WEAK SOLUTIONS OF FULLY NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

NEIL S. TRUDINGER

1. The PDEs

A general quasilinear second order PDE may be written in the form

$$\text{trace}\ a(\bullet, u, Du)D^2u = b(\bullet, u, Du) \tag{1.1}$$

where a and b are respectively $n\times n$ matrix-valued and scalar functions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, where Ω is a domain in $\mathbb{R}^n$ and Du, D^2u denote respectively the gradient and Hessian of the scalar function u, which is a classical solution if $u \in C^2(\Omega)$. The equation (**??**) is in divergence form if it is equivalent to an equation of the form

$$\text{div}\,\mathcal{A}(\bullet, u, Du) = \text{trace}\ D\mathcal{A}(\bullet, u, Du) = \mathcal{B}(\bullet, u, Du), \tag{1.2}$$

where $\mathcal{A}$ is an n-vector valued function and $\mathcal{B}$ a scalar function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$. We will use the notation (x, z, p) for points in $\Omega\times\mathbb{R}\times\mathbb{R}^n$. Here

$$D\mathcal{A} = [D_i\mathcal{A}^j] = [(D_{p_k}\mathcal{A}^j)D_{ik}u + D_uA^jD_iu + D_{x_i}\mathcal{A}^j] \tag{1.3}$$

denotes the Jacobian matrix of the vector field $\mathcal{A}$. The equation (**??**) is elliptic if the principal coefficient matrix $D_p\mathcal{A}$ is positive.

Fully nonlinear analogues of the equation (**??**) are generated by linear combinations of the minors of the matrix (**??**). The standard examples are given by

$$F_k[u] = F_k(\bullet, u, Du, D^2u) := S_k(D\mathcal{A}), \tag{1.4}$$

where for an $n\times n$ matrix a, $S_k(a)$ denotes the sum of its $k\times k$ principal minors, $k = 1, \ldots, n$. These operators enjoy a divergence form,

$$F_k[u] = \frac{1}{k}\,\text{div}\left(\frac{\partial S_k}{\partial a}(D\mathcal{A})Du\right) = \frac{1}{k}D_i(S_k^{ij}D_ju), \tag{1.5}$$

where $S_k^{ij} = \frac{\partial S_k}{\partial a_{ij}}$, by virtue of their k-homogeneity and the property, (see [**?**]),

$$D_i(S_k^{ij}(D\mathcal{A})) = 0, i = 1, \ldots, n. \tag{1.6}$$

Date: Lectures, IMS, Singapore, May-June, 2004.

Note that the divergence operator in (??) is applied to a second order vector field. In the special case $\mathcal{A} = Du$, $F_k[u]$ is the k^{th} elementary symmetric function of the eigenvalue of the Hessian matrix D^2u. When $\mathcal{A} = Du/\sqrt{1+|Du|^2}$, $F_k[u]$ is the k^{th} order mean curvature of the graph of u. The corresponding equations in these cases furnish the basic examples of Hessian and curvature equations.

In these lectures we will consider the two cases $\det D_p\mathcal{A} \neq 0$ and $D_p\mathcal{A} > 0$. The general case, $\det D_p\mathcal{A} \neq 0$ will only be treated for $k = n$, that is for equations of Monge-Ampère type,

$$\det[D\mathcal{A}(\bullet, u, Du)] = \mathcal{B}(\bullet, u, Du). \tag{1.7}$$

By multiplying through by the inverse matrix we have

$$\det[D_{ij}u + \mathcal{A}^j_u D_i u + \mathcal{A}^j_{x_i}] = \mathcal{B}/\det D_p\mathcal{A}, \tag{1.8}$$

that is an equation of the form

$$\det[D^2u - A(\bullet, u, Du)] = B(\bullet, u, Du), \tag{1.9}$$

which will be *elliptic* (degenerate elliptic) with respect to $u \in C^2(\Omega)$ if $D^2u > A$, $(D^2u \geq A)$. Note that the class of equations (??) is invariant under *coordinate changes* unlike the standard Monge-Ampère equation, where $A = 0$.

In the first part of these lectures we will consider weak solutions of Monge-Ampère equations of the general form (??), when the matrix A is generated by a cost function in an optimal transportation problem. In the remaining lectures, we consider operators of the type F_k, $k \leq n$, under the positivity condition $D\mathcal{A} > 0$, together with an extension to subelliptic operators.

2. c-normal mappings and c-convexity

Let $c : \Omega_0 \times \Omega_0^* \longrightarrow \mathbb{R}$, where Ω_0 and Ω_0^* are domains in $\mathbb{R}^n$. Typically we will take $\Omega_0 = \Omega_0^* = \mathbb{R}^n$ and assume this unless otherwise indicated. In accordance with the relation with optimal transportation we will refer to c as a *cost function*. We also assume that c is at least continuous with further smoothness assumed as required. The standard example, which gives the usual theory of normal mappings and convexity is the inner product,

$$c(x, y) = x \cdot y. \tag{2.1}$$

The *c-normal mapping* of a function $u \in C^0(\Omega)$, $(\Omega \subset \Omega_0^*)$ is defined as follows. For any $x_0 \in \Omega$, we set

$$\begin{aligned} \chi(x_0) &= \chi_u[u](x_0) \\ &= \{y \in \Omega_0^* \mid u(x) \geq u(x_0) + c(x,y) - c(x_0,y) \\ &\qquad \text{for all } x \in \Omega\} \end{aligned} \tag{2.2}$$

Note that $\chi(x_0)$ may be empty or contain more than one point. Following the usual case (**??**), we define the (lower) *contact set* Γ^- by

$$\Gamma^- = \Gamma_c^-[u] = \{x_0 \in \Omega \mid \chi_c[u](x_0) \neq \emptyset\}, \tag{2.3}$$

and call the function u *c-convex* if $\Gamma_c^-[u] = \Omega$. When u and c are differentiable at x_0, we clearly have

$$Du(x_0) = D_x c(x_0, \chi(x_0)), \tag{2.4}$$

and moreover, if u and c are twice differentiable at x_0, then

$$D^2 u(x_0) \geq D_x^2 c(x_0, \chi(x_0)). \tag{2.5}$$

It is readily shown that if c is semi-convex in x, with

$$D_x^2 c(x,y) \geq -c_0 I \tag{2.6}$$

for all $x \in \Omega$, $y \in \chi_c[u](\Omega)$ and u is c-convex, then u is also semi-convex, with $D^2 u \geq -c_0 I$, (in the sense of distributions). Consequently c-convex functions will be twice differentiable almost everywhere, by virtue of the fundamental corresponding result of Aleksandrov for convex functions. If c is (locally) uniformly Lipschitz continuous in x, uniformly with respect to y in $\Omega \times \chi_c[u](\Omega)$, then so also is u.

3. Aleksandrov's approach

The beautiful ideas of Aleksandrov, underlying the theory of normal mapping and Monge-Ampère measure, extend automatically to the general setting. We first employ his moving plane method to prove a height estimate and a basic monotonicity result for c convexity.

We begin with a function $u \in C^0(\overline{\Omega})$ and suppose there exists a point $y_0 \in \Omega_0^* - \chi_c[u](\Omega)$. Then

$$u(x) \geq c(x, y_0) - c_0 \tag{3.1}$$

for some constant c_0, (if $\Omega \subset \Omega_0$). By vertical translation upwards of the graph of $c(\bullet, y_0)$, we then find that there must exist a point $x_0 \in \partial\Omega$

such that

$$u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0) \tag{3.2}$$
$$\geq \inf_{\partial\Omega} u - \operatorname*{osc}_{\Omega} c(\bullet, y_0)$$
$$\geq \inf_{\partial\Omega} u - \inf_{y \in \Omega_0^* - \chi_c[u](\Omega)} \operatorname*{osc}_{\Omega} c(\bullet, y),$$

that is an *a priori* estimate from below for u!

Next let us suppose we have two functions $u, v \in C^0(\Omega)$ with $v \leq u$. Then if $y_0 \in \chi[u](x_0)$ for some $x_0 \in \Omega$, we vertically translate downwards the supporting graph, $x_{n+1} = u(x_0) + c(x, y_0) - c(x_0, y_0)$, to get a corresponding supporting graph to the graph of v, whence $y_0 \in \chi[v](\Omega)$, and consequently

$$\chi_c[u](\Omega) \subset \chi_c[v](\Omega). \tag{3.3}$$

4. Generalized Monge-Ampère measure

From now on let us assume that the cost function c satisfies the same conditions with respect to y as to x, so that for example we replace (**??**) by

$$D_x^2 c, D_y^2 c \geq -c_0 I \tag{4.1}$$

on $\Omega_0 \times \Omega_0^*$. Following still the development of the Aleksandrov theory, we introduce for a function $u \in C^0(\Omega)$ and density $g \geq 0, \in L^1_{\text{loc}}(\Omega_0^*)$, the generalized Monge-Ampère measure $\mu_c[u, g]$ by defining for any Borel set $e \subset \Omega$,

$$\mu(e) = \mu_c[u, g](e) = \int_{\chi(e)} g, \tag{4.2}$$

where as above $\chi(c) = \chi_c[u](e)$. To show that μ is well-defined, we must as in the standard case (**??**), prove that $\chi(e)$ is Borel measurable. As there, this issue reduces to showing that the image under χ of the set where χ^{-1} is multi-valued has Lebesgue measure zero, that is if

$$Y = \{y \in \Omega_0^* \mid \exists x_1 \neq x_2 \in \Omega \text{ with } y \in \chi(x_1) \cap \chi(x_2)\}, \tag{4.3}$$

then $|Y| = 0$. For this purpose we may introduce the c-transform, as a generalization of the Legendre transform in the case (**??**), by defining for $y \in \Omega_0^*$,

$$u^*(y) = \sup_{x \in \Omega} \{c(x, y) - u(x)\}. \tag{4.4}$$

The function u^* will clearly be c^*-convex,where $c^*(y, x) = c(x, y)$, and by virtue of (**??**), will then be semi-convex and twice differentiable a.e. At a point y_0 of differentiability we must have

$$Du^*(y_0) = c_y(x, y_0) \tag{4.5}$$

for any $x \in \Omega$ with $y_0 \in \chi_c[u](x)$. Assuming now that the mapping $c_y(\bullet, y) : \Omega \longrightarrow \mathbb{R}^n$ is one-to-one for each $y \in \Omega_0^*$, guarantees $|Y| = 0$ as desired.

With the aid of the generalized Monge-Ampère measure, we can develop further the results in the previous section. For example, if $g \equiv 1$, $\mu_c[u] := \mu_c[u, g]$, and $\mu_c[u](\Omega) < |\Omega_0^*|$, we can control the estimate (**??**) in terms of $\mu_c[u](\Omega)$ and Ω_0^*. In particular if $\Omega_0^* = \mathbb{R}^n$, we have

$$u(x) \geq \inf_{\partial\Omega} u - \sup_{|y|\leq r} \operatorname*{osc}_{\Omega} c(\bullet, y), \tag{4.6}$$

where $r = (\omega^{-n}\mu(\Omega))^{\frac{1}{n}}$. For general g, we require $\mu_c[u](\Omega) < \int_{\Omega^*} g$.

From (**??**), we infer the *monotonicity* property of the generalized Monge-Ampère measure, namely,

$$\mu_c[u, g](\Omega) \leq \mu_c[v, g](\Omega), \tag{4.7}$$

for any $u \geq v \in C^0(\overline{\Omega})$, $g \geq 0 \in L^1_{\text{loc}}(\Omega^*)$, $u = v$ on $\partial\Omega$.

5. Relation to the Monge-Ampère equation

We will say that the vector field $\mathcal{A} = \mathcal{A}(x, Du)$ is generated by a cost function c (satisfying the above properties) if

$$D_x c(x, \mathcal{A}(x, p)) = p, \tag{5.1}$$

for all $x \in \Omega$, $p \in D_x c(x, \Omega_0^*)$. Differentiating (**??**), we see that $D_{x,y}c$ is invertible with

$$\begin{aligned} (D_{x,y}c)^{-1} &= D_p\mathcal{A}, \\ D_{x,x}c &= -D_{x,y}c \; D_x\mathcal{A} \end{aligned} \tag{5.2}$$

Accordingly, with $B = f(x)\,\text{sign}(\det D_{x,y}c)/g(\mathcal{A}(x, Du))$, we may write equations (**??**), (**??**) as

$$\det[D^2u - D_x^2 c(x, \mathcal{A}(x, Du))] = |\det D_{x,y}c| f(x)/g(\mathcal{A}). \tag{5.3}$$

A c-convex function u is then called a *generalized solution* of (**??**) in Ω (in the sense of Aleksandrov-Bakelḿan) if

$$\mu_c[u, g] = f(x)dx. \tag{5.4}$$

Note that when $c(x, y) = x \cdot y$, $\mathcal{A} = Du$ and (**??**) reduces to the standard Monge-Ampère equation

$$\det D^2u = f(x)/g(Du). \tag{5.5}$$

Returning to our general form (**??**), we will say that it is generated by a cost function if there exists a cost function $c : \Omega_0 \times \Omega_0^* \longrightarrow \mathbb{R}$ satisfying

(5.6 i) $D_x c$, $D_y c$ are one-to-one for each fixed $x \in \Omega_0, y \in \Omega_0^*$ respectively;

(5.6 ii) $\det D_{x,y} c \neq 0$;

(5.6 iii) $A(x,p) = D_{x,x} c(x, [D_x c(x, \bullet)]^{-1} p$.

Note that unless $D_x c(x, \Omega_0^*) = \mathbb{R}^n$ for all $x \in \Omega$, (**??**) is not necessarily defined for all values of Du.

6. Weak continuity

7. Comparison principle

8. General operators F_k

9. Hessian measures

10. Extension to subelliptic operators