WEAK CONTINUITY

OF

NONLINEAR OPERATORS

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General Subharmonic Functions.

$$F \in C^0(\Gamma), \quad \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$$

$$\mathcal{F}[u] := F(\cdot, u, Du, D^2u), \quad u \in C^2(\Omega).$$

Definition.

$u : \Omega \to [-\infty, \infty)$ upper semi-continuous, $|u = -\infty| = 0$, is subharmonic w.r.t. $\mathcal{F}$ ($\mathcal{F}$-subharmonic) $\Leftrightarrow$ $\forall\, \omega \subset\subset \Omega$, $v \in C^2(\omega) \cap C^0(\bar{\omega})$ $\ni \mathcal{F}[v] \leq 0$ in ω, $u \leq v$ on $\partial\omega$, we have $u \leq v$ in ω.

Functions $u \in C^2(\Omega)$, $\mathcal{F}[u] \geq 0$ are subharmonic under appropriate ellipticity and monotonicity conditions.

Proposition. $u \in C^2(\Omega)$ is $\mathcal{F}$-subharmonic if

$$F(\cdot, u - c, Du, D^2u + \eta) > 0$$

$\forall$ const. $c \geq 0$, $\eta \in \mathbb{S}^n$, ≥ 0 with either $c > 0$ or $\eta > 0$.

Viscosity formulation.

u $\mathcal{F}$-subharmonic $\Rightarrow$ $\mathcal{F}[u] \geq 0$ in viscosity sense,

i.e. $\forall\ y \in \Omega$, $v \in C^2(\Omega)$ $\ni$ $u \leq v$ in Ω, $u(y) = v(y)$,

we have $\mathcal{F}[v](y) \geq 0$.

Conversely if $\mathcal{F}[u] \geq 0$ in viscosity sense,

F non-increasing in u and

$$F(\cdot, v, Dv, D^2v + \eta) \geq 0 \quad \forall\ \eta \geq 0, \left[v \in C^2(\Omega)\right]$$

$$\Rightarrow \quad F(\cdot, v, Dv, D^2v + \eta) > 0 \quad \text{for} \quad \eta > 0,$$

then u is $\mathcal{F}$-subharmonic.

In particular if $\mathcal{F}$ is elliptic and $F_z \leq 0$,

then $\mathcal{F}$-subharmonic $\Leftrightarrow$ $\mathcal{F}[u] \geq 0$ in viscosity sense.

Non-smooth F.

If F is not continuous in x-variables but at least measurable, we replace the space $C^2(\Omega)$ by a linear space V on which $\mathcal{F}$ is well defined in the distribution sense.

For quasilinear divergence operators of form,

$$\mathcal{F}[u] = \operatorname{div} A(\cdot, u, Du) + B(\cdot, u, Du),$$

V is typically a space of weakly differentiable functions.

In particular, if $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, satisfy standard structural conditions:

$$|A(x, z, p)| \leq a_0 |p|^{m-1} + a_1,$$

$$p \cdot A(x, z, p) \geq |p|^m - a_2$$

$$|B(x, z, p)| \leq b_0 |p|^{m-1} + b_1,$$

for constants $a_0, a_1, a_2, b_0, b_1 \geq 0$, $m > 1$, we take

$$V = W^{1,m}_{loc}(\Omega).$$

Programme (T. Wang)

- Let SH_q denote the set of $\mathcal{G}$-subharmonic functions, with respect to an operator $\mathcal{G}$, (and appropriate function space V).

- Extend $\mathcal{G}$ to $SH_{\mathcal{G}}$ as a measure $\mu_{\mathcal{G}} = \mu$ such that μ is weakly continuous with respect to a.e. convergence of functions in $SH_{\mathcal{G}}$.

- Solve appropriate boundary value problems, for PDE,

$$\mu_q[u] = \nu$$

where ν is a given measure on Ω.

- Develop potential theory for $\mathcal{G}$.

Special cases.

1. $\mathcal{F}[u] = \Delta u$, Laplacian

$SH_{\mathcal{F}} = \{\text{subharmonic functions}\} \subset L^1_{loc}(\Omega)$

$\mu_{\mathcal{F}} = \Delta u$ as non-negative distribution.

2. $\mathcal{F}[u] = \det D^2 u$, Monge-Ampère operator.

$SH_{\mathcal{F}} = \{\text{convex functions}\}$

$\mu = \mu_{\mathcal{F}}[u] =$ Monge-Ampère measure

ie. $\mu(E) = |\partial u(E)|$, ∂ subgradient

Aleksandrov: weak continuity,

$u_m \to u$ loc. unif $\Rightarrow \mu[u_m] \to \mu[u]$ weakly.

3 (<u>Hessian</u> <u>measures</u>)

$$\mathcal{G}_k[u] = \left[D^2u\right]_k, \quad k=1,\dots n, \quad (= S_k(D^2u))$$

= sum of $k\times k$ principal minors of D^2u

$$\Phi_k(\Omega) = SH\mathcal{G}_k = \{k\text{-convex functions}\} \subset L^1_{loc}(\Omega).$$

$$\Phi_k(\Omega)\cap C^2(\Omega) = \{u\in C^2(\Omega),\ \mathcal{G}_j[u]\geq 0,\ j=1,\dots k\}$$

<u>Theorem.</u> (T. Wang 1999) $u\in\Phi_k(\Omega) \Rightarrow \exists$ measure $\mu_k[u]$ $\ni$

(i) $\mu_k[u](E) = \int_E \mathcal{G}_k[u]$, $u\in C^2(\Omega)\cap\Phi_k(\Omega)$

(ii) $u_m\to u$ a.e. $\Rightarrow$

$\mu[u_m]\to\mu[u]$ weakly

<u>Remark.</u> Mollification $u_h\in\Phi_k(\Omega')$, $\to u$ $\forall\ \Omega'\subset\subset\Omega$.

$K = 1$, Laplacian, subharmonic fns.

n even, $= 2m$

k- Hessian operators,
k- convex functions.

$K = m$, complex Monge-Ampère operator in $\mathbb{C}^m$
plurisubharmonic functions
weak continuity [illegible]
- Cegrell.

$K = n$, Monge-Ampère operator, convex functions

Proofs of weak continuity.

Hessian measures

Two approaches
- (i) Integral gradient estimates
- (ii) Perron lifting.

(i) Uses integration by parts, motivated by complex Monge-Ampère case - formally $\sim k = n/2$, extends to sub elliptic case

(ii) Reduces to solutions of PDE, $F[u] = 0$, uses uniform convergence and comparison argument, extends to mean curvature case.

Subelliptic Hessian operators.

$X_1, \dots X_m$ vector fields in $\mathbb{R}^n$, ie

$$X_i = \sum_{j=1}^{n} b^{ij} D_j \quad , \quad i=1,\dots m, \quad b^{ij} \in C^\infty ,$$

symmetric Hessian

$$X_s^2 u = \left[\tfrac{1}{2} (X_i X_j + X_j X_i) u \right]_{i,j=1,\dots m.}$$

$$g_k[u] = \left[X_s^2 u \right]_k \quad , \quad k=1,\dots m$$

$u \in C^2(\Omega)$ k-convex w.r.t. $\{X_1, \dots X_m\}$ $\Leftrightarrow$

$g_j[u] \geq 0$, $j=1,\dots k$. $\Leftrightarrow$ u subharmonic

Hypotheses.

(i) X_i is anti-selfadjoint, ie $X_i^* = -X_i$, $i=1.m$

(ii) Hormander condition : $X_1, \dots X_m$ and commutators $[X_i, X_j] = X_i X_j - X_j X_i, \dots$ span $\mathbb{R}^n$;

(iii) Step 2, (or more generally second commutators from any two X_i vanish).

Theorem. (Trud) Mappings

$$u \mapsto \mathcal{F}_2[u] + \frac{3}{4}\sum_{i<j}\left([X_i, X_j]^{\varepsilon} u\right)^2$$

are weakly continuous w.r.t. a.e convergence, for u 2-convex,

Remarks. 1. Extends to $\Phi_k(\Omega) = \{$ a.e. limits of k-convex fns $\}$, $k=2$.

2. $\Phi_k(\Omega) \subset S_{\mathcal{F}_k}(\Omega)$ (subharmonic fns)

($=$ if e.g. X_i generate step 2 Carnot group).

3. Simple example, Heisenberg group H^ℓ,

$m = 2\ell$, $n = 2\ell+1$, $X_i u = \begin{cases} D_i u - \frac{1}{2} x_{i+\ell} D_{2\ell+1} u, & i = 1,..\ell \\ D_i u + \frac{1}{2} x_{i-\ell} D_{2\ell+1} u, & i = \ell+1,..2\ell. \end{cases}$

4. Expect extends to k even, with second term replaced by

$\left[[X_i, X_j] u \right]_k$ - sum of $k \times k$ prin. minors of commutator matrix.

Mean curvature measure.

$$F[u] = H_1[u] := \operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}}$$

$$= (1+|Du|^2)^{-3/2}\left\{(1+|Du|^2)\Delta u - D_i u D_j u D_{ij} u\right\}$$

Theorem (Dai - T - Wang, 2006).

For any H_1-subharmonic function u, in $\Omega \subset \mathbb{R}^n$, $\exists$ a sequence $\{u_m\} \subset C^2(\Omega) \cap SH_{H_1} \ni u_m \to u$ a.e. and a measure $\mu[u]$ such that

(i) $\mu[u] = H_1[u]\,dx$ if $u \in C^2(\Omega)$

and (ii) if $\{u_m\}$ is any sequence of H_1-subharmonic functions $\to u$ a.e., then

$$\mu[u_m] \to \mu[u] \quad \text{weakly}.$$

Interior gradient bound.

Theorem. (Bombieri, de Giorgi, Miranda 1969, $H_1 \equiv 0$, T 1973)

for $u \in C^2(\Omega)$, $y \in \Omega$, $u_0 = u(y)$, $d = \mathrm{dist}(y, \partial\Omega)$,

$$|Du(y)| \leq c_1 \exp\left\{ c_2 \sup_{\Omega} \left(\frac{u - u_0}{d} \right) \right\},$$

$c_1, c_2 \geq 0$ constants depending on n, $d \sup H_1[u]$, $d^2 \sup DH_1[u]$.

Proof. From differential inequality:

$$w = \log v, \qquad v = \sqrt{1 + |Du|^2}$$

$$\Delta_M w \geq |\delta_M w|^2 - D_\nu H_1$$

where

$$\begin{cases} \delta w = Dw - \nu(\nu \cdot Dw), & \text{tangential gradient} \\ \Delta_M w = \delta_i \delta_i w, & \text{Laplacian in} \\ & M = \text{graph } u \text{ in } \mathbb{R}^{n+1} \end{cases}$$

$$\nu = \left(\frac{Du}{\sqrt{1+|Du|^2}}, \frac{-1}{\sqrt{1+|Du|^2}} \right) \text{ normal to } M$$

In $\mathbb{R}^q$, $\Delta_M w = (\delta_{ij} - \nu_i \nu_j) D_{ij} w - H \nu . Dw$.

Harnack inequality.

Minimal surface equation.

$$H_1[u] = 0$$

Theorem (Dai, T, Wang 2006).

$u \in C^2(\Omega)$, ≥ 0, $H_1[u] = 0$ in Ω, $B_R(y) \subset B_{2R}(y) \subset \Omega$,

$$\sup_{B_R} u \leq C \inf_{B_R} u ,$$

C depends on n, R and $\psi(t) = |u > t|$.

Remarks. From Bombieri - de Giorgi - Miranda gradient estimate & Moser Harnack inequality, result follows with C depending on $\sup u$. Note that in C, ψ can be replaced by any $\psi \downarrow 0$ as $t \to \infty$ $\ni$ $\psi(t) \geq |u > t|$. True for $u \in W^{1,1}_{loc}(\Omega)$

Local bounds

Critical lemma : $u \in V = W^{1,1}_{loc}(\Omega)$ satisfies

$$Q[u] := \operatorname{div} A(\cdot, u, Du) + B(\cdot, u, Du) \geq 0$$

in Ω, where Q satisfies $(m=1)$ structure :

$$|A| \leq a_0 , \quad p \cdot A \geq |p| - a_1 \qquad B \leq b_0$$

$\forall \; x, z, p \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\quad a_0, a_1, b_0 \geq 0$ constant.

Then for $B_{2R}(y) \subset \Omega$, $B_R = B_R(y)$,

$$\sup_{B_R} u \leq C$$

where C depends on n, a_0, a_1, b_0, ψ and R.

Remarks. Can allow $a_0, a_1 \in L^q$, $q > n$, $b_0 \in L^n$ as well as dependence on z.

Dependence on R made precise through mapping $x \to \frac{x}{R}$, $u \to \frac{u}{R}$ and taking $R = 1$.

Proof (of critical lemma). Take $R = 1$.

From ψ we construct $\phi \in C^{0,1}[0,\infty)$, $\phi(0) \geq 1$, $\phi(\infty) = \infty$, $0 \leq \phi' \leq 1$ such that

$$\int_{B_2} \phi(u^+) < \infty$$

Test functions $v = \eta [\phi(u^+)]^\beta$ in weak form

$$\int A.Dv \leq \int Bv \quad , v \geq 0, \in W_0^{1,1}(B_2) \Rightarrow$$

$$\int \eta |D\phi^\beta| \leq (a_0 \sup|D\eta| + a_1\beta + b_0) \int_{\mathrm{supp}\,\eta} \phi^\beta ,$$

for $0 \leq \eta \leq 1$, $\eta \in C_0^1(\Omega)$, $\beta \geq 1$.

Result follows using Moser iteration.

Sobolev inequality $\Rightarrow$

$$\| \eta \phi^\beta \|_{\frac{n}{n-1}} \leq C(\beta + \sup |D\eta|) \| \phi^\beta \|_{1, \mathrm{supp}\,\eta} .$$

Fixing $\eta = 1$ on $\tilde{B}_{\nu+1}$, $\eta = 0$ outside $\tilde{B}_\nu$, $|D\eta| \leq 2^{\nu+2}$, $\tilde{B}_\nu = B_{1+2^{-\nu}}$

and taking β^{th} root $\Rightarrow$

$$\|\phi\|_{\beta\chi, \tilde{B}_{\nu+1}} \le \left[C\,(\beta + \sup|D\chi|)\right]^{1/\beta} \|\phi\|_{\beta, \tilde{B}_\nu}, \quad \chi = \frac{n}{n-1}$$

so that if $\beta = \chi^\nu$,

$$\|\phi\|_{\chi^{\nu+1}, \tilde{B}_{\nu+1}} \le C^{\nu\chi^{-\nu}} \|\phi\|_{\chi^\nu, \tilde{B}_\nu}$$

$\Rightarrow$ as $\nu \to \infty$,

$$\sup_{B_1} \phi \le C \int_{B_2} \phi .$$

Non-uniformly elliptic PDE

$$Q[u] := \operatorname{div} A(Du) = 0$$

$$|A| \leq a_0 \left(\frac{|p|}{1+|p|}\right)^{m-1}$$

$$p \cdot A \geq \frac{|p|^m}{(1+|p|)^{m-1}}$$

$$\frac{\Lambda}{\lambda}(a) \leq c_0 (1+|p|)^r ,$$

$a_0, c_0, r > 0$, $m > 1$ const.

λ, Λ = min, max eigenvalues of

a = symm. part of $D_p A > 0$.

No interior gradient bound!

Examples.

$$\int (1+|Du|^m)^{1/m} , \quad m > 1 , \quad r = m .$$

$m = 2$: mean curvature operator.

Harnack inequality : $u \in C^2(\Omega)$, ≥ 0

$$\sup_{B_R} u \leq C \inf_{B_R} u , \quad C = C(n, m, r, a_0, c_0, R, \psi).$$

Remark. C depending on $\sup u$ proved in T 1981,

$\Rightarrow$ result by critical lemma.

Proof (T 1981).

$$w = \log u/k \quad , \quad k = \text{const.}$$

Two parts:

(i) Estimate $\sup|w|$ in terms of

$\int |w|^p$ for large p.

Uses localized Aleksandrov maximum principle.

(ii) Test functions.

$$v = \eta^m u^{1-m} \left(|w|^\beta + \gamma \right), \quad \beta, \gamma \geq 0.$$

$$k = \exp\left(\frac{1}{B_\sigma} \int_{B_\sigma} \log u \right), \quad 1 < \sigma < 2.$$

Perron lifting

Mean curvature operator H_1

Assume $u \in SH_1(\Omega)$, $\omega \subset\subset \Omega$

Definition

u^ω is upper semicts regularization of

$$\tilde{u} = \sup \{ v \in SH_1(\Omega) \mid v \le u \text{ in } \Omega - \omega \}$$

Properties.

(i) $u^\omega \in SH_1(\Omega)$ and

$H_1[u^\omega] = 0$ smoothly in ω.

(ii) $u_j \to u$ a.e. Ω, $u_j, u \in SH_1(\Omega)$,

$\Rightarrow$ for almost all $r \in (0, r_0)$, $B_{r_0} \subset \Omega$,

$u_j^{B_r} \to u^{B_r}$ a.e. as $r \to r_0$.

Proofs use classical solvability of Dirichlet problem in balls, Harnack inequality.