

# Optimal Transportation and Nonlinear Partial Differential Equations

Neil S. Trudinger

Centre of Mathematics and its Applications  
Australian National University

26th Brazilian Mathematical Colloquium 2007

# Optimal Transportation and Nonlinear PDE.

- 1 Optimal Transportation.
- 2 Primary Examples.
- 3 Lagrange Multipliers.
- 4 Existence of Potentials.
- 5 Properties of Potentials.
- 6 Monge-Ampère equation.
- 7 Regularity.
- 8 Conditions on Cost Functions.
- 9 Specific Examples.
- 10 Domain Convexity.
- 11 Global Regularity.
- 12 Interior Regularity.
- 13 Sharpness of Conditions.
- 14 Transportation in Riemannian Manifolds.
- 15 Strategy of Proofs.
- 16 Apriori Estimates.
- 17 Ellipticity and  $c$ -convexity.

# 1. Optimal Transportation.

- Domains :  $\Omega, \Omega^* \subset \mathbb{R}^n$ , (or Riemannian manifold)  
 $\Omega$  : initial domain,  $\Omega^*$  : target domain.

- Densities :  $f, g \geq 0, \in L^1(\Omega), L^1(\Omega^*)$  resp.

- Mass balance :

$$\int_{\Omega} f = \int_{\Omega^*} g$$

- Mass preserving mappings :

$T : \Omega \rightarrow \Omega^*$ , Borel measurable,

$$\int_{T^{-1}(E)} f = \int_E g, \quad \forall \text{ Borel sets } E \subset \Omega^*,$$

$$\mathcal{T} = \mathcal{T}(f, \Omega; g, \Omega^*)$$

= set of mass preserving mappings.

- Cost function :  $c : \Omega \times \Omega^* \rightarrow \mathbb{R}$  continuous.
- Cost functional :

$$\mathcal{C}(T) = \int_{\Omega} c(x, Tx) f(x) dx.$$

- **Monge-Kantorovich Problem :**

Minimize  $\mathcal{C}$  over  $\mathcal{T}$

- Cost function :  $c : \Omega \times \Omega^* \rightarrow \mathbb{R}$  continuous.
- Cost functional :

$$\mathcal{C}(T) = \int_{\Omega} c(x, Tx) f(x) dx.$$

- **Monge-Kantorovich Problem :**

Minimize  $\mathcal{C}$  over  $\mathcal{T}$

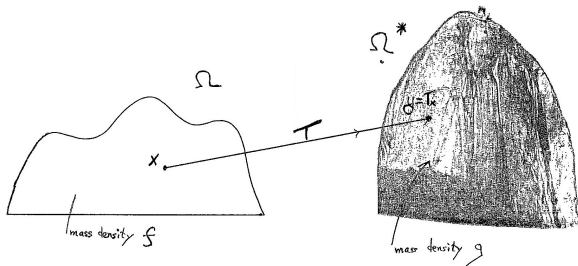


Figure: Mass Transportation.

## 2. Primary Examples.

- Monge Problem .

$$c(x, y) = |x - y|,$$

(work done in moving  $x$  to  $y$ ).

Existence:

Monge(1781), Appel(1887), Kantorovich(1942,48), Sudakov(1979), Evans–Gangbo(1999), Trudinger–Wang(2001), Caffarelli–Feldman–McCann(2002).

- Quadratic cost .

$$c(x, y) = \frac{1}{2} |x - y|^2.$$

Existence and uniqueness:

Knott–Smith(1984), Brenier(1987).

Regularity:

Caffarelli(1992,1996), Urbas(1997).

- Reference :

C. Villani, Topics in Optimal Transportation, Amer. Math. Soc., 2003.

### 3. Lagrange Multipliers.

Assume optimal mapping  $T$  is smooth diffeomorphism,  $f, g > 0$ .

- Mass preserving condition  $\Leftrightarrow$  constraint

$$(g \circ T)|\det DT| = f.$$

- Augmented cost functional

$$C_\lambda(T) = \int_{\Omega} \{f(x)c(x, Tx) + \lambda(x)g(Tx)\det DT(x)\} dx$$

- Euler-Lagrange equation.

$$\begin{aligned} \frac{\partial}{\partial x_i} [\lambda(x)g(Tx)(DT)^{ij}] \\ = f(x) \frac{\partial}{\partial y_j} c(x, Tx) + \lambda(x)\det DT \frac{\partial}{\partial y_j} g(Tx), \end{aligned}$$

$\Rightarrow$

$$c_y(x, Tx) = \frac{\partial}{\partial(Tx)} \lambda(x) = D\psi(Tx), \quad \psi = \lambda \circ T^{-1}.$$

- Interchanging  $\Omega$  and  $\Omega^* \Rightarrow$

$$c_x(x, Tx) = D\phi(x)$$

for some function  $\phi : \Omega \rightarrow \mathbb{R}$ .

- Functions  $\phi, \psi$  are called potentials.

They are related by

$$\phi(x) + \psi(Tx) = c(x, Tx) \quad (+\text{constant}).$$

- Under the hypothesis that the mappings  $c_x(x, \cdot)$  and  $c_y(\cdot, y)$  are invertible  $\forall x \in \Omega, y \in \Omega^*$ , (assumed henceforth), the optimal mapping  $T$  is uniquely determined by the potential  $\phi$  or  $\psi$ . Note that this is not the case for the Monge cost.



## 4. Existence of Potentials.

- **Kantorovich dual problem .**

Maximize

$$J(\phi, \psi) := \int_{\Omega} f\phi + \int_{\Omega^*} g\psi$$

over set

$$K = \{ \phi, \psi \in C^0(\mathbb{R}^n) \mid \phi(x) + \psi(y) \leq c(x, y), \forall x \in \Omega, y \in \Omega^* \}$$

- $J(\phi, \psi) \leq C(T) \quad \forall \phi, \psi \in K, T \in \mathcal{T}.$
- Direct method  $\Rightarrow$  solutions,  $\phi, \psi$ , semi-concave and self-dual, that is  $\psi = \phi^*, \phi = \psi^*$  where  $c$ -transforms  $\phi^*, \psi^*$  are defined by

$$\phi^* = \inf_x [c(x, y) - \phi(x)], \quad \psi^* = \inf_y [c(x, y) - \psi(y)].$$

- To construct optimal mapping  $T$  from Kantorovich potential  $\phi$ , set

$$T = Y(\cdot, D\phi) := c_x^{-1}(\cdot, D\phi)$$

- $\Rightarrow$

- 1  $T$  well-defined a.e.  $\Omega$ ,
- 2  $T$  mass preserving,
- 3  $J(\phi, \psi) = \mathcal{C}(T) \Rightarrow T$  optimal,
- 4  $T$  is unique a.e. on  $\{f > 0\}$ ,
- 5 if  $T^* = X(D\psi, \cdot) := c_y^{-1}(D\psi, \cdot)$ , then

$$T^* \circ T = I \text{ a.e. } \{f > 0\}, \quad T \circ T^* = I \text{ a.e. } \{g > 0\}.$$

- **Remark :**

This process originally carried out by Knott–Smith(1984), Brenier(1987) for quadratic case,  $c(x,y)=-x \cdot y$ , and extended to strictly convex  $c(x-y)$  by Caffarelli(1996), Gangbo–McCann(1996).

## 5. Properties of Potentials.

- $\phi, \psi$  are twice differentiable a.e.  $\Omega$ , a.e.  $\Omega^*$  resp.
- $\phi$  is  $c$ -concave on  $\Omega$ , that is  $\forall x_0 \in \Omega, \exists y_0 \in \mathbb{R}^n$  s.t.  $\forall x \in \Omega$ ,

$$\phi(x) \leq \phi(x_0) + c(x, y_0) - c(x_0, y_0).$$

Similarly,  $\psi$  is  $c^*$ -concave on  $\Omega^*$ ,  $c^*(x, y) = c^*(y, x)$ .

- $u = -\phi$  is  $c^-$ -convex on  $\Omega$ ,  $c^- = -c$ , and  $v = -\psi$  is  $c^{-,*}$ -convex on  $\Omega^*$ .
- $D^2\phi \leq D_x^2 c(\cdot, T)$  a.e.  $\Omega$ ,  $D^2u + D_x^2 c(\cdot, T) \geq 0$  a.e.  $\Omega$ .
- $c$ -normal mapping ,

$$\chi_u(x_0) = \{ y_0 \in \mathbb{R}^n \mid u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0), \forall x \in \Omega \}$$

satisfies

- (i)  $\chi_u = Y(\cdot, -Du)$  where  $u$  is differentiable,
- (ii)  $\emptyset \neq \chi_u(x) \subset c_x(x, \partial^- u(x))$  at singularities,

where  $\partial^- u$  denotes subdifferential of  $u$ .

## 6. Optimal Transportation Equation.

- Assume  $f, g, |\det c_{x,y}| > 0$ .
- Mass preserving condition  $\Rightarrow$  (for  $u = -\phi$ ),

$$|\det DY(\cdot, -Du)| = f/g(Y) \quad \text{a.e. } \Omega.$$

- $c^-$ -convexity  $\Rightarrow$

$$D^2u + D_x^2(c(\cdot, Y)) \geq 0 \quad \text{a.e. } \Omega.$$

$\Rightarrow$

- **Optimal transportation equation**, OTE.

$$\det[D^2u + D_x^2c(\cdot, Y)] = |\det c_{x,y}|f/g(Y) \quad \text{a.e. } \Omega.$$

- Conversely, if  $u \in C^2(\Omega)$  satisfies OTE,  $Y(\cdot, -Du)(\Omega) = \Omega^*$  and  $u$  is  $c^-$ -convex, then  $T = Y(\cdot, -Du)$  is optimal mapping for Monge-Kantorovich problem.

- The OTE is a PDE of Monge–Ampère type, MAE

$$\det [D^2u + A(\cdot, u, Du)] = B(\cdot, u, Du),$$

which is elliptic, (degenerate elliptic), with respect to a function  $u$  whenever

$$D^2u + A(\cdot, u, Du) > 0, \quad (\geq 0).$$

Consequently, a  $c^-$ -convex solution of the OTE is an elliptic solution.

- The OTE is a special case of the prescribed Jacobian equation, PJE.

$$\det DY(x, u, Du) = \psi(x, u, Du)$$

if the vector field  $Y : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $\det Y_p \neq 0$ .

## 7. Regularity.

- **Quadratic case** : (Caffarelli(1996), Urbas(1997)).

$$c(x, y) = \frac{1}{2}|x - y|^2$$

Domains  $\Omega, \Omega^*$  uniformly convex,  $C^\infty$

Densities  $f, g \in C^\infty(\bar{\Omega}), C^\infty(\bar{\Omega}^*)$  resp.  $\inf f, g > 0$

$\Rightarrow \exists$  unique (a.e.) optimal diffeomorphism  $T \in [C^\infty(\bar{\Omega})]^n$ , given by

$$T = Du$$

where potential  $u \in C^\infty(\bar{\Omega})$  is convex solution of the second boundary value problem,

$$\det(D^2u) = f/g(Du),$$

$$Du(\Omega) = \Omega^*$$

If  $\Omega, \Omega^*$  only convex,  $f, g \in C^\infty(\Omega), C^\infty(\Omega^*)$  resp. then  $T \in [C^\infty(\Omega)]^n, u \in C^\infty(\Omega)$  (Caffarelli, 1992).

- **General case :**

For what cost functions and domains are there smooth diffeomorphism solutions for smooth positive densities ?

- **Propaganda :** From Villani, 4.3 Open Problems.

"Without any doubt, the main open problem is to derive regularity estimates for more general transportation costs , .... At the moment essentially nothing is known concerning the smoothness of the solutions to these equations, beyond the regularity properties which automatically follow from  $c$ -concavity."

## 8. Conditions on Cost Functions.

- A1 For each  $p \in \mathbb{R}^n$ ,  $x \in \Omega$ ,  $q \in \mathbb{R}^n$ ,  $y \in \Omega^*$ ,  $\exists$  unique  $Y = Y(x, p)$ ,  $X = X(q, y)$  satisfying

$$c_x(x, Y) = -p, \quad c_y(X, y) = -q.$$

- A2  $|\det c_{x,y}| \geq c_0$  in  $\Omega \times \Omega^*$ ,  $c_0 =$  positive constant.
- A3w  $A(x, p) := c_{xx}(x, Y(x, p))$  is regular in sense that

$$\mathcal{F}(x, p; \xi, \eta) := -D_{p_k p_\ell} A_{ij}(x, p) \xi_i \xi_j \eta_k \eta_\ell \geq 0$$

for all  $x \in \Omega$ ,  $Y(x, p) \in \Omega^*$ ,  $\xi, \eta \in \mathbb{R}^n$ ,  $\xi \perp \eta$ .

- A3  $A$  is strictly regular in sense that

$$\mathcal{F}(x, p; \xi, \eta) \geq c_0 |\xi|^2 |\eta|^2, \quad \xi, \eta \in \mathbb{R}^n, \xi \perp \eta$$

- .
- Remark. A3w and A3 are symmetric and invariant under coordinate changes in  $x$  and  $y$ .



## 9. Specific Examples.

$$9.1 \quad c(x, y) = \begin{cases} \frac{1}{m}|x - y|^m, & m \neq 1, 0 \\ \log|x - y|, & m = 0 \end{cases}.$$

$$A(x, p) = A(p) = |p|^{\frac{m-2}{m-1}} I + (m-2)|p|^{-\frac{m}{m-1}} p \otimes p.$$

is *regular* only for quadratic case,  $m = 2$ .

$$-c \Rightarrow -A$$

- is *regular* for  $-2 \leq m < 1$ .
- is *strictly regular* for  $-2 < m < 1$  and
  - $|p|$  bounded above for  $-2 < m < 0$ ,
  - $|p|$  bounded away from 0 for  $0 < m < 1$ .
- vector field  $Y(x, p) = x \pm |p|^{\frac{2-m}{m-1}} p$ .

$$9.2 \quad c(x, y) = \sqrt{1 + |x - y|^2}$$

$$A(x, p) = \sqrt{1 - |p|^2} (I - p \otimes p)$$

- *strictly regular*,  
(for  $|p| \leq 1 - \delta$ ,  $\delta > 0$ , i.e.  $c$  bounded )
- *vector field*  $Y(x, p) = x + \frac{p}{\sqrt{1 - |p|^2}}$ .
- compare *Lorentzian* curvature.

$$9.3 \quad c(x, y) = \sqrt{1 - |x - y|^2}$$
$$A(x, p) = -\sqrt{1 + |p|^2} (I + p \otimes p)$$

- *strictly regular,*
- *vector field*  $Y(x, p) = x - \frac{p}{\sqrt{1 + |p|^2}}.$
- compare *Euclidean* curvature.

9.4  $M_f, M_g \subset \mathbb{R}^{n+1}$ , graphs of  $f, g \in C^2(\Omega), C^2(\Omega^*)$  resp.

$$\Omega, \Omega^* \subset \mathbb{R}^n \quad \ni \quad \sup_{\substack{x \in \Omega \\ y \in \Omega^*}} |\nabla f(x) \cdot \nabla g(y)| < 1,$$

$$c(x, y) = \frac{1}{2} |\hat{x} - \hat{y}|^2$$

where  $\hat{x} = (x, x_{n+1}) \in M_f, \hat{y} = (y, y_{n+1}) \in M_g$ .

$A$  (given implicitly in general) is

- *regular* if  $f, g$  are convex,
- *strictly regular* if  $f, g$  uniformly convex.

Examples of functions, ( $A$  strictly regular)

- $f = g = \sqrt{1 + |x|^2}, \quad \Omega, \Omega^* \subset \mathbb{R}^n,$
- $f = g = -\sqrt{1 - |x|^2}, \quad \Omega, \Omega^* \subset B_{1/\sqrt{2}}(0),$
- $f = g = \varepsilon|x|^2, \quad \Omega, \Omega^* \subset B_{\frac{1}{2\varepsilon}}(0).$

# 10. Domain Convexity.

- $\Omega$  is  $c$ -convex w.r.t.  $\Omega^* \Leftrightarrow$

$$c_y(\cdot, y)(\Omega) \text{ is convex, for all } y \in \Omega^*$$

- $\Omega$  is uniformly  $c$ -convex w.r.t.  $\Omega^* \Leftrightarrow$

$$c_y(\cdot, y)(\Omega) \text{ is uniformly convex, with respect to } y \in \Omega^*$$

- Analytic formulation :  $\Omega$  connected,  $C^2$ , is  $c$ -convex (uniformly  $c$ -convex) w.r.t.  $\Omega^* \Leftrightarrow$

$$\left[ D_i \gamma_j(x) + c^{k,\ell} c_{ij,k}(x, y) \gamma_\ell(x) \right] \tau_i \tau_j \geq 0 \quad (\delta_0)$$

for all  $x \in \partial\Omega$ ,  $y \in \Omega^*$  outer unit normal  $\gamma$ , unit tangent  $\tau$  to  $\partial\Omega$  (for some constant  $\delta_0 > 0$ ).

- With  $c^*(x, y) = c(y, x)$  we get analogous definitions for  $\Omega^*$ .

- Corresponds to usual convexity in quadratic case.
- Sufficiently small balls are uniformly  $c$ -convex.  
In example 9.4, The sublevel sets of  $f$  are  $c$ -convex.
- In analytic formulation,

$$c^{k,\ell} c_{ij,k}(x, y) = D_{p_k} A_{ij}(x, p), \quad y = Y(x, p)$$

$\Rightarrow$  barrier constructions.

- Invariant under coordinate changes

# 11. Global Regularity.

- **Theorem** (T–Wang 2007)

Cost function,  $c \in C^\infty$  satisfies A1, A2, A3w.

Domains,  $\Omega, \Omega^* \in C^\infty$ , uniformly  $c, c^*$ -convex.

Densities  $f, g \in C^\infty(\overline{\Omega}), C^\infty(\overline{\Omega^*})$  resp.,  $\inf f, g > 0$ .

$\Rightarrow$

There exists unique (a.e.) optimal diffeomorphism  $T \in [C^\infty(\overline{\Omega})]^n$ , given by

- $T = Y(\cdot, Du)$

where  $u \in C^\infty(\overline{\Omega})$  is an elliptic solution of OTE

- $\det[D^2u + D_x^2c(\cdot, Y(\cdot, Du))] = |\det c_{x,y}|f/g(Y)$

satisfying the second boundary condition

- $Y(\cdot, Du)(\Omega) = \Omega^*$ .

- Remark. Need only  $c \in C^{3,1}, f, g \in C^{1,1}(\overline{\Omega}), C^{1,1}(\overline{\Omega^*}), \Omega, \Omega^* \in C^{3,1}$   
 $\Rightarrow$  solution  $T \in C^{2,\alpha}(\overline{\Omega}), \forall \alpha < 1$ .

## 12. Interior Regularity.

- **Theorem** (Ma-T-Wang 2005, 2007)

Cost function,  $c \in C^\infty$  satisfies A1, A2, A3.

Domain  $\Omega^*$   $c^*$ -convex, w.r.t.  $\Omega$ .

Densities  $f, g \in C^\infty(\Omega), C^\infty(\Omega^*)$  resp.,  $\inf f, g > 0$ .

$\Rightarrow$

Optimal mapping  $T \in [C^\infty(\Omega)]^n$

- Remark.  $c \in C^{3,1}, f, g \in C^{1,1}(\Omega), C^{1,1}(\Omega^*)$  resp..

$\Rightarrow T \in [C^{2,\alpha}(\Omega)]^n, \forall \alpha < 1$ .

- **Theorem** (Loeper, 2007)

Cost function and domains as above.

Densities  $f \in L^p(\Omega), p > n, \inf g > 0$ .

$\Rightarrow$

Optimal mapping  $T \in [C^{0,\alpha}(\Omega)]^n$  for some  $\alpha > 0$ .



## 13. Sharpness of Conditions.

- A3w violated  $\Rightarrow \exists$  smooth positive densities  $f, g$  for which optimal transportation problem is not solvable with continuous mapping  $T$ . (Loeper 2007).
  
- $c^*$ -convexity of  $\Omega^*$  violated  
 $\Rightarrow$   
 $\exists$  smooth densities  $f, g > 0$  for which optimal transportation problem is not solvable with smooth mapping  $T$ . (Ma-T-Wang, 2005).

# 14. Transportation in Riemannian Manifolds.

## ■ Extrinsic costs.

$$c : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad M \rightsquigarrow \mathbb{R}^{n+1}.$$

Examples

- Light reflector problem,

$$M = S^n, \quad c(x, y) = -\log |x - y|.$$

Monge-Ampère equation :

$$\begin{aligned} \det \left\{ D^2 u + Du \otimes Du - \frac{1}{2} |Du|^2 g_0 + \frac{1}{2} g_0 \right\} \\ = \left[ \frac{1}{2} (1 + |Du|^2) \right]^n f/g(Y). \end{aligned}$$

strictly regular, (A3)

$$Y(x, p) = x - \frac{2}{1+|p|^2}(x + p), \quad p \in T_x^* S^n.$$

- Quadratic cost.

$$c(x, y) = \frac{1}{2} |x - y|^2, \quad \text{related to graph example.}$$

## ■ Intrinsic costs.

- Quadratic cost.

$$c(x, y) = \frac{1}{2} [d(x, y)]^2$$

where  $d$  is geodesic distance in  $\{M, g\}$ .

- $A$  regular (A3w)
  - $\Rightarrow$  sectional curvatures  $\geq 0$ , (Loeper 2007).
  - $\Rightarrow$  no regularity in hyperbolic manifolds.
- For sphere  $M = S^n$ ,  $A$  is strictly regular, (A3), (Loeper 2007).

## 15. Strategy of Proofs.

- **Global regularity.** Prove existence of classical elliptic solution of second boundary value problem. through a priori estimates and method of continuity. Show resulting solutions are  $c$ -convex and hence satisfy Kantorovich dual problem. Crucial estimates are obliqueness of second boundary conditions and global second derivative bounds.
- **Interior regularity.** Solve classical Dirichlet problem for OTE in sufficiently small balls with approximating densities and boundary data approximating Kantorovich potential. Interior regularity follows from interior estimates and comparison argument, utilizing equivalence of Kantorovich potential and generalized solution in sense of Aleksandrov and Bakel'man.

# 16. Apriori Estimates.

- 16.1 Obliqueness (T–Wang).

- $u$  smooth elliptic solution of second boundary value problem for OTE.

- $f, g \in C^{1,1}(\bar{\Omega})$ ,  $\inf f, g > 0$ .

- $\Omega, \Omega^*$  uniformly  $c, c^*$ -convex.

$$\Rightarrow c^{i,j} \gamma_i^*(Tu) \gamma_j \geq \delta_0 \quad \text{on } \partial\Omega,$$

for some positive constant  $\delta_0$ .

- 16.2 Global second derivatives (T–Wang).

- $A$  regular (i.e.  $c$  satisfies A1, A2, A3w)

$$\Rightarrow |D^2 u| \leq C(1 + \sup_{\partial\Omega} |D^2 u|).$$

- 16.3 Second boundary value problem (T–Wang).

- $A$  regular,
- $\Omega, \Omega^*$  uniformly  $c, c^*$ -convex

$$\Rightarrow |D^2 u| \leq C.$$

- 16.4 Interior second derivatives (Ma-T–Wang).
  - A strictly regular (i.e.  $c$  satisfies A1, A2, A3)
    - $\Rightarrow \sup_{\Omega'} |D^2 u| \leq C, \quad \Omega' \subset\subset \Omega.$
- 16.5 Hölder gradient estimate (Loeper).
  - A strictly regular,
  - $f \in L^p(\Omega), p > n, \inf g > 0$ 
    - $\Rightarrow [Du]_{0,\alpha;\Omega'} \leq C, \quad \Omega' \subset\subset \Omega \quad \text{for some } \alpha > 0.$

## 17. Ellipticity and $c$ -Convexity.

- $u \in C^2(\Omega)$ , degenerate elliptic w.r.t. OTE, i.e.

$$D^2u + D_x^2c(\cdot, T) \geq 0, \quad \text{in } \Omega,$$

for

$$T = Y(\cdot, Du)$$

$\Rightarrow u$  is  $c^-$ -convex if either

- $T(\Omega)$  is  $c^*$ -convex w.r.t.  $\Omega$  and  $T$  is one-to-one

or

- $c$  satisfies A3w and  $\Omega$  is  $c$ -convex w.r.t.  $T(\Omega)$ , (T-Wang).