ON THE LOCAL THEORY OF PRESCRIBED JACOBIAN EQUATIONS

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In honour of the 60th birthday of

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PRESCRIBED JACOBIAN PDE

$$Y : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad \in C^1, \ \Omega \subset \mathbb{R}^n$$
$$x, \quad u, \quad p \quad \to \quad Y(x,u,p)$$

Mapping $\quad Tu := Y(\cdot, u, Du), \quad u \in C^2(\Omega)$

$$\psi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad \geq 0$$

PJE:

$$\boxed{\left|\det [DTu]\right| = \psi(\cdot, u, Du)}$$

Assume $\quad \det Y_p \neq 0,$

$\Rightarrow$ Monge-Ampère equation :

MAE :

$$\det [D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du),$$
$$\begin{cases} A(\cdot,u,p) = -Y_p^{-1}(Y_x + Y_u \otimes p) \\ B(\cdot,u,p) = |\det Y_p|^{-1} \psi \end{cases}$$

for degenerate elliptic solution u,
$$D^2u \geq A(\cdot, u, Du).$$

SECOND BOUNDARY VALUE PROBLEM

$$Tu(\Omega) = \Omega^*, \quad \Omega^* \subset \mathbb{R}^n$$

If
$$\psi(x,u,p) = \frac{f(x)}{g \circ Y(x,u,p)}$$

for $f \geq 0$, $\in L^1(\Omega)$, $g > 0$, $\in L^1(\Omega^*)$, densities,

necessary condition for solution u with diffeomorphism Tu is

$$\int_\Omega f = \int_{\Omega^*} g,$$

mass balance condition.

Special case, $Y(x,u,p) = p$, $Tu = Du \Rightarrow$

standard Monge-Ampère equation:

$$\det D^2 u = \psi$$

Classical solvability due to Delanöe, Caffarelli, Urbas..

OPTIMAL TRANSPORTATION

Cost function, $\quad c : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \in C^2$

$$x, y \rightarrow c(x,y)$$

- $c_x(x, \cdot)$ one-to-one $\quad \forall x$
- $\det c_{x,y} \neq 0$

$$\Rightarrow \quad Y(x,p) = c_x^{-1}(x, \cdot)(p)$$

$$Y_p = [c_{x,y}]^{-1}$$

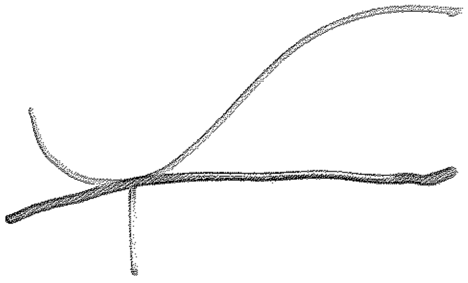
$$A(x,p) = c_{xx}(x, Y(x,p))$$

Note: no u dependence!

Special case. $\quad c(x,y) = x.y \Rightarrow Y(x,p) = p$

Remark. In general if A is symmetric, then Y is generated, at least locally, by cost function.

GEOMETRIC INTERPRETATION



$x_{n+1} = u(x)$, $u \in C^2(\Omega)$

$x_{n+1} = G(x, y_0, z_0) := c(x, y_0) - z_0$,

$y_0 \in \mathbb{R}^n$, $z_0 \in \mathbb{R}$

c - *affine support*

$$\Rightarrow \quad y_0 = Tu(x_0) = Y(x_0, Du(x_0))$$

$$D^2 u(x_0) \geq c_{xx}(x_0, Tu(x_0))$$

$$z_0 = c(x_0, y_0) - u(x_0)$$

$$:= Z(x_0, u(x_0), Du(x_0))$$

$$\Rightarrow \quad \{G_x, G\}(x_0, Y(x_0, Du(x_0)), Z(x_0, u(x_0), Du(x_0))$$
$$= \{Du(x_0), u(x_0)\}$$

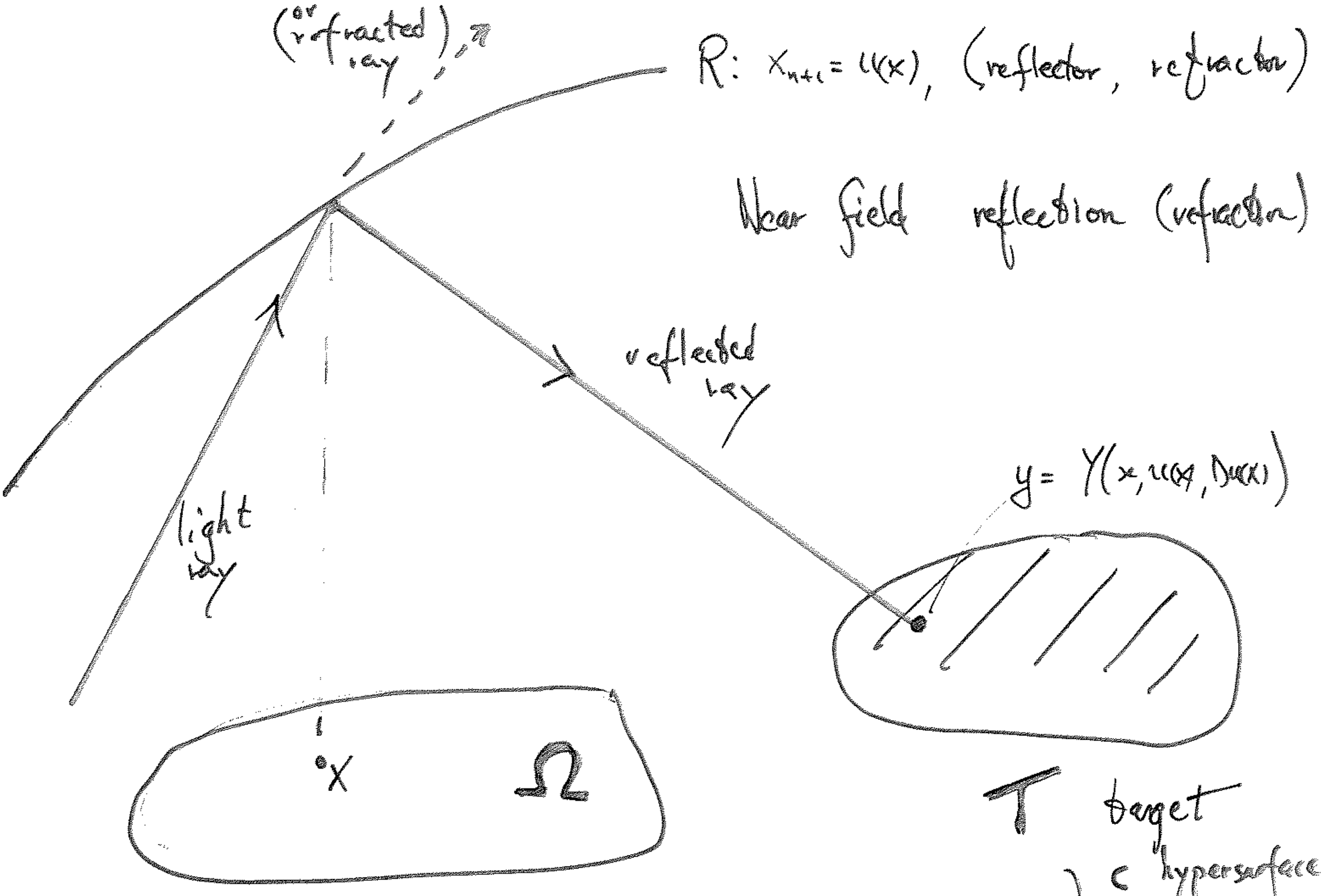
Note: $G_z = -1 < 0$.

IDEA : Extend to more general G !

Note : $TG(x, y_0, z_0) = y_0$, $\forall x \in \Omega$.

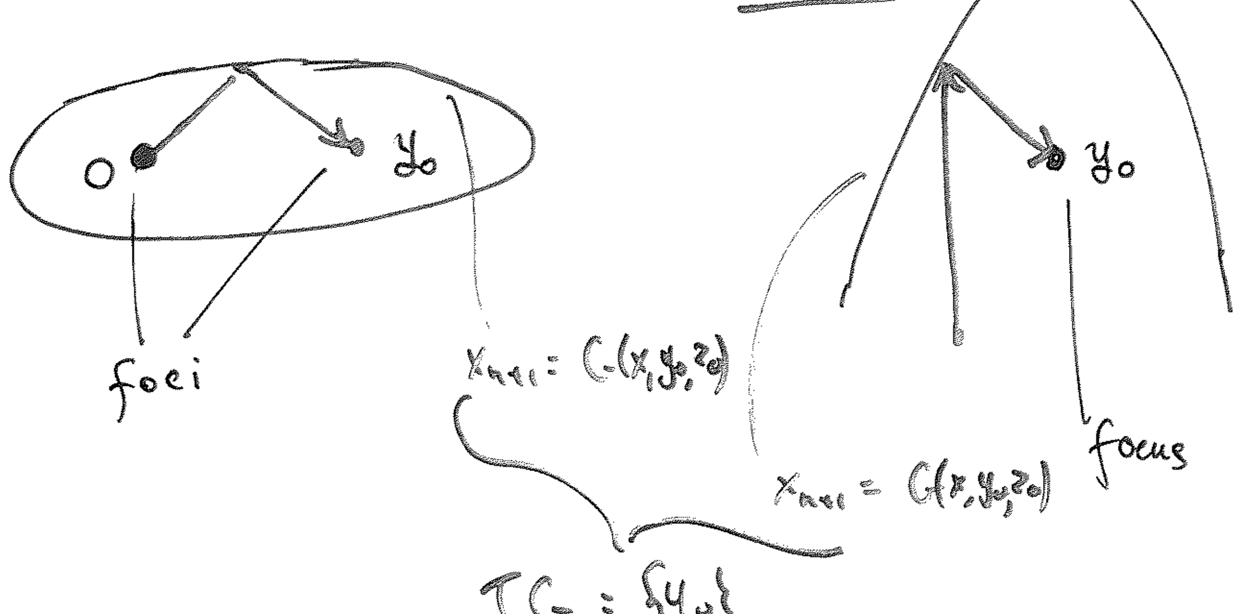
GEOMETRIC OPTICS

Near field reflection (refraction)



<u>Point source</u>, $\Omega \subset S^n$, <u>Beam</u> $\Omega \subset \mathbb{R}^n$

<u>Ellipsoid</u> (of revolution). <u>Paraboloid</u>



GENERATING FUNCTIONS

$$G : \mathcal{U} \times I \to \mathbb{R}, \quad \mathcal{U} \subset \underset{(x,}{\mathbb{R}^n} \times \underset{y)}{\mathbb{R}^n}, \quad \underset{z}{I} \subset \mathbb{R}, \text{ open interval,}$$

$$G \in C^2, \quad G_z \neq 0,$$

Projections: $\mathcal{U}_x^{*} = \{ y \in \mathbb{R}^n \mid (x,y) \in \mathcal{U} \}$, $\mathcal{U}_y = \{ x \in \mathbb{R}^n \mid (x,y) \in \mathcal{U} \}$,

$$\mathcal{U}^{(1)} = \cup\, \mathcal{U}_y, \quad \mathcal{U}^{(2)} = \cup\, \mathcal{U}_x^{*}$$

<u>Assumptions</u>:

G1: $\forall (x,y) \in \mathcal{U}$, $\exists$ open interval $I(x,y) \subset I \ni$ mapping $(G_x, G)(x,\cdot,\cdot)$ is one-to-one in $y \in \mathcal{U}_x^{*}$, $z \in I(x,y)$, for each $x \in \mathcal{U}^{(1)}$.

G2: $\forall (x,y) \in \mathcal{U}$, $z \in I(x,y)$,

$$\det E(x,y,z) \neq 0$$

where $E = E(x,y,z)$ is $n \times n$ matrix,

$$E = G_{x,y} - (G_z)^{-1} G_{x,z} \otimes G_y$$

G1 $\Rightarrow$ $Y = Y(x,u,p)$, $Z = Z(x,u,p)$ satisfying

$$\begin{cases} G_x(x,Y,Z) = p \\ G(x,Y,Z) = u \end{cases}.$$

Note: ① Jacobian det of mapping $(y,z) \to (G_x, G)(x,y,z)$

$= G_z \det E \neq 0$ by G2

$\Rightarrow$ Y, Z smooth

② $Y_p = E^{-1}$.

③ In MAE, we obtain

$$\begin{cases} A(x,u,p) = G_{xx}[x, Y(x,u,p), Z(x,u,p)], \\ B(x,u,p) = |\det E| \, \psi(x,u,p) \end{cases}$$

④ In OT case,

$G(x,y,z) = c(x,y) - z$, $G_z = -1$, $I = I(x,y) = \mathbb{R}$.

Further notation: $\mathcal{U}' \subset \mathcal{U}$,

$\Gamma(\mathcal{U}') = \{(x,y,z) \in \mathcal{U}' \times I \mid z \in I(x,y)\}$

$\Gamma = \Gamma(\mathcal{U})$.

DUALITY

Dual generating function $H : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$, $(x, y, u) \to H(x, y, u)$

$$G(x, y, H(x, y, u)) = u$$

well defined for $(x, y) \in \mathcal{U}$, $u \in G(x, y, \cdot)(I) =: J(x, y)$

- $H_x = -\dfrac{G_x}{G_z}$, $H_y = -\dfrac{G_y}{G_z}$, $H_u = \dfrac{1}{G_z}$ $\Rightarrow$

dual condition:

$G1^*$ $\quad Q := -\dfrac{G_y}{G_z}$ is one-to-one in x, $\forall y \in \mathcal{U}^{(2)}, z \in I(x, y)$.

- Jacobian matrix of $x \to Q(x, y, z)$ is $-E/G_z$

$\Rightarrow$ Jac. det $\neq E$ by G_z.

G-transform: $\quad u \in C^0(\Omega)$,

$$u^*_G(y) = v(y) = \sup_{x \in \Omega} H(x, y, u(x))$$

If $u \in C^2(\Omega)$, $Tu : \Omega \to \Omega^*$ is one-to-one and onto, then, (for local transform)

$$v(y) = H\left[T^{-1}y, y, u(T^{-1}y)\right],$$

$$\Rightarrow \quad Dv(y) = H_y = -\frac{G_y}{G_z}\left(T^{-1}y, y, v\right)$$

$$\Rightarrow \quad T^{-1}y = X(y, v, Dv) := \varphi^{-1}(\cdot, y, v)(Dv)$$

$\Rightarrow$ if u solves PJE, $(\psi = f/g \circ Y)$,

then v solves dual PJE

$$\det\left\{ D^2 v + \left[\left(\frac{G_y}{G_z}\right)_y + \left(\frac{G_y}{G_z}\right)_z \otimes Dv \right]\right\}$$
$$= \left(-\frac{1}{G_z}\right)^n |\det E| \; g/f \circ X$$

and u degenerate elliptic (elliptic) $\Longleftrightarrow$

v degenerate elliptic (elliptic).

<u>Note</u>: dual matrix A^*,

$$A^*(x, z, q) = -\left\{ \frac{G_{yy}}{G_z} - \frac{G_y \otimes G_{zy}}{G_z^2} - \frac{G_{zy} \otimes G_y}{G_z^2} + \frac{(G_y \otimes G_y)\, G_{zz}}{G_z^3} \right\}$$

is symmetric!

REGULARITY CONDITIONS

G3, (G3w):

$$D_{p_k p_l} A_{ij}(x, Y, Z)\, \xi_i \xi_j \eta_k \eta_l \quad >, (\geq), \quad 0$$

$\forall \quad (x, Y) \in \mathcal{U}, \quad Z \in I(x, Y), \quad \xi, \eta \in \mathbb{R}^n \Rightarrow \xi \cdot \eta = 0.$

Monotonicity :

G4, (G4w)

$$D_{z} A_{ij}\, \xi_i \xi_j \quad >, \; (\geq), \quad 0$$

$\forall \quad (x, Y) \in \mathcal{U}, \quad Z \in I(x, Y), \quad \xi \in \mathbb{R}^n.$

- Extend conditions A3, A3w in optimal transportation case

PARALLEL BEAM

$$\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^n \,, \quad I = (0,\infty)\,,$$

$$G(x,y,z) = \frac{1}{2}\left[\frac{1}{z} - z\,|x-y|^2\right]$$

satisfies $G1$, $G2$, $G1^\alpha$, $G3$, $G4$ with

$$G_z = -\frac{1}{2}\left(\frac{1}{z^2} + |x-y|^2\right), \quad G_x = z(x-y), \quad G_y = z(y-x),$$

$$E = \frac{z}{1+z^2|x-y|^2}\left\{(1+z^2|x-y|^2)\,I - z^2(x-y)\otimes(x-y)\right\}$$

$$\det E = \frac{z^n\,(1-z^2|x-y|^2)}{1+z^2|x-y|^2} > 0 \quad \text{for } z \in I(x,y),$$

$$Y = x + \frac{2u\,Du}{(1-|Du|^2)}\,, \quad Z = \frac{1-|Du|^2}{2u}\,,$$

$$A = -Z\,I\,, \quad I(x,y) = \left(0, \frac{1}{|x-y|}\right), \quad J(x,y) = G(x,y,\cdot)\,I(x,y) = (0,$$

$$H = \frac{1}{u + \sqrt{u^2 + |x-y|^2}}$$

MAE: $$\det\left\{D^2u + \frac{(1-|Du|^2)}{2u}\,I\right\} = \frac{(1-|Du|^2)^{n+1}}{(1+|Du|^2)(2u)^n}\,\psi$$

– well defined for $|Du|^2 < 1$, $u > 0$.

CONVEXITY NOTIONS

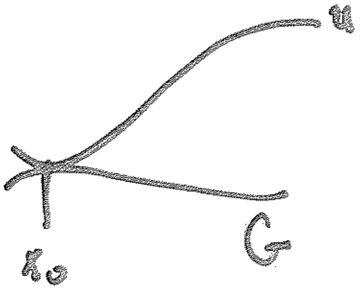
$\Omega \times \Lambda \subset \mathcal{U}$, $\Lambda \subset \mathbb{R}^n$

G generating fn. on $\mathcal{U} \times I$ satisfying G1 and G2.

$u \in C^0(\Omega)$ is <u>G-convex</u> in Ω $\Leftrightarrow$

for each $x_0 \in \Omega$, $\exists\, y_0 \in \Lambda$, $z_0 \in I(x_0, y_0)$ $\ni$

$$\begin{cases} u(x_0) = G(x_0, y_0, z_0), & \text{ie } z_0 = H(x_0, y_0, u(x_0)) \\ u(x) \geq G(x, y_0, z_0), & \forall\, x \in \Omega \end{cases}$$



u diff. at $x_0 \Rightarrow y_0 = Tu(x_0)$

u twice diff at $x_0 \Rightarrow$

$D^2 u(x_0) \geq G_{xx}(x_0, y_0, z_0)$,

ie u deg. ell. for MAE.

fns. of form $x \mapsto G(x, y_0, z_0)$ – <u>G-affine</u>,

$<$ <u>G-support</u> at x_0 if satisfies above condition.

$u \in C^2(\Omega)$ is <u>locally G-convex</u> if $D^2 u(x_0) \geq G_{xx}(x_0, y_0, z_0)$,

$y_0 = Tu(x_0)$, $z_0 = H(x_0, y_0, u(x_0))$ $\forall\, x_0 \in \Omega$.

DOMAIN CONVEXITY

Ω is <u>G-convex</u> with respect to $(y_0, z_0) \in \Lambda \times I$

if $(Q_0 =) \, Q(\cdot, y_0, z_0)(\Omega)$ is convex in $\mathbb{R}^n$

Ω^* $(\subset \Lambda)$ is <u>G^*-convex</u> with respect to (x_0, u_0)

$x_0 \in \Omega$, $u_0 \in J(x_0, y) \ \forall \ y \in \Omega^*$ if

$(P_0 =) \, G_x[x_0, \cdot, H(x_0, \cdot, u_0)]$ is convex in $\mathbb{R}^n$.

Note G^*-convexity corresponds to previous notion of

<u>Y^*-convexity</u> $\iff$ $\{p \in \mathbb{R}^n \mid \underset{\in \Omega^*}{Y(x_0, u_0, p)}\}$ convex.

For $u : \Omega \to \mathbb{R}$, we also define Ω^* to be

G^* (or Y^*)-convex w.r.t. u if Ω^* is

G^*-convex w.r.t. $\{(x, u(x)) \mid x \in \Omega\}$.

<u>Note</u>: When we apply G3w to convexity results

need to assume Γ is suff'tly large, eg $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^n$

is enough or more generally convex hulls of $Q(\Gamma(\Omega \times \Lambda))$,

$G_x[\Gamma(\Omega \times \Lambda)] \subset Q(\Gamma), \ G_x(\Gamma)$ resp. – enough for convex hulls of $Q_0, P_0 \subset Q, G_x(\Gamma)$ resp.

NORMAL MAPPINGS AND SECTIONS

$u \in C^0(\Omega)$, G-convex

G-normal mapping is multi-valued extension of T:

$$Tu(x_0) = \{y_0 \in \Omega \mid u(x) \geq G[x, y_0, z_0] \ \forall x \in \Omega\}$$

where $z_0 = H(x_0, y_0, u(x_0))$.

Note: $Tu(x_0) \subset Y(x_0, u(x_0), \partial u(x_0))$, ∂u subgradient.

for $y_0 \in Tu(x_0)$, $\sigma > 0$, G-section $S_\sigma = S_\sigma(x_0, y_0)$:

$$S_\sigma = \{x \in \Omega \mid u(x) < G(x, y_0, z_0 - \sigma)\}, \quad (\text{for } G_z < 0)$$

G-contact set S_0,

$$S_0 = \{x \in \Omega \mid u(x) = G(x, y_0, z_0)\}$$

CONVEXITY RESULTS

- Assume G1, G2, $G1^*$, G3w, G4w,
  $u \in C^2(\Omega)$ locally G-convex in Ω,
  Ω G-convex w.r.t. $Y, Z(\cdot, u, Du)(\Omega)$.
  Then u is G-convex in Ω.

- Assume G1, G2, $G1^*$, G3w, G4w,
  $u \in C^0(\Omega)$ G-convex in Ω,
  Then $Tu = Y(\cdot, u, \partial u)$ in Ω

- Assume G1, G2, $G1^*$, G3w, G4w,
  $u \in C^0(\Omega)$ G-convex in Ω, $x_0 \in \Omega$
  Ω G-convex w.r.t. $y_0 \in Tu(x_0)$, $z_0 = H(x_0, y_0, u(x_0))$.
  Then S_ϵ, S_0 are G-convex w.r.t. (y_0, z_0).

- Fundamental building blocks for regularity theory.
- Essentially sharp.

INDICATION OF PROOFS

Basic formulae.

$$q = Q(x, y_0, z_0), \qquad \text{coord. trans.}$$

$$E^{-1} = [E^{i,j}]$$

$$\boxed{D_{q_i q_j} = G_z^2 E^{i,r} E^{j,s} \{ D_{x_r x_s} - (D_{p_\ell} G_{x_r x_s}) D_{x_\ell} \} + E^{j,s} G_{x_s z} D_{q_i}}$$

$\Rightarrow$ For $h(x) = u(x) - G(x, y_0, z_0)$, $\quad u$ loc. G-convex

$$\boxed{D^2_{q_\gamma} h \geq -K |D_{q_\gamma} h|}$$ under G3w, G4w, $(\gamma \in \mathbb{R}^n, |\gamma| = 1)$.

$\Rightarrow$ Convexity Results.

$\Rightarrow$ G convexity ~ special case of Y-convexity

GENERALIZED SOLUTIONS

<u>Densities</u> $f, g \geq 0$, $\in L^1(\Omega), L^1(\Omega^*)$ resp, $\Omega \times \Omega^* \subset U$

<u>Mass balance condition</u> $\int f = \int g$

<u>Definition</u>. $u \in C^0(\Omega)$, G_convex is generalized solution of PJE with BC. $Tu(\Omega) = \Omega^*$ $\iff$

$$\int_{T^{-1}(\omega)} f = \int_\omega g \qquad \forall \text{ Borel } \omega \subset \Omega^*$$

<u>Remarks</u>: • $G1 \Rightarrow T^{-1}(\omega)$ is Borel meas. so well-defined.

• Extends Brenier solution in optimal transportation, Kochengin - Oliker weak solution in near field geometric optics

• Can take g = measure

• $G1, G1^* \Rightarrow$ equivalent to <u>Aleksandrov</u>

notion: $\displaystyle\int_\omega f = \int_{\Omega^* \cap T(\omega)} g \qquad \forall \omega \subset \Omega.$

EXISTENCE

Need further condition on G:

$G5$: $\exists\ m_0 \in [-\infty, \infty)$, $K_0 > 0 \ni (m_0, \infty) \subset J(x,y)$,

and $|G_x(x, y, z)| \leq K_0$

$\forall\ x \in \Omega,\ y \in \Omega^*$, $G(x, y, z) > m_0$

<u>Parallel</u> <u>beam</u>: $m_0 = 0$, $K_0 = 1$.

<u>Theorem</u>: For $x_0 \in \Omega$, u_0 sufficiently large, there exists generalized solution $u \ni u(x_0) = u_0$.

<u>Proof</u>. Uses approximation by piecewise G-affine functions with $g = \Sigma g_i \delta_{y_i}$, (Aleksandrov, Kochengin - Oliker).

INTERIOR REGULARITY

Assume : • densities , $f, g > 0$, smooth

• G satisfies G1, G2, $G1^*$, G3, G4w

• $u \in C^0(\Omega)$ generalized solution

• Ω^* is G^*-convex w.r.t. u

$\Rightarrow$ u smooth .

In particular $f, g \in C^2(\Omega) \Rightarrow u \in C^3(\Omega)$

$f, g \in L^\infty(\Omega) \Rightarrow u \in C^1(\Omega)$

Extends optimal transportation case, Ma-T.-Wang 2005 ;
Loeper 2009 .

Duality : • Ω is G-convex w.r.t. v

$\Rightarrow$ u diffeomorphism , (though!) ($G3 \sim G^*3$)

Question : G3 $\to$ G3w ? – Yes (at least in OT case
(sharp, Loeper) 2009 . {Figalli, Kim, McCann
Villani 2011)

Note : Does not include Karakhanyan - Wang (2010) need G4 with bound from below,