Lecture 9. Riemannian metrics

This lecture introduces Riemannian metrics, which define lengths of vectors and curves in the manifold.

9.1 Definition

There is one more crucial ingredient which we need to introduce for dealing with manifolds: Lengths and angles. Given a smooth manifold, since we know what it means for a curve in the manifold to be smooth, and we have a well-defined notion of the tangent vector to a curve, all we need in order to have a notion of distance on the manifold is a way of defining the *speed* of a curve — that is, the length of its tangent vector.

Definition 9.1.1 A **Riemannian metric** g on a smooth manifold M is a smoothly chosen inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ on each of the tangent spaces T_xM of M. In other words, for each $x \in M$, $g = g_x$ satisfies
(1). $g(u, v) = g(v, u)$ for all $u, v \in T_xM$;
(2). $g(u, u) \geq 0$ for all $u \in T_xM$;
(3). $g(u, u) = 0$ if and only if $u = 0$.

Furthermore, g is smooth in the sense that for any smooth vector fields X and Y, the function $x \mapsto g_x(X_x, Y_x)$ is smooth.

Locally, a metric can be described in terms of its coefficients in a local chart, defined by $g_{ij} = g(\partial_i, \partial_j)$. The smoothness of g is equivalent to the smoothness of all the coefficient functions g_{ij} in some chart.

Example 9.1.2 The standard inner product on Euclidean space is a special example of a Riemannian metric. $\mathbb{R}^n$ can be made a Riemannian manifold in many ways: Let f_{ij} be a bounded, smooth function for each i and j in $\{1, \ldots, n\}$, with $f_{ij} = f_{ji}$. Then for C sufficiently large, the functions $g_{ij} = C\delta_{ij} + f_{ij}$ are positive definite everywhere, and so define a Riemannian metric.

9.2 Existence of Riemannian metrics

Every smooth manifold carries a Riemannian metric (in fact, many of them). We will prove this using an argument very similar to that used in showing the existence of connections.

Choose an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$, and a subordinate partition of unity $\{\rho_\alpha\}$. On each of the regions V_α in $\mathbb{R}^n$, choose a Riemannian metric $g^{(\alpha)}$ (as in example 9.1.2). Then define

$$g(u,v) = \sum_\alpha \rho_\alpha g^{(\alpha)}\left(D\varphi_\alpha(u), D\varphi_\alpha(v)\right).$$

This is clearly symmetric; $g(u,u) \geq 0$; and $g(u,u) = 0$ iff $u = 0$. Furthermore it is smooth, and so defines a Riemannian metric on M.

9.3 Length and distance

Definition 9.3.1 Let $\gamma : [a,b] \to M$ be a (piecewise) smooth curve. Then the **length** $L[\gamma]$ of γ is defined by $L[\gamma] = \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}\,dt$. Given two points p and q in M, we define the **distance** from p to q by

$$d(p,q) = \inf\left\{ L[\gamma] \;\middle|\; \gamma : [a,b] \to M \text{ piecewise smooth, } \gamma(a) = p, \gamma(b) = q \right\}.$$

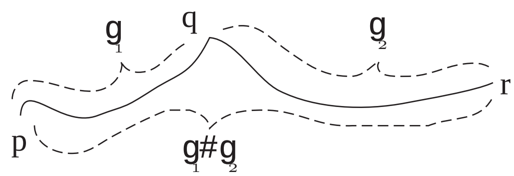
Proposition 9.3.2 *If M is a Riemannian manifold with metric g, then M is a metric space with the distance function d defined above. The metric topology agrees with the manifold topology.*

Proof. The symmetry of the distance function is immediate, as is its non-negativity. The triangle inequality is also easily established: For any curves $\gamma_1 : [a,b] \to M$ and $\gamma_2 : [a',b'] \to M$ with $\gamma_1(b) = \gamma_2(a')$, we can define the concatenation $\gamma_1\#\gamma_2 : [0, b+b'-a-a'] \to M$ by

$$\gamma_1\#\gamma_2(t) = \begin{cases} \gamma_1(t+a) & \text{for } 0 \leq t \leq b-a; \\ \gamma_2(t+a-b+a') & \text{for } b-a \leq t \leq b+b'-a-a'. \end{cases}$$

Thus $\gamma_1\#\gamma_2$ is a curve with length $L[\gamma_1] + L[\gamma_2]$. Given points p, q, and r in M, for any $\varepsilon > 0$ we can choose curves γ_1 joining p to q and γ_2 joining q to r, such that $L[\gamma_1] < d(p,q) + \varepsilon$ and $L[\gamma_2], d(q,r) + \varepsilon$. Therefore

$$d(p,r) \leq L[\gamma_1\#\gamma_2] = L[\gamma_1] + L[\gamma_2] < d(p,q) + d(q,r) + 2\varepsilon,$$



which gives the triangle inequality when $\varepsilon \to 0$.

We still need to check that $d(x, y) = 0$ only when $x = y$. Suppose we have $x \neq y$ with $d(x, y) = 0$. Choose a chart $\varphi : U \to V$ around x. Then we can choose $\delta > 0$ and $C > 0$ such that on $B_\delta(\varphi(x)) \subset V$, $g(u, u) \geq C|D\varphi(u)|^2$. Therefore for points z in $\varphi^{-1}(B_\delta(\varphi(x))$, we have $d(x, z) \geq C|\varphi(x) - \varphi(z)|$. So y cannot be in this set. But for y outside this set, any curve from x to y must first pass through $\varphi^{-1}(\partial B_\delta(\varphi(x)))$, and so has length at least $C\delta$, contradicting $d(x, y) = 0$.

The claim that the metric topology is equivalent to the manifold topology follows similarly: The metric restricted to charts is comparable to the Euclidean distance on the chart. □

9.4 Submanifolds

An important situation where a manifold can be given a Riemannian metric is when it is a submanifold of some Euclidean space $\mathbb{R}^N$. The inclusion into $\mathbb{R}^N$ gives a natural identification of tangent vectors to M with vectors in $\mathbb{R}^N$: Explicitly, if we write the inclusion as a map $F : M \to \mathbb{R}^N$, then we identify a vector $u \in TM$ with its image $DF(u)$ under the differential of the inclusion.

The inner product on $\mathbb{R}^N$ can then be used to induce a Riemannian metric on M, by defining

$$g(u, v) = \langle DF(u), DF(v) \rangle.$$

Similarly, if N is a Riemannian manifold with a metric h, and $F : M \to N$ is an immersion, then we can define the induced Riemannian metric on M by

$$g(u, v) = h(DF(u), DF(v)).$$

Many important Riemannian manifolds can be produced in this way, including the standard metrics on the spheres S^n (induced by the standard embedding in $\mathbb{R}^{n+1}$), and on cylinders.

9.5 Left-invariant metrics

Let G be a Lie group, and choose an inner product h on $T_eG \simeq \mathfrak{g}$. This can be extended to give a unique *left-invariant* Riemannian metric on G, by defining

$$\langle u, v\rangle_g = h\left((D_e l_g)^{-1}(u), (D_e l_g)^{-1}(v)\right).$$

Similarly, one can define right-invariant metrics; in general these are not the same.

Example 9.5.1 A metric on hyperbolic space. Recall that the hyperbolic plane $\mathbb{H}^2$ is upper half-plane, identified with the group of matrices of the form $\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$ for $y > 0$. If we choose an inner product at the identity $(0, 1)$ such that $(0, 1)$ and $(1, 0)$ are orthonormal, then the corresponding left-invariant Riemannian metric on $\mathbb{H}^2$ is the one for which the left-invariant vector fields $E_1 = (y, 0)$ and $E_2 = (0, y)$ are orthonormal. Thus in terms of the basis of coordinate tangent vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, the metric has the form $g_{ij} = y^{-2}\delta_{ij}$.

Example 9.5.2 Left-invariant metrics on S^3 Recall that S^3 is the group of unit length elements of the quaternions. The tangent space at the identity is the three- dimensional space spanned by i, j, and k, and any inner product on this space gives rise to a left-invariant metric on S^3. If i, j, and k are chosen to be orthonormal, the resulting metric is the standard metric on S^3 (i.e. it agrees with the induced metric from the inclusion of S^3 in $\mathbb{R}^4$). Other choices of inner product give deformed spheres called the *Berger spheres*.

9.6 Bi-invariant metrics

A more stringent requirement on a Riemannian metric on a Lie group is that it should be invariant under both left and right translations. Such a metric is called *bi-invariant*.

Every bi-invariant metric is left-invariant, and so can be constructed in a unique way from an inner product for T_eG. This raises the question: Which inner products on T_eG give rise to bi-invariant metrics?

For any g, and any $u, v \in T_eG$, we must have

$$\langle u, v\rangle_e = \langle D_e l_g(u), D_e l_g(v)\rangle_g = \langle (D_e r_g)^{-1} D_e l_g(u), (D_e r_g)^{-1} D_e l_g(v)\rangle_e.$$

The maps $\mathrm{Ad}_g = (D_e r_g)^{-1} D_e l_g : T_eG \to T_eG$ are isomorphisms of T_eG for each $g \in G$, and give a representation of G on the vector space T_eG, since $\mathrm{Ad}_g\mathrm{Ad}_h = \mathrm{Ad}_{gh}$ for all g and h in G. Then the requirement that the inner product give rise to a bi-invariant metric is the same as requiring that it be invariant under the representation Ad of G. If G is commutative, then Ad_g is the identity map for every g, so this requirement is vacuous. In some cases there may be no Ad-invariant inner product on T_eG, but it can be shown that any compact Lie group carries at least one.

Exercise 9.6.1 Show that the adjoint action of S^3 on its Lie algebra $\mathbb{R}^3$ gives a homomorphism $\rho : S^3 \to SO(3)$ (compare Exercise 5.3.2 and the remark following it). Deduce that the only bi-invariant metric on S^3 is the standard one.

9.7 Semi-Riemannian metrics

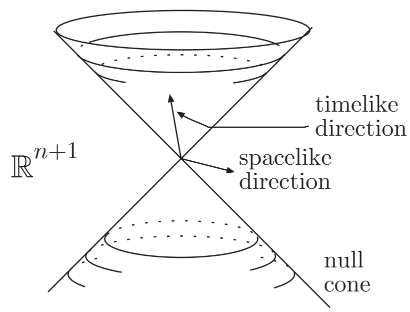
It is sometimes useful to consider a generalisation of Riemannian manifolds which drops the requirement of positivity: A semi- Riemannian manifold is a smooth manifold together with a smoothly defined symmetric bilinear form g_x on each tangent space T_xM, which is non-singular: $g_x(u,v) = 0$ for all v implies $u = 0$. If M is connected, then the signature of M is constant: One can choose a basis $\{e_1, \ldots, e_n\}$ at each point of M such that g_x takes the form

$$g_x = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & -1 \end{bmatrix}$$

with 1 occuring k times and -1 occuring $(n-k)$ times, independent of x. Of particular interest is the case of *Lorenzian* manifolds, in which $k = n-1$; when $n = 4$ these arise as spacetimes in general relativity.

In a semi-Riemannian manifold one can divide vectors into three categories: Spacelike vectors, for which $g(v,v) > 0$; timelike vectors, which have $g(v,v) < 0$; and null vectors, for which $g(v,v) = 0$. A submanifold M of a Lorenzian manifold is similarly called spacelike, timelike, or null if all of its tangent vectors are spacelike, timelike, or null respectively. If M is a spacelike submanifold of a semi-Riemannian manifold, then the induced metric makes M a Riemannian manifold.

Example 9.7.1 Minkowski space. Let $M = \mathbb{R}^{n+1}$ with variables $x^1, \ldots, x^n, t$, with the constant semi-Riemannian metric $g_{ij} = \mathrm{diag}(1, \ldots, 1, -1)$. The is the Minkowski space $\mathbb{R}^{n,1}$. The geometry of the Minkowski space $\mathbb{R}^{3,1}$ is the subject of special relativity. The *null cone* of directions with zero length is the right circular cone $|x|^2 = t^2$. Directions which point from the origin above the upper surface or below the lower surface of this cone are timelike, while those that point between the two surfaces are spacelike. In particular, a hypersurface given by the graph of a function u over $\mathbb{R}^n \times \{0\}$ is spacelike if the gradient of u is everywhere less than 1 in magnitude.



Example 9.7.2 Hyperbolic space. The n-dimensional hyperbolic space arises as a spacelike hypersurface in the Minkowski space $\mathbb{R}^{n,1}$: Let $\mathbb{H}^n = \{(x,t) \in \mathbb{R}^{n,1} : \ t = \sqrt{1+|x|^2}\}$. Thus $\mathbb{H}^n$ is one component of the set of vectors of length -1 in $\mathbb{R}^{n,1}$; thus it is an analogue of the sphere in this setting. We can also describe $\mathbb{H}^n$ as the open unit ball in $\mathbb{R}^n$ with a particular Riemannian metric: We identify $\mathbb{H}^n$ with the unit ball by stereographic projection – given a point $z \in \mathbb{H}^n$, we consider the line from z to the point $(0,-1)$, and let $\xi(z)$ be the point of intersection of this line with the plane $\mathbb{R}^n \times \{0\}$. with the unique point on the line from the origin to z which has $t = 1$. As z ranges over $\mathbb{H}^n$, $\xi(z)$ ranges over the unit ball in $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}$. This should be though of as being analogous to stereographic projection from the north pole in the usual sphere; here we are doing stereographic projection from the point $(0,-1)$. Explicitly, this gives $z = \left(\frac{2x}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2}\right)$.

Exercise 9.7.3 Show that the induced Riemannian metric on the unit ball in $\mathbb{R}^n$ by the stereographic projection described above is

$$g_{ij} = \frac{4}{(1-|x^2|)^2}\delta_{ij}.$$

Exercise 9.7.4 Show that the map $(x,y) \mapsto \left(\frac{2x}{|x|^2+(y-1)^2}, \frac{1-|x|^2-y^2}{|x|^2+(y-1)^2}\right)$ for $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \simeq \mathbb{R}^n$ diffeomorphically maps the open unit ball to the upper half space, and induces from the metric on the unit ball in exercise 9.7.3 the following metric on the upper half-space:

$$g_{ij} = \frac{1}{y^2}\delta_{ij}.$$

In the case $n = 2$, show that this is the same as the left-invariant metric on $\mathbb{H}^2$ from example 9.5.1.