{ij} = g(\nabla_{e_k} e_i, e_j)\omega_k$, where $\nabla$ is the Levi-Civita connection of $g$. We use the identity

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

from section 13.7. Applying this with $\omega = \omega_i$, $X = e_j$ and $Y = e_k$, we find (noting $\omega_i(e_j) = \delta_{ij}$) that

$$d\omega_i(e_k, e_l) = -\omega_i([e_k, e_l]) = -\omega_i(\nabla_{e_k} e_l - \nabla_{e_l} e_k) = -\Gamma_{kli} + \Gamma_{lki}$$

and

$$\omega_{ij} \wedge \omega_j(e_k, e_l) = g(\nabla_{e_k} e_i, e_j)\delta_{jl} - g(\nabla_{e_l} e_i, e_j)\delta_{jk} = \Gamma_{kil} - \Gamma_{lik}.$$

The fact that these two agree follows from the compatibility of the connection with the metric, which gives

$$0 = e_k \delta_{ij} = \Gamma_{kij} + \Gamma_{kji}.$$

The same identity shows that $\omega_{ij} + \omega_{ji} = 0$. □

The 1-forms $\omega_{ij}$ are called the *connection 1-forms* corresponding to the frame $\{e_i\}$. Once these have been computed, we can compute the curvature as follows:

**Proposition 19.3** *For any orthonormal frame* $\{e_i\}$*, the curvature 2-form*

$$\Omega_{ij} = -\frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l$$

*can be computed from the connection 1-forms as follows:*

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}.$$

*Proof.* From the proof of the previous proposition, we have $\omega_{ij}(e_k) = g(\nabla_{e_k} e_i, e_j)$. Taking the exterior derivative, we find

$$\begin{aligned}
d\omega_{ij}(e_k, e_l) &= e_k \omega_{ij}(e_l) - e_l \omega_{ij}(e_k) - \omega_{ij}([e_k, e_l]) \\
&= e_k g(\nabla_{e_l} e_i, e_j) - e_l g(\nabla_{e_k} e_i, e_j) - g(\nabla_{[e_k, e_l]} e_i, e_j) \\
&= g(\nabla_{e_k} \nabla_{e_l} e_i, e_j) + g(\nabla_{e_l} e_i, \nabla_{e_k} e_j) \\
&\quad - g(\nabla_{e_l} \nabla_{e_k} e_i, e_j) - g(\nabla_{e_k} e_i, \nabla_{e_l} e_j) \\
&\quad - g(\nabla_{[e_k, e_l]} e_i, e_j) \\
&= -R_{ijkl} + \omega_{ip}(e_l)\omega_{jp}(e_k) - \omega_{ip}(e_k)\omega_{jp}(e_l) \\
&= (\Omega_{ij} + \omega_{ip} \wedge \omega_{pj})(e_k, e_l)
\end{aligned}$$

□

An important special case to keep in mind is the following: If $n = 2$, then we have the simple set of equations

$$\begin{aligned}
d\omega_1 &= \omega_{12} \wedge \omega_2 \\
d\omega_2 &= -\omega_{12} \wedge \omega_1 \\
\Omega_{12} &= d\omega_{12}.
\end{aligned}$$

In this case it is very easy to find the connection 1-form $\omega_{12}$, since if we write $\omega_{12} = a\omega_1 + b\omega_2$, then

$$d\omega_1(e_1, e_2) = a$$

and

$$d\omega_2(e_1, e_2) = -b$$

so that

$$\omega_{12} = d\omega_1(e_1, e_2)\omega_1 - d\omega_2(e_1, e_2)\omega_2.$$

The connection 2-form is also particularly simple in this case, since the curvature tensor has only one component up to symmetries: $R_{1212} = K$, the Gauss curvature. Thus we have

$$\Omega_{12} = -K\omega_1 \wedge \omega_2.$$

*Example 19.4* We will compute the curvatures of the Riemannian metric $g_{ij} = f^2 \delta_{ij}$ on a region of $\mathbb{R}^n$, where $f = f(x^n)$.

Here we have an obvious orthonormal frame given by $e_i = f^{-1}\partial_i$, and the corresponding basis of 1-forms $\omega_i = f dx^i$. Computing exterior derivatives, we find

$$d\omega_i = d(f dx^i) = f' dx^n \wedge dx^i.$$

This gives the equations

$$\omega_{ij} \wedge \omega_j = f'/f^2 \omega_n \wedge \omega_i$$

for $i < n$, and

$$\omega_{nj} \wedge \omega_j = 0.$$

Now define 1-forms $\eta_{ij}$ by taking $\eta_{ij} = \omega_{ij}$ for $i < j < n$, $\eta_{in} = \omega_{in} + f'/f^2 \omega_i$ for $i < n$, and requiring $\eta_{ij} + \eta_{ji} = 0$. Then the equations read:

$$\eta_{ij} \wedge \omega_j = 0.$$

Therefore by Cartan's Lemma, $\eta_{ij} = 0$ everywhere, and we deduce that $\omega_{ij} = 0$ for $i < j < n$ and $\omega_{in} = -f'/f^2 \omega_i$.

Taking exterior derivatives, we find:

$$d\omega_{ij} = 0$$

for $i < j < n$, and

$$d\omega_{in} = -(f'/f)'/f^2 \omega_n \wedge \omega_i.$$

Also we have

$$\omega_{ik} \wedge \omega_{kn} = 0$$

since the sum over $k$ has either $k = n$, hence $\omega_{kn} = 0$, or $k < n$, hence $\omega_{ik} = 0$. On the other hand we have

$$\omega_{ik} \wedge \omega_{kj} = \omega_{in} \wedge \omega_{nj} = -(f')^2/f^4 \omega_i \wedge \omega_j.$$

Combining these identities, we find

$$\Omega_{ij} = (f')^2/f^2 \omega_i \wedge \omega_j$$

for $i < j < n$, and

$$\Omega_{in} = (f'/f)'/f^2 \omega_i \wedge \omega_n.$$

This shows that

$$R_{inin} = -(f'/f)'/f^2,$$

for $1 \leq i \leq n-1$, and and

$$R_{ijij} = -(f')^2/f^4$$

for $1 < i < j < n$, while (except for symmetries) all other curvature components are zero.

An important special case of this example is where $f(x) = x^{-1}$. Then we find $(f'/f)'/f^2 = 1$ and $(f')^2/f^4 = 1$, and therefore all sectional curvatures of this metric are equal to $-1$.