MULTIPLE-PRECISION ZERO-FINDING METHODS AND THE COMPLEXITY OF ELEMENTARY FUNCTION EVALUATION

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Abstract

We consider methods for finding high-precision approximations to simple zeros of smooth functions. As an application, we give fast methods for evaluating the elementary functions $\log(x)$, $\exp(x)$, $\sin(x)$ etc. to high precision. For example, if x is a positive floating-point number with an n-bit fraction, then (under rather weak assumptions) an n-bit approximation to $\log(x)$ or $\exp(x)$ may be computed in time asymptotically equal to $13M(n)\log_2 n$ as $n \to \infty$, where $M(n)$ is the time required to multiply floating-point numbers with n-bit fractions. Similar results are given for the other elementary functions, and some analogies with operations on formal power series are mentioned.

Comments

Only the Abstract is given here. The full paper appeared as [3]. This paper gives the quadratically convergent "Gauss-Legendre" algorithm [2] for the computation of π (discovered independently by Salamin [6]), and gives algorithms for computing elementary functions which are simpler and faster than those of [5]. It also gives a fast algorithm for computing the first n terms in an arbitrary power $F(x)^q$ of a formal power series $F(x)$ in one variable. A related paper (written earlier) is [4].

Errata

Page 152, line -10: replace "performe" by "perform".
Page 172, line 7: replace "β_i" by "β".

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