

ON THE REGULARIZING LEVENBERG-MARQUARDT SCHEME IN BANACH SPACES

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Abstract. By making use of duality mappings and the Bregman distance, we propose a regularizing Levenberg-Marquardt scheme to solve nonlinear inverse problems in Banach spaces, which is an extension of the one proposed in [6] in Hilbert space setting. The method consists of two components: an outer Newton iteration and an inner scheme. The inner scheme involves a family of convex minimization problems in Banach spaces from which a suitable criterion is used to select one to produce the increments. The outer iteration is then terminated by a discrepancy principle. Under certain conditions, we establish the convergence of the method.

Key words. Nonlinear inverse problems in Banach spaces, Levenberg-Marquardt scheme, discrepancy principle, Bregman distance, duality mapping, convergence

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1. Introduction. In this paper we consider the nonlinear operator equations

$$F(x) = y \tag{1.1}$$

arising from nonlinear inverse problems in Banach spaces, where $F : D(F) \subset \mathcal{X} \rightarrow \mathcal{Y}$ is a nonlinear Fréchet differentiable operator between two Banach spaces \mathcal{X} and \mathcal{Y} with domain $D(F)$. A characteristic property of such equations is their ill-posedness in the sense that their solutions are not stable with respect to the perturbation of data. Due to errors in measurement, one never has exact data but only noisy data are available in practical applications. Therefore, it is an important issue to construct stable approximate solutions of (1.1) from noisy data.

Many methods have been developed for solving nonlinear inverse problems in Hilbert spaces, see [1, 6, 7, 10, 11, 14, 16] and the references therein. However, methods in Hilbert spaces may not produce good results since they tend to smooth the solutions and thus destroy the special structure in the exact solution. On the other hand, formulating inverse problems in Hilbert spaces may restrict the consideration to smaller spaces which place extra constraints on the exact solution. Thus, in some applications, it is more natural to formulate inverse problems and develop stable algorithms in the framework of Banach spaces.

Due to its variational formulation, Tikhonov regularization can be easily adapted to solve nonlinear inverse problems in Banach spaces and some convergence analysis, including the derivation of convergence rates, has been carried out, see [15] and the references therein. Since the numerical realization requires to solve several non-convex minimization problems, Tikhonov regularization in general is rather expensive. Due to their straightforward implementation, the development of iterative regularization methods in Banach spaces has received more and more attention in recent years. By making use of duality mappings and the Bregman distance, several iterative regularization methods in Hilbert spaces, including the iteratively regularized Gauss-Newton method, the nonlinear Landweber iteration and some variants, have been extended in [8, 12, 13] to the Banach space setting.

Motivated by the inexact Newton methods in [4] for well-posed problems, Hanke proposed in [6] his regularizing Levenberg-Marquardt scheme to solve nonlinear inverse

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problems in Hilbert spaces. His scheme has been proved to be an efficient method and has stimulated a lot of successive work; in particular, a family of inexact Newton regularizations have been proposed in [16] in Hilbert space setting. In this paper, by using the duality mappings and the Bregman distance, we will propose an extension of the regularizing Levenberg-Marquardt scheme to the Banach space setting. The method consists of two components: an outer Newton iteration and an inner scheme providing increments. The inner scheme involves a family of convex minimization problems in Banach spaces from which we can select one by a suitable criterion to produce the increment. The outer iteration is then terminated by a discrepancy principle.

This paper is organized as follows. In Section 2 we will briefly review some basic geometric aspects of Banach spaces. In Section 3, by making use of duality mappings and Bregman distance, we will formulate the regularizing Levenberg-Marquardt scheme in Banach spaces and show that the method is well-defined. Finally in Section 4 we will show that our method indeed is a regularization method by establishing the convergence result.

2. Preliminaries. Let \mathcal{X} and \mathcal{Y} be two Banach spaces whose norms are denoted by $\|\cdot\|$. We will use \mathcal{X}^* and \mathcal{Y}^* to denote their dual spaces respectively. Given $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$ we will write $\langle x^*, x \rangle = x^*(x)$ for the duality pair. We will use “ \rightarrow ” and “ \rightharpoonup ” to denote the strong convergence and the weak convergence respectively. By $L(\mathcal{X}, \mathcal{Y})$ we will denote for the space of all continuous linear operators from \mathcal{X} to \mathcal{Y} . For any $T \in L(\mathcal{X}, \mathcal{Y})$, we will use $T^* : \mathcal{Y}^* \rightarrow \mathcal{X}^*$ to denote its dual, i.e. $\langle T^*y^*, x \rangle = \langle y^*, Tx \rangle$ for any $x \in \mathcal{X}$ and $y^* \in \mathcal{Y}^*$.

We first review some geometric aspects of Banach spaces. A Banach space \mathcal{X} is called strictly convex if for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ and $\|x_1\| = \|x_2\| = 1$ there holds $\|x_1 + x_2\| < 2$, and it is called uniformly convex if $\delta_{\mathcal{X}}(\epsilon) > 0$ for all $0 < \epsilon \leq 2$, where $\delta_{\mathcal{X}}(\epsilon)$ is the modulus of convexity of \mathcal{X} defined by

$$\delta_{\mathcal{X}}(\epsilon) := \inf \{2 - \|x + \bar{x}\| : \|x\| = \|\bar{x}\| = 1, \|x - \bar{x}\| \geq \epsilon\}.$$

A Banach space \mathcal{X} is called smooth if for every $x \neq 0$ there is a unique $x^* \in \mathcal{X}^*$ such that $\|x^*\| = 1$ and $\langle x^*, x \rangle = \|x\|$, and it is called uniformly smooth if $\lim_{\tau \rightarrow 0} \frac{\rho_{\mathcal{X}}(\tau)}{\tau} = 0$, where $\rho_{\mathcal{X}}(\tau)$ is the modulus of smoothness of \mathcal{X} defined by

$$\rho_{\mathcal{X}}(\tau) := \sup \{\|x + \bar{x}\| + \|x - \bar{x}\| - 2 : \|x\| = 1, \|\bar{x}\| \leq \tau\}, \quad \tau \geq 0$$

It is clear that a uniformly convex Banach space must be strictly convex and a uniformly smooth Banach space must be smooth. Moreover, uniformly smooth or uniformly convex Banach spaces are reflexive.

Given $1 < p < \infty$, the set-valued mapping $J_p : \mathcal{X} \rightarrow 2^{\mathcal{X}^*}$ defined by

$$J_p(x) = \{x^* \in \mathcal{X}^* : \|x^*\| = \|x\|^{p-1} \text{ and } \langle x^*, x \rangle = \|x\|^p\}$$

is called the duality mapping with gauge function $t \rightarrow t^{p-1}$. J_p in general is multi-valued and equals the subdifferential of the convex functional $x \rightarrow \|x\|^p/p$. However, in certain Banach spaces, J_p can be single-valued and admit some nice properties. The following lemma collects some important facts which will be used in this paper.

LEMMA 2.1. *Let \mathcal{X} be a Banach space.*

(a) *If \mathcal{X} is strictly convex, then every duality mapping J_p of \mathcal{X} is strictly monotone, i.e. $\langle x_1^* - x_2^*, x_1 - x_2 \rangle > 0$ for all $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ and $x_1^* \in J_p(x_1)$, $x_2^* \in J_p(x_2)$.*

(b) If \mathcal{X} is uniformly convex, then for any sequence $\{x_n\} \subset \mathcal{X}$ satisfying $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ there holds $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

(c) If \mathcal{X} is smooth, then every duality mapping J_p of \mathcal{X} is single valued.

(d) If \mathcal{X} is uniformly smooth, then every J_p is uniformly continuous on bounded subsets of \mathcal{X} .

The proof of these results can be found in [2] where one can find more interesting facts on the duality mappings together with examples of uniformly smooth and uniformly convex Banach spaces including the sequence spaces l^p , the Lebesgue spaces L^p and the Sobolev spaces $W^{k,p}$ with $1 < p < \infty$.

In order to formulate the method in Banach spaces and study the convergence property, instead of the norm it is more convenient to use the Bregman distance. When \mathcal{X} is smooth, the Bregman distance is defined as

$$\Delta_p(\bar{x}, x) := \frac{1}{p} \|\bar{x}\|^p - \frac{1}{p} \|x\|^p - \langle J_p(x), \bar{x} - x \rangle. \quad (2.1)$$

It is easy to show for any $x, x_1, x_2 \in \mathcal{X}$ that

$$\Delta_p(x, x_1) - \Delta_p(x, x_2) = -\Delta_p(x_1, x_2) + \langle J_p(x_1) - J_p(x_2), x_1 - x_2 \rangle. \quad (2.2)$$

Let q be the number conjugate to p , i.e. $1/p + 1/q = 1$. Then, by using the properties of the duality mapping J_p and the Young's inequality we have

$$\Delta_p(\bar{x}, x) \geq \frac{1}{p} \|\bar{x}\|^p + \frac{1}{q} \|x\|^p - \|x\|^{p-1} \|\bar{x}\| \geq 0.$$

Thus the Bregman distance is nonnegative. The Bregman distance is in general not a metric since it does not satisfy the symmetry and the triangle inequality. However, in a smooth and uniformly convex Banach space the Bregman distance can be used to get information with respect to the norm. By making use of the characterization of uniformly convex Banach spaces in [18], the following result has been proved in [17].

LEMMA 2.2. *Let \mathcal{X} be a smooth and uniformly convex Banach space. Then for any $x \in \mathcal{X}$ and sequence $\{x_n\} \subset \mathcal{X}$ the following hold:*

(a) *The boundedness of $\{\Delta_p(x_n, x)\}$ implies the boundedness of $\{\|x_n\|\}$.*

(b) *$\lim_{n \rightarrow \infty} \|x_n - x\| = 0 \iff \lim_{n \rightarrow \infty} \Delta_p(x, x_n) = 0 \iff \lim_{n \rightarrow \infty} \Delta_p(x_n, x) = 0$.*

(c) *$\{x_n\}$ is a Cauchy sequence if and only if $\Delta_p(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$.*

3. The method. We consider the equation (1.1) arising from nonlinear inverse problems in Banach spaces, where $F : D(F) \subset \mathcal{X} \rightarrow \mathcal{Y}$ is a nonlinear operator between two Banach spaces \mathcal{X} and \mathcal{Y} with domain $D(F)$. Such equations are ill-posed in general. Due to the error in measurements, instead of y the only available data is an approximation y^δ satisfying

$$\|y^\delta - y\| \leq \delta \quad (3.1)$$

with a given small noise level $\delta > 0$. We will use the noisy data y^δ to construct a stable approximate solution to the equation (1.1). We will work under the following conditions.

ASSUMPTION 3.1. (a) \mathcal{X} and \mathcal{Y} are Banach spaces with \mathcal{X} being uniformly convex and uniformly smooth and \mathcal{Y} being smooth.

(b) F is Fréchet differentiable over $D(F)$ and the map $x \rightarrow F'(x)$ is continuous from $D(F) \subset \mathcal{X}$ to $L(\mathcal{X}, \mathcal{Y})$, where $F'(x)$ denotes the Fréchet derivative of F at x .

(c) The equation (1.1) has a solution x^\dagger . There is a number $\rho > 0$ such that

$$B_\rho(x^\dagger, \Delta_p) := \{x \in \mathcal{X} : \Delta_p(x, x^\dagger) \leq \rho\} \subset D(F)$$

and there is a constant $0 \leq \eta < 1$ such that

$$\|F(\bar{x}) - F(x) - F'(x)(\bar{x} - x)\| \leq \eta \|F(\bar{x}) - F(x)\|$$

for all $x, \bar{x} \in B_\rho(x^\dagger, \Delta_p)$.

Now we are ready to formulate the method for solving (1.1) stably. We first take an initial guess $x_0^\delta := x_0 \in B_\rho(x^\dagger, \Delta_p)$. Assume that x_n^δ is the current iterate. Then the linearized equation around x_n^δ is

$$F'(x_n^\delta)(x - x_n^\delta) = y^\delta - F(x_n^\delta).$$

For each $\alpha > 0$ we define $x_n(\alpha, y^\delta)$ to be the minimizer of the convex minimization problem

$$\min_{x \in \mathcal{X}} \left\{ \frac{1}{r} \|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x - x_n^\delta)\|^r + \alpha \Delta_p(x, x_n^\delta) \right\}, \quad (3.2)$$

where $1 < p, r < \infty$. We then define $\alpha_n(y^\delta) > 0$ to be the root of the equation

$$\|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\| = \mu \|y^\delta - F(x_n^\delta)\| \quad (3.3)$$

with some $0 < \mu < 1$ and define $x_{n+1}^\delta := x_n(\alpha_n(y^\delta), y^\delta)$. The iteration is then terminated by the discrepancy principle

$$\|y^\delta - F(x_{n_\delta}^\delta)\| \leq \tau \delta < \|y^\delta - F(x_n^\delta)\|$$

for some $\tau > 1$, which outputs an integer n_δ . We will use $x_{n_\delta}^\delta$ to approximate a solution of (1.1).

We remark that when \mathcal{X} and \mathcal{Y} are Hilbert spaces and $p = r = 2$, our method reduces to the regularizing Levenberg-Marquardt scheme proposed in [6] and each minimizer $x_n(\alpha, y^\delta)$ can be written explicitly. In the general Banach space setting, $x_n(\alpha, y^\delta)$ does not have an explicit formula. This increases the difficulty in convergence analysis. By making use of the duality mapping and the Bregman distance, in this section we will show that our method is well-defined, and in Section 4 we will show that $x_{n_\delta}^\delta$ indeed converges to a solution of (1.1) as $\delta \rightarrow 0$.

LEMMA 3.1. *Let Assumption 3.1 (a) and (b) hold. Then for each $\alpha > 0$, $x_n(\alpha, y^\delta)$ is uniquely determined and satisfies the equation*

$$\begin{aligned} & \alpha (J_p(x_n(\alpha, y^\delta)) - J_p(x_n^\delta)) \\ & = F'(x_n^\delta)^* J_r (y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)), \end{aligned} \quad (3.4)$$

where $J_r : \mathcal{Y} \rightarrow \mathcal{Y}^*$ denotes the duality mapping with the gauge function $t \rightarrow t^{r-1}$. Moreover, the map $\alpha \rightarrow x_n(\alpha, y^\delta)$ is continuous over $(0, \infty)$, and the function

$$\alpha \rightarrow \|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\| \quad (3.5)$$

is continuous and monotonically increasing over $(0, \infty)$.

Proof. Recall that any bounded sequence in a reflexive Banach space has a weak convergent subsequence. By using the weak lower semi-continuity of the norms in

Banach spaces, it is easy to show the existence of $x_n(\alpha, y^\delta)$ for each $\alpha > 0$. Since the smoothness of \mathcal{X} and \mathcal{Y} guarantees the differentiability of the functionals $x \rightarrow \|x\|^p/p$ and $y \rightarrow \|y\|^r/r$ over \mathcal{X} and \mathcal{Y} respectively, as a minimizer $x_n(\alpha, y^\delta)$ must satisfy the equation (3.4). In view of the uniform convexity of \mathcal{X} , the uniqueness of $x_n(\alpha, y^\delta)$ follows easily. The continuity of the map $\alpha \rightarrow x_n(\alpha, y^\delta)$ follows from the same argument in [9, Section 2]. Finally the monotonicity of (3.5) follows from a standard argument in [3]. \square

LEMMA 3.2. *Let Assumption 3.1 hold with $0 \leq \eta < 1$. Let $\eta < \mu < 1$ and $\tau > (1 + \eta)/(\mu - \eta)$. Assume that $x_n^\delta \in B_\rho(x^\dagger, \Delta_p)$ is well-defined with $0 \leq n < n_\delta$. Then for any $\alpha > 0$ such that*

$$\|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\| \geq \mu \|y^\delta - F(x_n^\delta)\| \quad (3.6)$$

there hold

$$\Delta_p(x_*, x_n(\alpha, y^\delta)) \leq \Delta_p(x_*, x_n^\delta) \quad (3.7)$$

and

$$\frac{1}{\alpha} \|y^\delta - F(x_n^\delta)\|^r \leq C_0 (\Delta_p(x_*, x_n^\delta) - \Delta_p(x_*, x_n(\alpha, y^\delta))), \quad (3.8)$$

where x_* denotes any solution of (1.1) in $B_\rho(x^\dagger, \Delta_p)$ and $C_0 = \tau/[\tau\mu - (1 + \eta + \tau\eta)]$.

Proof. For simplicity of presentation, we write $x_n(\alpha) := x_n(\alpha, y^\delta)$ and $T_n := F'(x_n^\delta)$. By using the identity (2.2) and the nonnegativity of the Bregman distance, we obtain

$$\Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \leq \langle J_p(x_n(\alpha)) - J_p(x_n^\delta), x_n(\alpha) - x_* \rangle.$$

Since $x_n(\alpha)$ satisfies the equation (3.4), we have

$$\begin{aligned} \Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \\ \leq \frac{1}{\alpha} \langle J_r(y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)), T_n(x_n(\alpha) - x_*) \rangle. \end{aligned}$$

We can write

$$T_n(x_n(\alpha) - x_*) = [y^\delta - F(x_n^\delta) - T_n(x_* - x_n^\delta)] - [y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)].$$

Then, by using the property of the duality mapping J_r , we obtain

$$\begin{aligned} \Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \\ \leq \frac{1}{\alpha} \|y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)\|^{r-1} \|y^\delta - F(x_n^\delta) - T_n(x_* - x_n^\delta)\| \\ - \frac{1}{\alpha} \|y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)\|^r. \end{aligned}$$

By using (3.1) and Assumption 3.1 (c), we have

$$\|y^\delta - F(x_n^\delta) - T_n(x_* - x_n^\delta)\| \leq (1 + \eta)\delta + \eta \|y^\delta - F(x_n^\delta)\|.$$

Since $n < n_\delta$, we have $\|y^\delta - F(x_n^\delta)\| > \tau\delta$. Thus

$$\|y^\delta - F(x_n^\delta) - T_n(x_* - x_n^\delta)\| \leq \frac{1 + \eta + \tau\eta}{\tau} \|y^\delta - F(x_n^\delta)\|.$$

Therefore

$$\begin{aligned} & \Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \\ & \leq \frac{1 + \eta + \tau\eta}{\tau\alpha} \|y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)\|^{r-1} \|y^\delta - F(x_n^\delta)\| \\ & \quad - \frac{1}{\alpha} \|y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)\|^r. \end{aligned}$$

In view of the inequality (3.6), we thus obtain

$$\Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \leq -\frac{c_0}{\alpha} \|y^\delta - F(x_n^\delta) - T_n(x_n(\alpha) - x_n^\delta)\|^r.$$

where $c_0 := 1 - (1 + \eta + \tau\eta)/(\tau\mu)$. According to the conditions on μ and τ , we have $c_0 > 0$. Thus, in view of (3.6) again, it follows that

$$\Delta_p(x_*, x_n(\alpha)) - \Delta_p(x_*, x_n^\delta) \leq -\frac{c_0\mu}{\alpha} \|y^\delta - F(x_n^\delta)\|^r.$$

This implies (3.7) and (3.8) immediately. \square

LEMMA 3.3. *Let Assumption 3.1 hold with $0 \leq \eta < 1/3$. Let $\eta < \mu < 1 - 2\eta$ and $\tau > (1 + \eta)/(\mu - \eta)$. Then x_n^δ are well-defined for all $0 \leq n \leq n_\delta$ and the method terminates after $n_\delta < \infty$ iterations with $n_\delta = O(1 + |\log \delta|)$. Moreover, for any solution x_* of (1.1) in $B_\rho(x^\dagger, \Delta_p)$ there hold*

$$\Delta_p(x_*, x_{n+1}^\delta) \leq \Delta_p(x_*, x_n^\delta) \quad (3.9)$$

and

$$\frac{1}{\alpha_n(y^\delta)} \|y^\delta - F(x_n^\delta)\|^r \leq C_0 (\Delta_p(x_*, x_n^\delta) - \Delta_p(x_*, x_{n+1}^\delta)) \quad (3.10)$$

for all $0 \leq n < n_\delta$.

Proof. We will show by induction that x_n^δ are well-defined for all $0 \leq n \leq n_\delta$. The case $n = 0$ is trivial. Now we assume that x_n^δ is well-defined for some $0 \leq n < n_\delta$ and show that x_{n+1}^δ is also well-defined.

We first show the existence of $\alpha_n := \alpha_n(y^\delta)$. By the minimality of $x_n(\alpha, y^\delta)$ it is easy to see that $\Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \rightarrow 0$ as $\alpha \rightarrow \infty$. This and Lemma 2.2 (b) imply that $x_n(\alpha, y^\delta) \rightarrow x_n^\delta$ as $\alpha \rightarrow \infty$. Consequently, the left hand side of (3.3) is greater than its right hand side for large α . If (3.3) does not have a positive root, then (3.6) holds for all $\alpha > 0$. Consequently, it follows from (3.8) that

$$\|y^\delta - F(x_n^\delta)\|^r \leq C_0 \alpha \Delta_p(x_*, x_n^\delta)$$

for all $\alpha > 0$. Taking $\alpha \rightarrow 0$ yields $y^\delta = F(x_n^\delta)$ which is absurd since $n < n_\delta$ implies $\|y^\delta - F(x_n^\delta)\| > \tau\delta$.

Next we will show that α_n is uniquely determined. Let $\alpha'_n > 0$ be another root of (3.3). Then

$$\|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\| = \mu \|y^\delta - F(x_n^\delta)\| \quad (3.11)$$

for $\alpha = \alpha'_n$ and α_n . This together with the minimality of $x_n(\alpha_n, y^\delta)$ and $x_n(\alpha'_n, y^\delta)$ implies that

$$\Delta_p(x_n(\alpha_n, y^\delta), x_n^\delta) = \Delta_p(x_n(\alpha'_n, y^\delta), x_n^\delta). \quad (3.12)$$

In view of (3.11) and (3.12), it follows that $x_n(\alpha_n, y^\delta)$ is the minimizer of the minimization problem (3.2) with $\alpha = \alpha_n$ and α'_n . Therefore we have from (3.4) that

$$\alpha (J_p(x_n(\alpha_n, y^\delta)) - J_p(x_n^\delta)) = F'(x_n^\delta)^* J_r (y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_n(\alpha_n, y^\delta) - x_n^\delta))$$

for $\alpha = \alpha_n$ and α'_n , which implies that $J_p(x_n(\alpha_n, y^\delta)) - J_p(x_n^\delta) = 0$. Since \mathcal{X} is strictly convex, it follows from Lemma 2.1 (a) that the duality mapping J_p is strictly monotone. Therefore $x_n(\alpha_n, y^\delta) = x_n^\delta$ and thus it follows from (3.11) that

$$\|y^\delta - F(x_n^\delta)\| = \mu \|y^\delta - F(x_n^\delta)\|.$$

Since $0 < \mu < 1$, this forces $y^\delta = F(x_n^\delta)$ which is a contradiction.

Since $\alpha_n := \alpha_n(y^\delta) > 0$ is uniquely determined, x_{n+1}^δ is therefore well-defined. The inequalities (3.9) and (3.10) follow from Lemma 3.2.

Finally we show that n_δ is finite by a standard argument from [6]. By Assumption 3.1 (c) and the definition of x_{n+1}^δ we have for all $0 \leq n < n_\delta$ that

$$\begin{aligned} \|y^\delta - F(x_{n+1}^\delta)\| &\leq \|y^\delta - F(x_n^\delta) - F'(x_n^\delta)(x_{n+1}^\delta - x_n^\delta)\| \\ &\quad + \|F(x_{n+1}^\delta) - F(x_n^\delta) - F'(x_n^\delta)(x_{n+1}^\delta - x_n^\delta)\| \\ &\leq \mu \|y^\delta - F(x_n^\delta)\| + \eta \|F(x_{n+1}^\delta) - F(x_n^\delta)\| \\ &\leq (\mu + \eta) \|y^\delta - F(x_n^\delta)\| + \eta \|y^\delta - F(x_{n+1}^\delta)\|. \end{aligned}$$

This implies that $\|y^\delta - F(x_{n+1}^\delta)\| \leq \frac{\mu + \eta}{1 - \eta} \|y^\delta - F(x_n^\delta)\|$ and hence

$$\|y^\delta - F(x_n^\delta)\| \leq \left(\frac{\mu + \eta}{1 - \eta} \right)^n \|y^\delta - F(x_0)\|, \quad 0 \leq n < n_\delta. \quad (3.13)$$

If $n_\delta = \infty$, then we must have $\|y^\delta - F(x_n^\delta)\| > \tau\delta$ for all n . But the inequality (3.13) implies $\|y^\delta - F(x_n^\delta)\| \rightarrow 0$ as $n \rightarrow \infty$ since $(\mu + \eta)/(1 - \eta) < 1$. Therefore $n_\delta < \infty$. Now we take $n = n_\delta - 1$ in (3.13) and obtain $(\frac{\mu + \eta}{1 - \eta})^{n_\delta - 1} \|y^\delta - F(x_0)\| > \tau\delta$. This implies $n_\delta = O(1 + |\log \delta|)$. \square

4. Convergence analysis. In this section we will show that $x_{n_\delta}^\delta$ converges to a solution of (1.1) as $\delta \rightarrow 0$. We start with the consideration on the approximate solutions $\{x_n\}$ corresponding to the noise-free case which are defined as follows: We take the same x_0 as the initial guess. Assume x_n is the current iterate. If $F(x_n) = y$, then we define $x_{n+1} := x_n$; otherwise, for each $\alpha > 0$ we define $x_n(\alpha, y)$ to be the minimizer of the minimization problem

$$\min_{x \in \mathcal{X}} \left\{ \frac{1}{r} \|y - F(x_n) - F'(x_n)(x - x_n)\|^r + \alpha \Delta_p(x, x_n) \right\},$$

take $\alpha_n(y)$ to be the root of the equation

$$\|y - F(x_n) - F'(x_n)(x_n(\alpha, y) - x_n)\| = \mu \|y - F(x_n)\|$$

and define $x_{n+1} := x_n(\alpha_n(y), y)$. We will show that $\{x_n\}$ converges to a solution of (1.1) as $n \rightarrow \infty$.

LEMMA 4.1. *Let Assumption 3.1 hold with $0 \leq \eta < 1/3$ and let $\eta < \mu < 1 - 2\eta$. Then x_n is well-defined for all n and converges to a solution x_* of (1.1) in $B_\rho(x^\dagger, \Delta_p)$ as $n \rightarrow \infty$. If x^\dagger is the unique solution of (1.1) in $B_\rho(x^\dagger, \Delta_p)$, then $x_n \rightarrow x^\dagger$ as $n \rightarrow \infty$.*

Proof. By the similar proof of Lemma 3.3, it is easy to show that $\{x_n\}$ is well-defined and for all n there hold

$$\Delta_p(x^\dagger, x_{n+1}) \leq \Delta_p(x^\dagger, x_n), \quad (4.1)$$

$$\frac{1}{\alpha_n(y)} \|y - F(x_n)\|^r \leq \frac{1}{\mu - \eta} (\Delta_p(x^\dagger, x_n) - \Delta_p(x^\dagger, x_{n+1})) \quad (4.2)$$

and

$$\|y - F(x_{n+1})\| \leq \frac{\mu + \eta}{1 - \eta} \|y - F(x_n)\|. \quad (4.3)$$

Now we show that $\{x_n\}$ is a Cauchy sequence. For $0 \leq l < m < \infty$ we have from (2.2) that

$$\Delta_p(x_m, x_l) = \Delta_p(x^\dagger, x_l) - \Delta_p(x^\dagger, x_m) + \langle J_p(x_m) - J_p(x_l), x_m - x^\dagger \rangle. \quad (4.4)$$

Let $T_n := F'(x_n)$ and recall that x_{n+1} satisfies the equation

$$\alpha_n(y) (J_p(x_{n+1}) - J_p(x_n)) = T_n^* J_r (y - F(x_n) - T_n(x_{n+1} - x_n)),$$

we have, with the help of the property of the duality mapping J_r , that

$$\begin{aligned} \langle J_p(x_m) - J_p(x_l), x_m - x^\dagger \rangle &= \sum_{n=l}^{m-1} \langle J_p(x_{n+1}) - J_p(x_n), x_m - x^\dagger \rangle \\ &= \sum_{n=l}^{m-1} \frac{1}{\alpha_n(y)} \langle J_r(y - F(x_n) - T_n(x_{n+1} - x_n)), T_n(x_m - x^\dagger) \rangle \\ &\leq \sum_{n=l}^{m-1} \frac{1}{\alpha_n(y)} \|y - F(x_n) - T_n(x_{n+1} - x_n)\|^{r-1} \|T_n(x_m - x^\dagger)\|. \end{aligned}$$

By the triangle inequality $\|T_n(x_m - x^\dagger)\| \leq \|T_n(x_n - x^\dagger)\| + \|T_n(x_m - x_n)\|$ and Assumption 3.1 (c), we have

$$\begin{aligned} \|T_n(x_m - x^\dagger)\| &\leq (1 + \eta) (\|y - F(x_n)\| + \|F(x_n) - F(x_m)\|) \\ &\leq (1 + \eta) (2\|y - F(x_n)\| + \|y - F(x_m)\|). \end{aligned}$$

Since $(\mu + \eta)/(1 - \eta) < 1$, the inequality (4.3) implies that $\{\|y - F(x_n)\|\}$ monotonically decreases to 0. Thus

$$\|T_n(x_m - x^\dagger)\| \leq 3(1 + \eta)\|y - F(x_n)\|, \quad 0 \leq n \leq m.$$

Therefore

$$\begin{aligned} &\langle J_p(x_m) - J_p(x_l), x_m - x^\dagger \rangle \\ &\leq 3(1 + \eta) \sum_{n=l}^{m-1} \frac{1}{\alpha_n(y)} \|y - F(x_n) - T_n(x_{n+1} - x_n)\|^{r-1} \|y - F(x_n)\| \\ &= 3(1 + \eta) \mu^{r-1} \sum_{n=l}^{m-1} \frac{1}{\alpha_n(y)} \|y - F(x_n)\|^r. \end{aligned}$$

In view of (4.2), we obtain with $c_1 := 3(1 + \eta)\mu^{r-1}/(\mu - \eta)$ that

$$\langle J_p(x_m) - J_p(x_l), x_m - x_l \rangle \leq c_1 (\Delta_p(x^\dagger, x_l) - \Delta_p(x^\dagger, x_m)).$$

Hence, in view of (4.4), it follows

$$\Delta_p(x_m, x_l) \leq (1 + c_2) (\Delta_p(x^\dagger, x_l) - \Delta_p(x^\dagger, x_m)).$$

This together with the monotonicity result (4.1) implies that $\Delta_p(x_m, x_l) \rightarrow 0$ as $l, m \rightarrow \infty$. Therefore, it follows from Lemma 2.2 (c) that $\{x_n\}$ is a Cauchy sequence and thus $x_n \rightarrow x_*$ as $n \rightarrow \infty$ for some $x_* \in B_p(x^\dagger, \Delta_p) \subset \mathcal{X}$. Since $F(x_n) \rightarrow y$ as $n \rightarrow \infty$ and F is continuous, we have $F(x_*) = y$. The proof is therefore complete. \square

We define n_* to be the first integer such that $F(x_{n_*}) = y$; if such an integer does not exist, we take $n_* = \infty$. If $n_* < \infty$, then the definition of $\{x_n\}$ implies $x_n = x_{n_*}$ for all $n \geq n_*$, and thus $x_{n_*} = x_*$, where x_* is the solution of (1.1) given by Lemma 3.3.

LEMMA 4.2. *Let the conditions in Lemma 3.3 hold. Then for each fixed $0 \leq n \leq n_*$ there holds $x_n^\delta \rightarrow x_n$ as $y^\delta \rightarrow y$.*

Proof. We will show this result by induction. It is trivial when $n = 0$ since $x_0^\delta = x_0$. Now we assume that the conclusion holds for some $n < n_*$ and show that the conclusion holds also for $n + 1$.

As the first step, we will show that $x_n(\alpha, y^\delta) \rightarrow x_n(\alpha_0, y)$ if $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$ for some $\alpha_0 > 0$. We will adapt the arguments from [5, 9]. For simplicity of exposition, we set $g_n^\delta := y^\delta - F(x_n^\delta)$ and $g_n := y - F(x_n)$. Then by the induction hypothesis $x_n^\delta \rightarrow x_n$ and the continuity of F we have $g_n^\delta \rightarrow g_n$ as $y^\delta \rightarrow y$. By the minimality of $x_n(\alpha, y^\delta)$ we have

$$\alpha \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \leq \frac{1}{r} \|y^\delta - F(x_n^\delta)\|^r.$$

This implies the boundedness of $\{\Delta_p(x_n(\alpha, y^\delta), x_n^\delta)\}$ and hence the boundedness of $\{\|x_n(\alpha, y^\delta)\|\}$ by Lemma 2.2 (a). Since \mathcal{X} is reflexive, by taking a subsequence if necessary, we may assume that $x_n(\alpha, y^\delta) \rightharpoonup \bar{x}_n$ as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$ for some $\bar{x}_n \in \mathcal{X}$. Since $x_n^\delta \rightarrow x_n$ and $x \rightarrow F'(x)$ is continuous, we have $F'(x_n^\delta) \rightarrow F'(x_n)$. This together with the weak convergence of $x_n(\alpha, y^\delta)$ to \bar{x}_n implies

$$g_n^\delta - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta) \rightharpoonup g_n - F'(x_n)(\bar{x}_n - x_n)$$

as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$. Recall that the uniform smoothness of \mathcal{X} implies the continuity of J_p , see Lemma 2.1 (d), we therefore have $\langle J_p(x_n^\delta), x_n(\alpha, y^\delta) - x_n^\delta \rangle \rightarrow \langle J_p(x_n), \bar{x}_n - x_n \rangle$ as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$. Thus, by the weak lower semi-continuity of the norms in Banach spaces, we can obtain

$$\Delta_p(\bar{x}_n, x_n) \leq \liminf \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \quad (4.5)$$

and

$$\|g_n - F'(x_n)(\bar{x}_n - x_n)\| \leq \liminf \|g_n^\delta - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\|. \quad (4.6)$$

Therefore, in view of the minimality of $x_n(\alpha, y^\delta)$ and the induction hypothesis $x_n^\delta \rightarrow$

x_n , we obtain

$$\begin{aligned}
& \frac{1}{r} \|g_n - F'(x_n)(\bar{x}_n - x_n)\|^r + \alpha_0 \Delta_p(\bar{x}_n, x_n) \\
& \leq \liminf \left\{ \frac{1}{r} \|g_n^\delta - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\|^r + \alpha \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \right\} \\
& \leq \limsup \left\{ \frac{1}{r} \|g_n^\delta - F'(x_n^\delta)(x_n(\alpha_0, y) - x_n^\delta)\|^r + \alpha \Delta_p(x_n(\alpha_0, y), x_n^\delta) \right\} \\
& = \frac{1}{r} \|g_n - F'(x_n)(x_n(\alpha_0, y) - x_n)\|^r + \alpha_0 \Delta_p(x_n(\alpha_0, y), x_n).
\end{aligned}$$

By the minimality of $x_n(\alpha_0, y)$ and its uniqueness, we must have $\bar{x}_n = x_n(\alpha_0, y)$ and thus $x_n(\alpha, y^\delta) \rightarrow x_n(\alpha_0, y)$ as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$.

Next we will show that

$$\Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \rightarrow \Delta_p(x_n(\alpha_0, y), x_n) \quad (4.7)$$

as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$. Let

$$a := \limsup \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \quad \text{and} \quad b := \Delta_p(x_n(\alpha_0, y), x_n).$$

According to (4.5), it suffices to show that $a \leq b$. Assume to the contrary that $a > b$. Then we can find subsequences $\{\alpha_j\}$ and $\{y^{\delta_j}\}$ with $\alpha_j \rightarrow \alpha_0$ and $y^{\delta_j} \rightarrow y$ as $j \rightarrow \infty$ such that

$$\Delta_p(x_n(\alpha_j, y^{\delta_j}), x_n^{\delta_j}) \geq \limsup \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) - \frac{a-b}{4} \quad (4.8)$$

and

$$\begin{aligned}
& \frac{1}{r} \|g_n^{\delta_j} - F'(x_n^{\delta_j})(x_n(\alpha_j, y^{\delta_j}) - x_n^{\delta_j})\|^r \\
& \geq \liminf \frac{1}{r} \|g_n^\delta - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\|^r - \frac{\alpha_0(a-b)}{4}.
\end{aligned} \quad (4.9)$$

Therefore, by using (4.6), (4.8) and (4.9), we obtain

$$\begin{aligned}
& \frac{1}{r} \|g_n - F'(x_n)(x_n(\alpha_0, y) - x_n)\|^r + \alpha_j \Delta_p(x_n(\alpha_0, y), x_n) \\
& \leq \liminf \frac{1}{r} \|g_n^\delta - F'(x_n^\delta)(x_n(\alpha, y^\delta) - x_n^\delta)\|^r + \alpha_j \limsup \Delta_p(x_n(\alpha, y^\delta), x_n^\delta) \\
& \quad - \alpha_j(a-b) \\
& \leq \frac{1}{r} \|g_n^{\delta_j} - F'(x_n^{\delta_j})(x_n(\alpha_j, y^{\delta_j}) - x_n^{\delta_j})\|^r + \alpha_j \Delta_p(x_n(\alpha_j, y^{\delta_j}), x_n^{\delta_j}) \\
& \quad - \frac{(3\alpha_j - \alpha_0)(a-b)}{4}
\end{aligned}$$

Since $x_n^{\delta_j} \rightarrow x_n$, we have for large j that

$$\begin{aligned}
& \frac{1}{r} \|g_n^{\delta_j} - F'(x_n^{\delta_j})(x_n(\alpha_0, y) - x_n^{\delta_j})\|^r + \alpha_j \Delta_p(x_n(\alpha_0, y), x_n^{\delta_j}) \\
& \leq \frac{1}{r} \|g_n^{\delta_j} - F'(x_n^{\delta_j})(x_n(\alpha_j, y^{\delta_j}) - x_n^{\delta_j})\|^r + \alpha_j \Delta_p(x_n(\alpha_j, y^{\delta_j}), x_n^{\delta_j}) - \frac{\alpha_0(a-b)}{4}.
\end{aligned}$$

Since $\alpha_0(a - b) > 0$, this contradicts to the minimality of $x_n(\alpha_j, y^{\delta_j})$. We therefore obtain (4.7). With the help of $x_n^\delta \rightarrow x_n$, the continuity of J_p and the fact $x_n(\alpha, y^\delta) \rightarrow x_n(\alpha_0, y)$, then we can conclude that $\|x_n(\alpha, y^\delta)\| \rightarrow \|x_n(\alpha_0, y)\|$ as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$. Since \mathcal{X} is uniformly convex, it follows from Lemma 2.1 (b) that $\|x_n(\alpha, y^\delta) - x_n(\alpha_0, y)\| \rightarrow 0$ as $\alpha \rightarrow \alpha_0$ and $y^\delta \rightarrow y$.

As the second step, we will show that $\alpha_n(y^\delta) \rightarrow \alpha_n(y)$ as $y^\delta \rightarrow y$. Let

$$\underline{\alpha}_n := \liminf \alpha_n(y^\delta), \quad \bar{\alpha}_n := \limsup \alpha_n(y^\delta).$$

Then $0 \leq \underline{\alpha}_n \leq \bar{\alpha}_n \leq \infty$. If $\underline{\alpha}_n < \alpha_n(y)$, then there exists a sequence $\{y^{\delta_j}\}$ satisfying $y^{\delta_j} \rightarrow y$ such that $\alpha_n(y^{\delta_j}) \leq (\underline{\alpha}_n + \alpha_n(y))/2$. By the monotonicity of the function (3.5) and the definition of $\alpha_n(y^\delta)$ we have for all $(\underline{\alpha}_n + \alpha_n(y))/2 \leq \alpha \leq \alpha_n(y)$ that

$$\|y^{\delta_j} - F(x_n^{\delta_j}) - F'(x_n^{\delta_j})(x_n(\alpha, y^{\delta_j}) - x_n^{\delta_j})\| \geq \mu \|y^{\delta_j} - F(x_n^{\delta_j})\|$$

Now we take $j \rightarrow \infty$, use the induction hypothesis $x_n^\delta \rightarrow x_n$ and the fact $x_n(\alpha, y^\delta) \rightarrow x_n(\alpha, y)$. It then follows that

$$\|y - F(x_n) - F'(x_n)(x_n(\alpha, y) - x_n)\| \geq \mu \|y - F(x_n)\|$$

for all $(\underline{\alpha}_n + \alpha_n(y))/2 \leq \alpha \leq \alpha_n(y)$. Since the above inequality becomes equality at $\alpha = \alpha_n(y)$, and since the function

$$\alpha \rightarrow \|y - F(x_n) - F'(x_n)(x_n(\alpha, y) - x_n)\|$$

is monotonically increasing, we must have

$$\|y - F(x_n) - F'(x_n)(x_n(\alpha, y) - x_n)\| = \mu \|y - F(x_n)\|$$

for all $(\underline{\alpha}_n + \alpha_n(y))/2 \leq \alpha \leq \alpha_n(y)$. This is a contradiction, since $F(x_n) \neq y$ implies that $\alpha_n(y)$ is the unique root of the above equation. Therefore $\underline{\alpha}_n \geq \alpha_n(y)$. By a similar argument we can show that $\bar{\alpha}_n \leq \alpha_n(y)$. Hence $\underline{\alpha}_n = \bar{\alpha}_n = \alpha_n(y)$ which implies $\lim \alpha_n(y^\delta) = \alpha_n(y)$.

Combining the above two steps, we therefore obtain $x_{n+1}^\delta = x_n(\alpha_n(y^\delta), y^\delta) \rightarrow x_n(\alpha_n(y), y) = x_{n+1}$ as $y^\delta \rightarrow y$. The proof is thus complete. \square

Now we can conclude this paper by giving the convergence result on the regularizing Levenberg-Marquardt scheme in Banach spaces.

THEOREM 4.3. *Let Assumption 3.1 hold with $0 \leq \eta < 1/3$. Let $\eta < \mu < 1 - 2\eta$ and $\tau > (1 + \eta)/(\mu - \eta)$. Then the regularizing Levenberg-Marquardt scheme in Banach spaces is well-defined and terminates after $n_\delta < \infty$ iteration with $n_\delta = O(1 + |\log \delta|)$. Moreover $x_{n_\delta}^\delta$ converges to a solution of (1.1) in $B_\rho(x^\dagger, \Delta_p)$ as $\delta \rightarrow 0$. If x^\dagger is the unique solution of (1.1) in $B_\rho(x^\dagger, \Delta_p)$, then $x_{n_\delta}^\delta \rightarrow x^\dagger$ as $\delta \rightarrow 0$.*

Proof. It remains to show the convergence of $x_{n_\delta}^\delta$ since other parts have been proved in Lemma 3.3. Let x_* be the limit of $\{x_n\}$ which exists by Lemma 4.1. We will show that $x_{n_\delta}^\delta \rightarrow x_*$ as $\delta \rightarrow 0$.

Assume first that $\{y^{\delta_j}\}$ is a sequence satisfying $\|y^{\delta_j} - y\| \leq \delta_j$ with $\delta_j \rightarrow 0$ such that $n_{\delta_j} \rightarrow n_0$ as $j \rightarrow \infty$ for some finite integer n_0 . We may assume $n_{\delta_j} = n_0$ for all j . Let n_* be the first integer such that $F(x_{n_*}) = y$ as we defined before. If $n_0 < n_*$, then it follows from Lemma 4.2 that $x_{n_0}^{\delta_j} \rightarrow x_{n_0}$. But from the definition of $n_0 = n_{\delta_j}$ we have

$$\|F(x_{n_0}^{\delta_j}) - y^{\delta_j}\| \leq \tau \delta_j.$$

Letting $j \rightarrow \infty$ gives $F(x_{n_0}) = y$ which contradicts the definition of n_* . Therefore $n_0 \geq n_*$ and hence $F(x_{n_*}) = y$. This implies that $x_{n_*} = x_*$. Consequently Lemma 4.2 implies $x_{n_{\delta_j}}^{\delta_j} \rightarrow x_*$ as $j \rightarrow \infty$. By using the monotonicity result in Lemma 3.3 we obtain

$$\Delta_p(x_*, x_{n_{\delta_j}}^{\delta_j}) = \Delta_p(x_*, x_{n_0}^{\delta_j}) \leq \Delta_p(x_*, x_{n_*}^{\delta_j})$$

which together with Lemma 2.2 (b) shows that $\Delta_p(x_*, x_{n_{\delta_j}}^{\delta_j}) \rightarrow 0$ and hence $x_{n_{\delta_j}}^{\delta_j} \rightarrow x_*$ as $j \rightarrow \infty$.

Assume next that $\{y^{\delta_j}\}$ is a sequence satisfying $\|y^{\delta_j} - y\| \leq \delta_j$ with $\delta_j \rightarrow 0$ such that $n_{\delta_j} \rightarrow \infty$ as $j \rightarrow \infty$. If n_* is finite, we can use the same argument in the above case to show that $x_{n_{\delta_j}}^{\delta_j} \rightarrow x_*$ as $j \rightarrow \infty$. So we may assume that $n_* = \infty$. Let $\epsilon > 0$ be an arbitrary number. Since $x_n \rightarrow x_*$, we may pick an integer $n(\epsilon)$ such that $\Delta_p(x_*, x_{n(\epsilon)}) < \epsilon/2$. Since Lemma 4.2 implies that $x_{n(\epsilon)}^{\delta_j} \rightarrow x_{n(\epsilon)}$ and since the uniform smoothness of \mathcal{X} implies that J_p is continuous, we can take an integer $j(\epsilon)$ such that $n_{\delta_j} \geq n(\epsilon)$ and $|\Delta_p(x_*, x_{n(\epsilon)}^{\delta_j}) - \Delta_p(x_*, x_{n(\epsilon)})| < \epsilon/2$ for all $j \geq j(\epsilon)$. Consequently, by using the monotonicity result in Lemma 3.3 it follows that

$$\Delta_p(x_*, x_{n_{\delta_j}}^{\delta_j}) \leq \Delta_p(x_*, x_{n(\epsilon)}^{\delta_j}) \leq \Delta_p(x_*, x_{n(\epsilon)}) + |\Delta_p(x_*, x_{n(\epsilon)}^{\delta_j}) - \Delta_p(x_*, x_{n(\epsilon)})| < \epsilon$$

for all $j \geq j(\epsilon)$. Since $\epsilon > 0$ is arbitrary, we must have $\Delta_p(x_*, x_{n_{\delta_j}}^{\delta_j}) \rightarrow 0$ as $j \rightarrow \infty$. Consequently, by Lemma 2.2 (b), $x_{n_{\delta_j}}^{\delta_j} \rightarrow x_*$ as $j \rightarrow \infty$. \square

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