Lecture 4. Tangent vectors

4.1 The tangent space to a point

Let M^n be a smooth manifold, and x a point in M. In the special case where M is a submanifold of Euclidean space $\mathbb{R}^N$, there is no difficulty in defining a space of tangent vectors to M at x: Locally M is given as the zero level-set of a submersion $G : U \to \mathbb{R}^{N-n}$ from an open set U of $\mathbb{R}^N$ containing x, and we can define the tangent space to be $\ker(D_x G)$, the subspace of vectors which map to 0 under the derivative of G. Alterntively, if we describe M locally as the image of an embedding $\varphi : U \to \mathbb{R}^N$ from an open set U of $\mathbb{R}^n$, then we can take the tangent space to M at x to be the subspace $\operatorname{rng}\left(D_{\varphi^{-1}(x)}\varphi\right) = \{D_{\varphi^{-1}(x)}\varphi(u) :\ u \in \mathbb{R}^n\}$, the image subspace of the derivative map.

If M is an abstract manifold, however, then we do not have any such convenient notion of a tangent vector.

From calculus on $\mathbb{R}^n$ we have several complementary ways of thinking about tangent vectors: As k-tuples of real numbers; as 'directions' in space, such as the tangent vector of a curve; or as directional derivatives.

I will give three alternative candidates for the tangent space to a smooth manifold M, and then show that they are equivalent:

First, for $x \in M$ we define $T_x M$ to be the set of pairs (φ, u) where $\varphi : U \to V$ is a chart in the atlas for M with $x \in U$, and u is an element of $\mathbb{R}^n$, modulo the equivalence relation which identifies a pair (φ, u) with a pair (η, w) if and only if u maps to w under the derivative of the transition map between the two charts:

$$D_{\varphi(x)}\left(\eta \circ \varphi^{-1}\right)(u) = w. \tag{4.1}$$

Remark. The basic idea is this: We think of a vector as being an 'arrow' telling us which way to move inside the manifold. This information on which way to move is encoded by viewing the motion through a chart φ, and seeing which way we move 'downstairs' in the chart (this corresponds to a vector in n-dimensional space according to the usual notion of a velocity vector). The equivalence relation just removes the ambiguity of a choice of chart through which to follow the motion.

Another way to think about it is the following: We have a local description for M using charts, and we know what a vector is 'downstairs' in each chart.

We want to define a space of vectors T_xM 'upstairs' in such a way that the derivative map $D_x\varphi$ of the chart map φ makes sense as a linear operator between the vector spaces T_xM and $\mathbb{R}^n$, and so that the chain rule continues to hold. Then we would have for any vector $v \in T_pM$ vectors $u = D_x\varphi(v) \in \mathbb{R}^n$, and $w = D_x\eta(v) \in \mathbb{R}^n$. Writing this another way (implicitly assuming the chain rule holds) we have

$$D_{\varphi(x)}\varphi^{-1}(u) = v = D_{\eta(x)}\eta^{-1}(w).$$

The chain rule would then imply $D_{\varphi(x)}\left(\eta \circ \varphi^{-1}\right)(u) = w$.

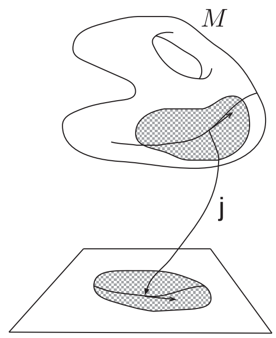


Fig.1: A tangent vector to M at x is implicitly defined by a curve through x

The second definition expresses even more explicitly the idea of a 'velocity vector' in the manifold: We define M_x to be the space of smooth paths in M through x (i.e. smooth maps $\gamma : I \to M$ with $\gamma(0) = x$) modulo the equivalence relation which identifies any two curves if they agree to first order (as measured in some chart): $\gamma \sim \sigma \Leftrightarrow (\varphi\circ\gamma)'(0) = (\varphi\circ\sigma)'(0)$ for some chart $\varphi : U \to V$ with $x \in U$. The equivalence does not depend on the choice of chart: If we change to a chart η, then we have

$$(\eta\circ\gamma)'(0) = \left((\eta\circ\varphi^{-1})\circ(\varphi\circ\gamma)\right)'(0) = D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)(\varphi\circ\gamma)'(0)$$

and similarly for sigma, so $(\eta\circ\gamma)'(0) = (\eta\circ\sigma)'(0)$.

Finally, we define D_xM to be the space of derivations at x. Here a derivation is a map v from the space of smooth functions $C^\infty(M)$ to $\mathbb{R}$, such that for any real numbers c_1 and c_2 and any smooth functions f and g on M,

$$\begin{aligned} v(c_1f + c_2g) &= c_1v(f) + c_2v(g) \qquad \text{and} \\ v(fg) &= f(x)v(g) + g(x)v(f) \end{aligned} \tag{4.2}$$

The archetypal example of a derivation is of course the directional derivative of a function along a curve: Given a smooth path $\gamma : I \to M$ with $\gamma(0) = x$, we can define

$$v(f) = \frac{d}{dt} (f \circ \gamma) \Big|_{t=0}, \tag{4.3}$$

and this defines a derivation at x. We will see below that all derivations are of this form.

Proposition 4.1.1 *There are natural isomorphisms between the three spaces M_x, T_xM, and D_xM.*

$$\begin{tikzcd} TM_x \arrow[rr, "a"] & & M_x \arrow[dl, "b"] \\ & D_xM \arrow[ul, "c"] & \end{tikzcd}$$

Proof. First, we write down the isomorphisms: Given an equivalence class $[(\varphi, u)]$ in T_xM, we take $\alpha([(\varphi, u)])$ to be the equivalence class of the smooth path γ defined by

$$\gamma(t) = \varphi^{-1}(\varphi(x) + tu). \tag{4.4}$$

The map α is well-defined, since if (η, w) is another representative of the same equivalence class in T_xM, then α gives the equivalence class of the curve $\sigma(t) = \eta^{-1}(\eta(x) + tw)$, and

$$\begin{aligned} (\varphi \circ \sigma)'(0) &= \left((\varphi \circ \eta^{-1}) \circ (\eta \circ \sigma)\right)'(0) \\ &= D_{\eta(x)} \left(\varphi \circ \eta^{-1}\right) (\eta \circ \sigma)'(0) \\ &= D_{\eta(x)} \left(\varphi \circ \eta^{-1}\right)(w) \\ &= u \\ &= (\varphi \circ \gamma)'(0), \end{aligned}$$

so $[\sigma] = [\gamma]$.

Given an element $[\gamma] \in M_x$, we take $\beta([\gamma])$ to be the natural derivation v defined by Eq. (4.3). Again, we need to check that this is well-defined: Suppose $[\sigma] = [\gamma]$. Then

$$\begin{aligned} \frac{d}{dt} (f \circ \gamma) \Big|_{t=0} &= D_{\varphi(x)} \left(f \circ \varphi^{-1}\right) (\varphi \circ \gamma)'(0) \\ &= D_{\varphi(x)} \left(f \circ \varphi^{-1}\right) (\varphi \circ \sigma)'(0) \\ &= \frac{d}{dt} (f \circ \sigma) \Big|_{t=0} \end{aligned}$$

for any smooth function f.

Finally, given a derivation v, we choose a chart φ containing x, and take $\chi(v)$ to be the element of T_xM given by taking the equivalence class $[(\varphi, u)]$ where $u = \big(v(\varphi^1), \dots, v(\varphi^n)\big)$. Here φ^i is the ith component function of the chart φ.

There is a technicality involved here: In the definition, derivations were assumed to act on smooth functions defined on all of M. However, φ^i is defined only on an open set U of M. In order to overcome this difficulty, we will extend φ_i (somewhat arbitrarily) to give a smooth map on all of M. For concreteness, we can proceed as follows (the functions we construct here will prove useful later on as well):

$$\xi(z) = \begin{cases} \exp\left(\frac{1}{z^2-1}\right) & \text{for } -1 < z < 1; \\ 0 & \text{for } |z| \geq 1. \end{cases} \tag{4.5}$$

and

$$\rho(z) = \frac{\int_{-1}^{z} \xi(z')dz'}{\int_{-1}^{1} \xi(z')dz'}. \tag{4.6}$$

Then

- ξ is a C^∞ function on all of $\mathbb{R}$, with $\xi(z)$ equal to zero whenever $|z| \geq 1$, and $\xi(z) > 0$ for $|z| < 1$;
- ρ is a C^∞ function on all of $\mathbb{R}$, which is zero whenever $z \leq -1$, identically 1 for $z > 1$, and strictly increasing for $z \in (-1, 1)$.

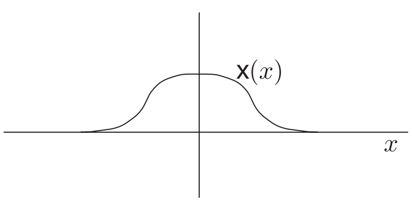


Fig. 2: The 'bump function' or 'cut-off function' ξ.

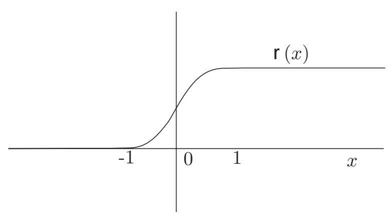


Fig. 3: A C^∞ 'ramp function' ρ.

Now, given the chart $\varphi : U \to V$ with $x \in U$, choose a number r sufficiently small to ensure that the closed ball of radius $4r$ about $\varphi(x)$ is contained in V. Then define a function $\tilde{\rho}$ on M by

$$\tilde{\rho}(y) = \begin{cases} \rho\left(3 - \frac{|\varphi(y)-\varphi(x)|}{r}\right) & \text{for } y \in U; \\ 0 & \text{for all other } y \in M. \end{cases} \tag{4.7}$$

Exercise 4.1 Prove that $\tilde{\rho}$ is a smooth function on M.

Note that this construction gives a function $\tilde{\rho}$ which is identically equal to 1 on a neighbourhood of x, and identically zero in the complement of a larger neighbourhood.

Now we can make sense of our definition above:

Definition 4.1.2 If $f : U \to \mathbb{R}$ is a smooth function on an open set U of M containing x, we define $v(f) = v(\tilde{f})$, where $\tilde{f}$ is any smooth function on M which agrees with f on a neighbourhood of x.

For this to make sense we need to check that there is some smooth function $\tilde{f}$ on M which agrees with f on a neighbourhood of x, and that the definition does not depend on which such function we choose.

Without loss of generality we suppose f is defined on U as above, and define $\tilde{\rho}$ as above. Then we define

$$\tilde{f}(y) = \begin{cases} f(y)\tilde{\rho}(y) & \text{for } y \in U; \\ 0 & \text{for } y \in M \backslash U. \end{cases} \tag{4.8}$$

$\tilde{f}$ is smooth, and agrees with f on the set $\varphi^{-1}(B_{2r}(\varphi(x)))$.

Next we need to check that $v(\tilde{f})$ does not change if we choose a different function agreeing with f on a neighbourhood of x.

Lemma 4.1.3 *Suppose f and g are two smooth functions on M which agree on a neighbourhood of x. Then $v(f) = v(g)$.*

Proof. Without loss of generality, assume that f and g agree on an open set U containing x, and construct a 'bump' function $\tilde{\rho}$ in U as above. Then we observe that $\tilde{\rho}(f - g)$ is identically zero on M, and that $v(0) = v(0.0) = 0.v(0) = 0$. Therefore

$$0 = v(\tilde{\rho}(f - g)) = \tilde{\rho}(x)v(f - g) + (f(x) - g(x))v(\tilde{\rho}) = v(f) - v(g)$$

since $f(x) - g(x) = 0$ and $\tilde{\rho}(x) = 1$. □

This shows that the definition of $v(f)$ makes sense, and so our definition of $\chi(v)$ makes sense. However we still need to check that $\chi(v)$ does not depend on the choice of a chart φ. Suppose we instead use another chart η. Then we have in a small region about x,

$$\eta^i(y) = \eta^i(x) + \sum_{j=1}^{n} G^i_j(y)(\varphi^j(y) - \varphi^j(x)) \tag{4.9}$$

for each $i = 1, \ldots, n$, where G^i_j is a smooth function on a neighbourhood of x for which $G^i_j(x) = \frac{\partial}{\partial z^j}\left(\eta^i \circ \varphi^{-1}\right)\Big|_{\varphi(x)}$. To prove this, consider the Taylor expansion for $\eta^i \circ \varphi^{-1}$ on the set V, where φ is the chart from U to V.

Now apply v to Eq. (4.9). Note that the first term is a constant.

Lemma 4.1.4 *$v(c) = 0$ for any constant c.*

Proof.

$$v(1) = v(1.1) = 1.v(1) = 1.v(1) = 2v(1) \Longrightarrow v(1) = 0 \Longrightarrow v(c) = cv(1) = 0.$$

□

v applied to η^i gives

$$\begin{aligned} v(\eta^i) &= \sum_{j=1}^n G^i_j(x)v(\varphi^j) + (\varphi^j(x) - \varphi^j(x))v(G^i_j) \\ &= \sum_{j=1}^n \frac{\partial}{\partial z^j}\left(\eta^i \circ \varphi^{-1}\right)\Big|_{\varphi(x)} v(\varphi^j). \end{aligned}$$

Therefore we have

$$\begin{aligned} \sum_{i=1}^n v(\eta^i)e_i &= \sum_{i,j=1}^n \frac{\partial}{\partial z^j}\left(\eta^i \circ \varphi^{-1}\right)\Big|_{\varphi(x)} v(\varphi^j)e_i \\ &= D_{\varphi(x)}\left(\eta \circ \varphi^{-1}\right)\left(\sum_{i=1}^n v(\varphi^i)e_i\right) \end{aligned}$$

and so $[(\varphi, \sum_{i=1}^n v(\varphi^i)e_i)] = [(\eta, \sum_{i=1}^n v(\eta^i)e_i)]$ and χ is independent of the choice of chart.

In order to prove the proposition, it is enough to show that the three triple compositions $\chi \circ \beta \circ \alpha$, $\beta \circ \alpha \circ \chi$, and $\alpha \circ \chi \circ \beta$ are just the identity map on each of the three spaces.

We have

$$\begin{aligned} \chi \circ \beta \circ \alpha([(\varphi, u)]) &= \chi \circ \beta\left([t \mapsto \varphi^{-1}\left(\varphi(x) + tu\right)]\right) \\ &= \chi\left(f \mapsto D_{\varphi(x)}\left(f \circ \varphi^{-1}\right)(u)\right) \\ &= \left[\left(\varphi, \sum_{i=1}^n D_{\varphi(x)}\left(\varphi^i \circ \varphi^{-1}\right)(u)e_i\right)\right] \\ &= [(\varphi, u)]. \end{aligned}$$

Similarly we have

$$\begin{aligned} \alpha \circ \chi \circ \beta([\sigma]) &= \alpha \circ \chi\left(f \mapsto \frac{d}{dt}\left(f \circ \gamma\right)\Big|_{t=0}\right) \\ &= \alpha\left(\left[\left(\varphi, \sum_{i=1}^n \left(\varphi \circ \gamma\right)'(0)\right)\right]\right) \\ &= \left[t \mapsto \varphi^{-1}\left(\varphi(x) + t\left(\varphi \circ \gamma\right)'(0)\right)\right], \end{aligned}$$

and this curve is clearly in the same equivalence class as γ.

Finally, we have

$$\begin{aligned}
\beta \circ \alpha \circ \chi(v) &= \beta \circ \alpha \left(\left[\left(\varphi, \sum_{i=1}^{n} v(\varphi^i) e_i \right) \right] \right) \\
&= \beta \left(\left[t \mapsto \varphi^{-1} \left(\varphi(x) + t \sum_{i=1}^{n} v(\varphi^i) e_i \right) \right] \right) \\
&= \left(f \mapsto \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \left(f \circ \varphi^{-1} \right) \Big|_{\varphi(x)} v(\varphi^i) \right).
\end{aligned}$$

We need to show that this is the same as v. To show this, we note (using the Taylor expansion for $f \circ \varphi^{-1}$) that

$$f(y) = f(x) + \sum_{i=1}^{n} G_i(y) \left(\varphi^i(y) - \varphi^i(x) \right)$$

for y in a sufficiently small neighbourhood of x, where G_i is a smooth function with $G_i(x) = \frac{\partial}{\partial z^i} \left(f \circ \varphi^{-1} \right) |_{\varphi(x)}$. Applying v to this expression, we find

$$v(f) = \sum_{i=1}^{n} G_i(x) v(\varphi^i) = \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \left(f \circ \varphi^{-1} \right) \Big|_{\varphi(x)} v(\varphi^i) \tag{4.10}$$

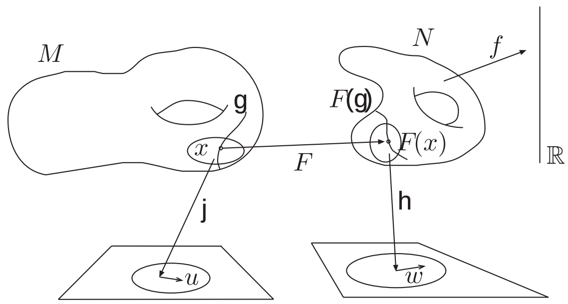
and the right hand side is the same as $\beta \circ \alpha \circ \gamma v(f)$.

Having established the equivalence of the three spaces M_x, T_x, and $D_x M$, I will from now on keep only the notation $T_x M$ (the tangent space to x at M) while continuing to use all three different notions of a tangent vector.

4.2 The differential of a map

Definition 4.2.1 Let $f : M \to \mathbb{R}$ be a smooth function. Then we define the differential $d_x f$ of f at the point $x \in M$ to be the linear function on the tangent space $T_x M$ given by $(d_x f)(v) = v(f)$ for each $v \in T_x M$ (thinking of v as a derivation). Let $F : M \to N$ be a smooth map between two manifolds. Then we define the differential $D_x F$ of F at $x \in M$ to be the linear map from $T_x M$ to $T_{F(x)} N$ given by $((D_x F)(v))(f) = v(f \circ F)$ for any $v \in T_x M$ and any $f \in C^\infty(M)$.

It is useful to describe the differential of a map in terms of the other representations of tangent vectors. If v is the vector corresponding to the equivalence class $[(\varphi, u)]$, then we have $v : f \mapsto D_{\varphi(x)}(f \circ \varphi^{-1})(u)$, and so by the definition above, $D_x F(v)$ sends a smooth function f on N to $v(f \circ F)$:



$$\begin{aligned} D_xF(v) : f \mapsto & D_{\varphi(x)}\left(f \circ F \circ \varphi^{-1}\right)(u) \\ & = D_{\eta(F(x))}\left(f \circ \eta^{-1}\right) \circ D_{\varphi(x)}\left(\eta \circ F \circ \varphi^{-1}\right)(u) \end{aligned}$$

which is the vector corresponding to $\left[\left(\eta, D_{\varphi(x)}\left(\eta \circ F \circ \varphi^{-1}\right)(u)\right)\right]$.

Alternatively, if we think of a vector v as the tangent vector of a curve γ, then we have $v : \ f \mapsto (f \circ \gamma)'(0)$, and so $D_xF(v) : \ f \mapsto (f \circ F \circ \gamma)'(0)$, which is the tangent vector of the curve $F \circ \gamma$. In other words,

$$D_xF([\gamma]) = [F \circ \gamma]. \tag{4.11}$$

In most situations we can use the differential of a map in exactly the same way as we use the derivative for maps between Euclidean spaces. In particular, we have the following results:

Theorem 4.2.2 The Chain Rule *If $F : M \to N$ and $G : N \to P$ are smooth maps between manifolds, then so is $G \circ F$, and*

$$D_x\left(G \circ F\right) = D_{F(x)}G \circ D_xF.$$

Proof. By Eq. (4.11),

$$D_x(G \circ F)([\gamma]) = [G \circ F \circ \gamma] = D_{F(x)}G([F \circ \gamma]) = D_{F(x)}G\left(D_xF([\gamma])\right).$$

□

Theorem 4.2.3 The inverse function theorem *Let $F : M \to N$ be a smooth map, and suppose D_xF is an isomorphism for some $x \in M$. Then there exists an open set $U \subset M$ containing x and an open set $V \subset N$ containing $F(x)$ such that $F\Big|_U$ is a diffeomorphism from U to V.*

Proof. We have $D_xF([(\varphi, u)]) = [\eta, D_{\varphi(x)}\left(\eta \circ F \circ \varphi^{-1}\right)(u)]$, so D_xF is an isomorphism if and only if $D_{\varphi(x)}\left(\eta \circ F \circ \varphi^{-1}\right)$ is an isomorphism. The result follows by applying the usual inverse function theorem to $\eta \circ F \circ \varphi^{-1}$.

Theorem 4.2.4 The implicit function theorem (surjective form) *Let $F : M \to N$ be a smooth map, with D_xF surjective for some $x \in M$. Then there exists a neighbourhood U of x such that $F^{-1}(F(x)) \cap U$ is a smooth submanifold of M.*

Theorem 4.2.5 The implicit function theorem (injective form) *Let $F : M \to N$ be a smooth map, with D_xF injective for some $x \in M$. Then there exists a neighbourhood U of x such that $F\Big|_U$ is an embedding.*

These two theorems follow directly from the corresponding theorems for smooth maps between Euclidean spaces.

4.3 Coordinate tangent vectors

Given a chart $\varphi : U \to V$ with $x \in U$, we can construct a convenient basis for T_xM: We simply take the vectors corresponding to the equivalence classes $[(\varphi, e_i)]$, where $e_1, \ldots, e_n$ are the standard basis vectors for $\mathbb{R}^n$. We use the notation $\partial_i = [(\varphi, e_i)]$, suppressing explicit mention of the chart φ. As a derivation, this means that $\partial_i f = \frac{\partial}{\partial z^i}\left(f \circ \varphi^{-1}\right)\Big|_{\varphi(x)}$. In other words, ∂_i is just the derivation given by taking the ith partial derivative in the coordinates supplied by φ. It is immediate from Proposition 4.1.1 that $\{\partial_1, \ldots, \partial_n\}$ is a basis for T_xM.

4.4 The tangent bundle

We have just constructed a tangent space at each point of the manifold M. When we put all of these spaces together, we get the *tangent bundle* TM of M:

$$TM = \{(p, v) : \, p \in M, v \in T_pM\}.$$

If M has dimension n, we can endow TM with the structure of a $2n$-dimensional manifold, as follows: Define $\pi : TM \to M$ to be the projection which sends (p, v) to p. Given a chart $\varphi : U \to V$ for M, we can define a chart $\tilde{\varphi}$ for TM on the set $\pi^{-1}(U) = \{(p, v) \in TM : \, p \in U\}$, by

$$\tilde{\varphi}(p, v) = \big(\varphi(p), v(\varphi^1), \ldots, v(\varphi^n)\big) \in \mathbb{R}^{2n}.$$

Thus the first n coordinates describe the point p, and the last n give the components of the vector v with respect to the basis of coordinate tangent vectors $\{\partial_i\}_{i=1}^n$, since by Eq. (4.10),

$$v(f)=\sum_{i=1}^n \frac{\partial}{\partial z^i}\left(f\circ\varphi^{-1}\right)\Big|_{\varphi(x)} v(\varphi^i)=\sum_{i=1}^n v(\varphi^i)\partial_i(f) \tag{4.12}$$

for any smooth f, and hence $v=\sum_{i=1}^n v(\varphi^i)\partial_i$. For convenience we will often write the coordinates on TM as $(x^1,\dots x^n,\dot{x}^1,\dots,\dot{x}^n)$

To check that these charts give TM a manifold structure, we need to compute the transition maps. Suppose we have two charts $\varphi: U\to V$ and $\eta: W\to Z$, overlapping non-trivially. Then $\tilde{\eta}\circ\tilde{\varphi}^{-1}$ first takes a $2n$-tuple $(x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n)$ to the element $\left(\varphi^{-1}(x^1,\dots,x^n),\sum_{i=1}^n \dot{x}^i\partial_i^{(\varphi)}\right)$ of TM, then maps this to $\mathbb{R}^{2n}$ by $\tilde{\eta}$. Here we add the superscript (φ) to distinguish the coordinate tangent vectors coming from the chart φ from those given by the chart η. The first n coordinates of the result are then just $\eta\circ\varphi^{-1}(x^1,\dots,x^n)$. To compute the last n coordinates, we need to $\partial_i^{(\varphi)}$ in terms of the coordinate tangent vectors $\partial_j^{(\eta)}$: We have

$$\begin{aligned}
\partial_i^{(\varphi)} f &= D_{\varphi(x)}\left(f\circ\varphi^{-1}\right)(e_i)\\
&= D_{\varphi(x)}\left(\left(f\circ\eta^{-1}\right)\circ\left(\eta\circ\varphi^{-1}\right)\right)(e_i)\\
&= D_{\eta(x)}\left(f\circ\eta^{-1}\right)\circ D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)(e_i)\\
&= D_{\eta(x)}\left(f\circ\eta^{-1}\right)\left(\sum_{j=1}^n D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)_i^j e_j\right)\\
&= \sum_{j=1}^n D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)_i^j D_{\eta(x)}\left(f\circ\eta^{-1}\right)(e_j)\\
&= \sum_{j=1}^n D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)_i^j \partial_j^{(\eta)} f
\end{aligned}$$

for every smooth function f. Therefore

$$\sum_{i=1}^n \dot{x}^i\partial_i^{(\varphi)}=\sum_{i,j=1}^n D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)_i^j \partial_j^{(\eta)}$$

and

$$\tilde{\eta}\circ\tilde{\varphi}^{-1}(x,\dot{x})=\left(\eta\circ\varphi^{-1}(x), D_{\varphi(x)}\left(\eta\circ\varphi^{-1}\right)(\dot{x})\right).$$

Since $\eta\circ\varphi^{-1}$ is smooth by assumption, so is its matrix of derivatives $D(\eta\circ\varphi^{-1})$, so $\tilde{\eta}\circ\tilde{\varphi}^{-1}$ is a smooth map on $\mathbb{R}^{2n}$.

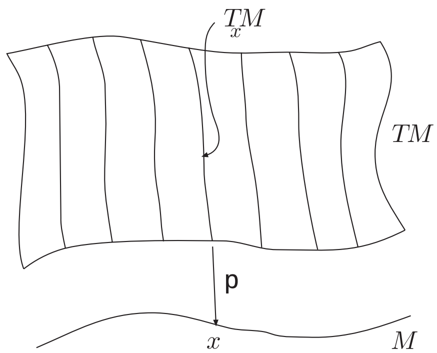


Fig.4: Schematic depiction of the tangent bundle TM as a 'bundle' of the fibres T_xM over the manifold M.

4.5 Vector fields

A vector field on M is given by choosing a vector in each of the tangent spaces T_xM. We require also that the choice of vector varies smoothly, in the following sense: A choice of V_x in each tangent space T_xM gives us a map from M to TM, $x \mapsto V_x$. A smooth vector field is defined to be one for which the map $x \mapsto V_x$ is a smooth map from the manifold M to the manifold TM.

In order to check whether a vector field is smooth, we can work locally. In a chart φ, the vector field can be written as $V_x = \sum_{i=1}^n V_x^i \partial x_i$, and this gives us n functions $V_x^1, \ldots, V_x^n$. Then it is easy to show that V is a smooth vector field if and only if these component functions are smooth as functions on M. That is, when viewed through a chart the vector field is smooth, in the usual sense of an n-tuple of smooth functions.

Our notion of a tangent vector as a derivation allows us to think of a vector field in another way:

Proposition 4.5.1 *Smooth vector fields are in one-to-one correspondence with derivations $V : C^\infty(M) \to C^\infty(M)$ satisfying the two conditions*

$$\begin{aligned} V(c_1 f_1 + c_2 f_2) &= c_1 V(f_1) + c_2 V(f_2) \\ V(f_1 f_2) &= f_1 V(f_2) + f_2 V(f_1) \end{aligned}$$

for any constants c_1 and c_2 and any smooth functions f_1 and f_2.

Proof : Given such a derivation V, and $p \in M$, the map $V_p : C^\infty(M) \to \mathbb{R}$ given by $V_p(f) = (V(f))(p)$ is a derivation at p, and hence defines an element of T_pM. The map $p \mapsto V_p$ from M to TM is therefore a vector field, and it remains to check smoothness. The components of V_p with respect to the coordinate tangent basis $\partial_1, \ldots, \partial_n$ for a chart $\varphi : U \to V$ is given by

$$V_p^i = V_p(\varphi^i)$$

which is by assumption a smooth function of p for each i (since φ^i is a smooth function and V maps smooth functions to smooth functions – here one should really multiply φ^i by a smooth cut-off function to convert it to a smooth function on the whole of M). Therefore the vector field is smooth.

Conversely, given a smooth vector field $x \mapsto V_x \in T_xM$, the map

$$(V(f))(x) = V_x(f)$$

satisfies the two conditions in the proposition and takes a smooth function to a smooth function. □

A common notation is to refer to the space of smooth vector fields on M as $\mathcal{X}(\mathcal{M})$. Over a small region of a manifold (such as a chart), the space of smooth vector fields is in $1:1$ correspondence with n-tuples of smooth functions. However, when looked at over the whole manifold things are not so simple. For example, a theorem of algebraic topology says that there are no continuous vector fields on the sphere S^2 which are everywhere non-zero ("the sphere has no hair"). On the other hand there are certainly nonzero functions on S^2 (constants, for example).