Fully Nonlinear PDEs in Geometry

CBMS Lectures

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Lecture 1

Introduction

The term "fully nonlinear" describes partial differential equations which are *nonlinear* in the highest order derivatives of a prospective solution. The oldest and most famous example is the Monge-Ampère equation in two variables, $(x, y) \in \mathbb{R}^2$, which has the form,

$$u_{xx}\, u_{yy} - (u_{xy})^2 = \psi \tag{1.1}$$

where ψ is a given function in $\mathbb{R}^2$. Here and throughout all functions, unless otherwise indicated, are real valued and subscripts on functions denote partial derivatives. Letting

$$Du = (u_i), \quad D^2u = [u_{ij}],\ i, j = 1, \dots, n, \tag{1.2}$$

denote respectively the gradient vector and Hessian matrix of second derivatives of a function u of n variables, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we may write the n-dimensional Monge-Ampère equation in the form

$$\det D^2u = \psi \tag{1.3}$$

where ψ is a given function in $\mathbb{R}^n$.

We will mostly be concerned with *second order* partial differential equations on domains $\Omega \subset \mathbb{R}^n$, such as the Monge-Ampère equation. Letting $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the linear space of $n \times n$ symmetric real matrices, a general second order partial differential operator F may be written as

$$F[u] = F(\cdot, u, Du, D^2u) \tag{1.4}$$

where $F : \Gamma \to \mathbb{R}$, $(x, z, p, r) \mapsto F(x, z, p, r)$. Provided there is no confusion we use the same letter F for operator and function. We call a function $u \in C^2(\Omega)$ a *classical solution* of the equation,

$$F[u] = 0 \quad \text{in } \Omega, \tag{1.5}$$

if (1.5) is satisfied in the normal pointwise sense. As with linear equations, we may call a twice weakly differentiable function u, (ie $u \in W^1_{\text{loc}}(\Omega)$) a *strong solution* of (1.5) if the equation is satisfied almost everywhere in Ω. (For relevant function space information, the reader is referred to [3].)

Applications of fully nonlinear equations in geometry largely involve elliptic and parabolic equations. If the function F is differentiable with respect to r at a point $\gamma = (x, z, p, r) \in \Gamma$ and the matrix

$$F_r(\gamma) = \left[F^{ij}(\gamma)\right] := \left[F_{r_{ij}}(\gamma)\right] > 0, \ (\geqslant 0), \tag{1.6}$$

then F is called *elliptic*, (*degenerate elliptic*) at γ. The operator F (and corresponding equation (1.5) are called *elliptic*, (*degenerate elliptic*), with respect to a function $u \in C^2(\Omega)$, if F is elliptic (degenerate elliptic) on the set

$$\Gamma_u = \left\{ (x, u(x), Du(x), D^2u(x)) \,\middle|\, x \in \Omega \right\}.$$

For the Monge-Ampère operator,

$$F[u] := \det D^2 u, \tag{1.7}$$

we have

$$F_r(D^2u) = \left[U^{ij}\right], \tag{1.8}$$

the cofactor matrix of $D^2u = [u_{ij}]$, so that F is elliptic (degenerate elliptic) with respect to u if and only if $D^2u > 0$, $(\geqslant 0)$, in Ω, that is u is *locally uniformly convex*, (*locally convex*). Consequently if the Monge-Ampère equation (1.3) is elliptic, (degenerate elliptic), the inhomogeneous term must be positive, (non-negative). Moreover, for $n = 2$, the converse is true in the sense that $f > 0$ implies (1.3) is elliptic for either $\pm u$.

While parabolic equations are special cases of degenerate elliptic equations, it is normal to distinguish a "time variable" t. Taking a domain $\tilde\Omega \subset \mathbb{R}^{n+1}(x,t)$ and letting $\tilde\Gamma = \tilde\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R}$, $\tilde F : \tilde\Omega \to \mathbb{R}$, $(x,t), z, p, r, s \mapsto \tilde F(x,t,z,p,r,s)$, we then consider operators of the form

$$\tilde F[u] := F(\cdot, u, Du, D^2u, u_t) \tag{1.9}$$

where $Du = (u_i)$, $D^2u = [u_{ij}]$, $i, j = 1, \ldots, n$, denote the spatial gradient and Hessian respectively. The operator (1.9) is *parabolic* whenever $\tilde F_r > 0$, $\tilde F_s < 0$. As examples we see that the following two time dependent Monge-Ampère equations,

$$u_t \det D^2u = \psi \tag{1.10}$$

$$\log\det D^2u - u_t = \psi \tag{1.11}$$

are parabolic whenever $D^2u > 0$, $u_t < 0$ and $D^2u > 0$ respectively.

Examples

(i) General Monge-Ampère equations

The general Monge-Ampère equation is

$$\det D^2u = \psi(\cdot, u, Du) \tag{1.12}$$

where now ψ is a given function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$. The most important example, where ψ depends on Du, is the prescribed Gauss curvature equation,

$$\det D^2u = K(\cdot, u)\left(1+|Du|^2\right)^{(n+2)/2}. \tag{1.13}$$

In (1.3) K is given on $\Gamma\times\mathbb{R}$ and the equation means that K is the Gauss curvature of the graph of u (with respect to an upwards directed normal).

(ii) Hessian equations

The Monge-Ampère equation is a special case of a Hessian equation,

$$F[u] := f(\lambda(D^2u)) = \psi(\cdot, u, Du), \tag{1.14}$$

where f is a given symmetric function of n variables and $\lambda = (\lambda_1, \dots, \lambda_n)$ denote the eigenvalues of the Hessian matrix D^2u. Typical examples of functions f are the elementary symmetric functions,

$$S_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}, \quad k = 1, \dots, n, \tag{1.15}$$

and their quotients

$$S_{k,\ell}(\lambda) = \frac{S_k}{S_\ell}(\lambda), \quad 1 \leqslant \ell < k \leqslant n \tag{1.16}$$

restricted to the positivity set of the denominator, S_ℓ. The case $k = 1$ is the well known Poisson equation. We will discuss the ellipticity of these operators later in conjunction with other properties of the functions S_k. At this stage, we leave it to the reader to check that all these operators are at least elliptic on locally uniformly convex functions.

(iii) Curvature equations

The general form of a curvature equation in Euclidean space is

$$F[u] := f(\kappa[u]) = \psi(\cdot, u, Du) \tag{1.17}$$

where now $\kappa = (\kappa_1, \dots, \kappa_n)$ denote the principal curvatures of the graph of u and again f is a given symmetric function of n variables. Since $\kappa_1, \dots, \kappa_n$ are the eigenvalues of the Hessian D^2u with respect to the metric $I + Du \otimes Du$, we see that (1.17) is a second order equation of the form (1.5). The Gauss curvature equation (1.13) corresponds to the case $f(\kappa) = S_n(\kappa) = \prod \kappa_i$. Other important examples are the mean curvature, $S_1(\kappa)$, yielding a quasilinear elliptic equation, the scalar curvature $S_2(\kappa)$ and the harmonic curvature $S_{n,1}(\kappa)$.

Geometric invariance

As proposed by Felix Klein (Erlangen 1872), geometry is the study of quantities invariant under groups of transformations. Before proceeding to more examples, let us take stock of the invariance properties of the above examples.

The Monge-Ampère operator is invariant under unimodular affine transformations in $\mathbb{R}^n$ of the independent variables, that is affine transformations of the form

$$x \mapsto Tx + y \tag{1.18}$$

where $y \in \mathbb{R}^n$ and T is a non-singular matrix with $\det T = 1$. The remaining Hessian operators are invariant under the subgroups of rigid motions, that is affine mappings (1.18) where T is orthogonal. The curvature operators are invariant under rigid motions in $\mathbb{R}^{n+1}$, involving jointly the independent and dependent variables. This means that under such a transformation the resultant solution hypersurface, although not necessarily a graph anymore, will satisfy $F[v] = F[u]$ for any local graph representation v. We will see presently that we cannot find equations possessing affine invariance in $\mathbb{R}^{n+1}$, unless we go up to fourth order.

Further examples

(iv) Conformal Hessian equations

For $n \geqslant 3$, $u > 0$, we consider

$$F[u] := f(\lambda(A^u)) = \psi(x, u), \tag{1.19}$$

where again f is a symmetric function and $\lambda = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of the conformal Hessian

$$A^u = u\, D^2u - \frac{1}{2}|Du|^2 I. \tag{1.20}$$

The operator (1.19) is invariant under *conformal* mappings $T : \mathbb{R}^n \to \mathbb{R}^n$, that is mappings which preserve angles between curves. The conformal Liouville theorem characterizes conformal mappings in $\mathbb{R}^n$, $(n \geqslant 3)$, as Moebius transformations, that is mappings of the form

$$Tx = y + \frac{\alpha\, A(x - z)}{|x - z|^\theta}, \tag{1.21}$$

where $y, z \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, A is an orthogonal matrix and $\theta = 0$ or 2. If T is a conformal mapping and $v = J^{-1/n}\, u \circ T$, where J denotes the Jacobian determinant of T, then $F[v] = F[u]$. In the case $f = S_1$, we obtain the equation

$$u\,\Delta u - \frac{n}{2}|Du|^2 = \psi(x, u), \tag{1.22}$$

which, for $\psi \equiv$ constant, is equivalent to the celebrated Yamabe equation in $\mathbb{R}^n$.

On Riemannian manifolds $M = M^n$, we define

$$A^u = u\,D^2 u - \frac{1}{2}|Du|^2 g_0 + u^2\, S_{g_0} \tag{1.23}$$

where g_0 is the given initial Riemannian metric and S_{g_0} is the Schouten tensor of g_0, namely

$$S_{g_0} = \frac{1}{n-2}\left(\mathrm{Ric}_{g_0} - \frac{R_{g_0}}{2(n-1)}\, g_0\right), \tag{1.24}$$

where Ric_{g_0} and R_{g_0} denote respectively the Ricci tensor and scalar curvature of g. The tensor A^u then becomes the Schouten tensor of the conformal metric $g = u^{-2} g_0$.

(v) Geometric optics

The case when M is a sphere S^n and $f = S_n$ above has some analytic resemblance to a Monge-Ampère equation arising in the design of reflector antennae, namely

$$F[u] := \det\left(D^2 u - \frac{|Du|^2}{2u}\, g_0 + \frac{1}{2}\, u g_0\right) = \psi(x, u, Du) \tag{1.25}$$

where the function ψ is determined by the illumination densities on the input and output domains $\Omega, \Omega^* \subset S^n$.

(vi) Affine maximal surface equation

The affine maximal surface equation is a nonlinear *fourth order* equation, which may be written as a system of two fully nonlinear second order equations. Letting, as before, $[U^{ij}]$ denote the cofactor matrix of the Hessian matrix D^2u of a function u, which is non-degenerate in the sense that $\det D^2u \neq 0$, we may write this equation in the form,

$$U^{ij}\, D_{ij}w = 0 \tag{1.26}$$

where $w = |\det D^2u|^{(n+1)/(n+2)}$. If we differentiate out, we obtain the equation

$$u^{ij}u^{k\ell}u_{ijk\ell} = \left(\frac{2n+3}{n+2}\right) u^{ij}u^{k\ell}u^{rs}u_{ik\ell}\, u_{jrs} \tag{1.27}$$

where $[u^{ij}]$ denotes the inverse matrix of D^2u. The equation (1.26) is elliptic provided either $\pm D^2u > 0$, that is u is locally uniformly convex or concave. In affine geometry, equation (1.26) means that the affine mean curvature of the graph of u vanishes and it is invariant with respect to affine transformations in $\mathbb{R}^{n+1}$. More generally any operator determined by a symmetric function of the affine principal curvatures will be a fourth order partial differential operator with this invariance. As would be expected, the theory of Monge-Ampère type equations plays a critical role in the treatment of the affine maximal surface equation.

(vii) Optimal transportation

The fundamental problem of optimal transportation in Euclidean space $\mathbb{R}^n$ may be formulated as follows. Given two domains, $\Omega, \Omega^* \subset \mathbb{R}^n$ and corresponding densities $f, g \geqslant 0, \in L^1(\Omega)$, $L^1(\Omega^*)$ respectively with $\int_\Omega f = \int_{\Omega^*} g$, we let $\mathcal{I}$ denote the set of measure preserving transformations from Ω to Ω^*, that is $T : \Omega \to \Omega^* \ \in \mathcal{I}$ provided T is Borel measurable and

$$\int_{T^{-1}(E)} f = \int_E g \tag{1.28}$$

for any Borel set $E \subset \Omega^*$. For a given *cost function* $c : \Omega \times \Omega^* \to \mathbb{R}$, which for our purposes here we take to be smooth, we consider the problem of determining a transformation $T \in \mathcal{I}$ which minimizes the *cost functional*

$$C(T) = \int_\Omega c(x, Tx)\, f(x)\, dx. \tag{1.29}$$

In the original Monge problem, the cost c, given by

$$c(x,y) = |x-y|, \tag{1.30}$$

corresponded to the work done in moving a mass distribution from Ω to Ω^*.

Under the hypothesis that for each $y \in \mathbb{R}^n$, there exists a mapping $p(\cdot, y) : \Omega \to \Omega^*$ such that

$$c_x(x, p(x,y)) = y \tag{1.31}$$

together with

$$\det c_{x,y}(x,y) \neq 0 \tag{1.32}$$

for all $x, y \in \Omega \times \Omega^*$, a smooth optimal map T (if it exists) can be realised as

$$T(x) = p(x, Du(x)), \tag{1.33}$$

where u is a *potential* function satisfying the Monge-Ampère type equation

$$\det\left(D_x^2 c(x, T(x)) - D^2 u\right) = |\det c_{x,y}| \frac{f(x)}{g(T(x))}, \tag{1.34}$$

together with the generalized convexity condition

$$D^2 u \leqslant D_x^2 c(x, T(x)). \tag{1.35}$$

In the special case of a quadratic cost function

$$c(x,y) = -x \cdot y, \tag{1.36}$$

by replacing u by $-u$, we obtain the Monge-Ampère equation

$$\det D^2 u = \frac{f(x)}{g(Du)}. \tag{1.37}$$

(viii) Subelliptic operators

We may replace the partial derivatives ∂_i, $i = 1, \ldots, n$, by vector fields X_i, $i = 1, \ldots, m$, $m \leqslant n$, in the general form (1.4), which do not necessarily commute, with $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^m \times (\mathbb{R}^m \times \mathbb{R}^m)$. To illustrate, we will give an example of a Monge-Ampère operator on the Heisenberg group, $\mathbb{H}^1$. We consider $\mathbb{R}^3(x, y, t)$ and the vector fields X_1, X_2 defined by

$$\begin{aligned} X_1 u &= u_x + \tfrac{1}{2} y u_t, \\ X_2 u &= u_y - \tfrac{1}{2} x u_t, \end{aligned} \tag{1.38}$$

with commutator

$$[X_1, X_2]u = u_t. \tag{1.39}$$

The *horizontal Monge-Ampère equation* is

$$F_{\mathbb{H}}[u] := X_1^2 u\, X_2^2 u - \tfrac{1}{4}\left[(X_1X_2 + X_2X_1)u\right]^2 = \psi(x, u, Xu), \tag{1.40}$$

that is $F_{\mathbb{H}}$ is the determinant of the symmetrical horizontal Hessian, $\frac{1}{2}[X_iX_j + X_jX_i]$. The corresponding horizontal Monge-Ampère equations on more general Lie groups with graded Lie algebras, are readily formulated. Incidentally, it turns out that the operator

$$F[u] = F_{\mathbb{H}} + \tfrac{3}{4}u_t^2 \tag{1.41}$$

may be a credible alternative Monge-Ampère candidate.

(ix) Complex Monge-Ampère equation

Replacing $\mathbb{R}^n$ by its complex analogue $\mathbb{C}^n$, the *complex Monge-Ampère equation* is,

$$F_{\mathbb{C}}[u] := \det[u_{z_i\bar{z}_j}] = \psi(z, u, Du) \tag{1.42}$$

where

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.43}$$

As an equation in $\mathbb{R}^{2n}(x, y)$, equation (1.42) takes the form

$$F_{\mathbb{C}}[u] = \tfrac{1}{4}\det[u_{x_ix_j} + u_{y_iy_j} + i(u_{x_iy_j} - u_{x_jy_i})] = \psi, \tag{1.44}$$

and agrees with the Poisson equation for $n = 1$. Note that for real valued u, $F_{\mathbb{C}}[u]$ is also real so that equation (1.44) may also be written in the form (1.3), (1.4) with $n \to 2n$.

Lecture 2

Linear elliptic equations

In the general theory of fully nonlinear equations, which are elliptic with respect to at least prospective solutions, underlying *linear* structures play a crucial role. For second order equations, linear equations and inequalities satisfied by solutions and derivatives up to second order already provide much information before even taking account of the geometric or physical context.

We illustrate by considering a simple form of the general equation (1.3), (1.4) namely

$$F[u] = F(D^2u) = \psi(x), \tag{2.1}$$

which includes for example the standard Monge-Ampère and Hessian equations. First, by writing (2.1) in the form,

$$\left(\int_0^1 F^{ij}(t\,D^2u)dt\right) u_{ij} = \psi(x) + F(0), \tag{2.2}$$

we see that a solution u of (2.1) is also a solution of a linear equation of the form

$$Lu := a^{ij}u_{ij} = \tilde{\psi}, \tag{2.3}$$

which is elliptic if F is elliptic with respect to tu, $0 \leqslant t \leqslant 1$. Any results for equations of the form (2.3), which do not depend on properties beyond ellipticity, automatically apply to solutions of (2.1).

Next, by differentiating (2.1) with respect to x_k, $k = 1, \ldots, n$, we obtain

$$F^{ij}(D^2u)\,u_{ijk} = \psi_k \tag{2.4}$$

so that first derivatives u_k also satisfy linear equations of the form (2.3). Differentiating again, we obtain

$$F^{ij}(D^2u)\,u_{ijk\ell} + F^{ij,pq}(D^2u)\,u_{ijk}\,u_{pq\ell} = \psi_{k\ell}, \tag{2.5}$$

where

$$F^{ij,pq} = \frac{\partial^2 F}{\partial r_{ij}\,\partial r_{pq}},$$

which is not so nice. However if the function F is concave, (or convex), and we take derivatives in the same direction $\xi \in S^n$, we obtain linear inequalities

$$Lw := a^{ij}\, D_{ij} w \geqslant,\ (\leqslant)\ \psi_{\xi\xi}, \tag{2.6}$$

for the pure second derivatives $w = u_{\xi\xi}$, only "half" as much information as before. As we will show presently, if we now return to equation (2.1), regarding it as a functional relation between second derivatives, we can compensate for the missing information.

The simplest and most fundamental estimate for the elliptic equation (2.3) is the *weak maximum principle*, namely if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies

$$Lu \geqslant \tilde{\psi} \tag{2.7}$$

in Ω then

$$u \leqslant \max_{\partial\Omega} u + \frac{d^2}{2n} \sup_{\Omega} \frac{\tilde{\psi}}{T}, \tag{2.8}$$

where $d = \operatorname{diam}\Omega$ and $T = \operatorname{trace} A$, $A = [a^{ij}]$, which holds more generally for degenerate elliptic L with $\tilde{\psi}/T$ bounded. From this we have the well known *comparison principle*, namely if $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfy $F[u] \geqslant F[v]$ in Ω and F is degenerate elliptic with respect to $tu + (1-t)v$ for all $0 \leqslant t \leqslant 1$, then $u \leqslant v$ in Ω.

The Bernstein-Liouville Theorem

About ninety years ago, Serge Bernstein proved a remarkable Liouville Theorem for linear homogeneous elliptic equations of the form (2.3) in two variables. His immediate application was to prove the famous Bernstein Theorem for two dimensional minimal graphs, but recently two further geometric applications have appeared in conjunction with the application of the theory of fully nonlinear equations, namely to affine maximal surfaces and to mean curvature flow.

The Bernstein-Liouville Theorem may be formulated as follows.

Theorem 2.1. *Let $u \in C^2(\mathbb{R}^2)$ be a solution of an elliptic equation,*

$$Lu :\ a u_{xx} + 2b\, u_{xy} + c\, u_{yy} = 0, \tag{2.9}$$

subject to only the ellipticity condition, $b^2 < ac$. Then if $u = o(r)$ as $r = \sqrt{x^2 + y^2} \to \infty$, u must be constant.

There was a slight gap, of a topological nature, in the original proof of Bernstein, which was pointed out and repaired by Hopf in 1949. We present here the Bernstein proof, incorporating a correction along the lines of that proposed by Mickle, also in 1949. So far, we are unaware of any different approaches to that of Bernstein.

Proof. The equation (2.1) may be expressed in an equivalent fully nonlinear form,

$$\det D^2 u = u_{xx}\, u_{yy} - u_{xy}^2 \leqslant 0, \tag{2.10}$$

with equality if and only if $D^2 u = 0$. To see this, write (2.9) as

$$\alpha\lambda_1 + \beta\lambda_2 = 0, \tag{2.11}$$

where $\lambda_1 \leqslant \lambda_2$ are the eigenvalues of the Hessian matrix $D^2 u$, and α, β are positive, by virtue of the ellipticity of (2.9). Multiplying through by either λ_1 or λ_2, we obtain,

$$\det D^2 u = \lambda_1 \lambda_2 \leqslant 0.$$

Conversely, if $\lambda_1\lambda_2 \leqslant 0$ and $\lambda_1, \lambda_2 \not\equiv 0$ we have

$$\lambda_2(\lambda_1) + (-\lambda_1)\lambda_2 = 0,$$

which is equivalent to an elliptic equation of the form (2.9).

More generally we consider functions $u \in C^2(\mathbb{R}^2)$ satisfying $\det D^2 u \leqslant 0$, that is whose graphs have non-negative Gauss curvature, under the condition $u = o(r)$ as $r \to \infty$. First, we observe that such functions will continue to satisfy the weak maximum principle, that is for any bounded domain $\Omega \subset \mathbb{R}^2$,

$$\min_{\partial\Omega} u \leqslant u \leqslant \max_{\partial\Omega} u. \tag{2.12}$$

Now let us assume that $\det D^2 u \not\equiv 0$, so that there will be some neighbourhood in $\mathbb{R}^2$ where $\det D^2 u < 0$ and $Du \not\equiv 0$. By choosing appropriate coordinates, we can thus assume, at the origin,

$$u = u_x = 0, \quad u_y = q_0 > 0, \quad u_{xx}\, u_{yy} - u_{xy}^2 < 0. \tag{2.13}$$

Now let, for $r > 0$,

$$N(r) = \sup_{x^2+y^2<r^2} |u(x, y)|. \tag{2.14}$$

By the maximum principle, (2.12), N is non-decreasing. Under the hypothesis $u = o(r)$, we have

$$N(r) < q_0 r - 1$$

for $r \geqslant$ some $r_0 > 0$, so that we can define an infinite strip $S = \{|y| < N^*(r)\}$, where

$$N^*(r) = \frac{1}{q_0}\{N(\max(r, r_0)) + 1\},$$

whose upper and lower boundaries are the two lines $L^{\pm}$, with equations

$$y = \pm N^*(r). \tag{2.15}$$

Next we define,

$$v(x, y) = u(x, y) - q_0 y, \tag{2.16}$$

so that, at the origin,

$$v = v_x = v_y = 0, \quad v_{xx}v_{yy} - v_{xy}^2 < 0, \tag{2.17}$$

and $v < 0, > 0$ on the lines $L^{\pm}$ respectively. Since the origin O is a saddle point for v, there must exist two straight line segments $\ell^{\pm}$, intersecting at O on which O is a strict minimum and maximum respectively. Consequently there must exist four distinct components of the set where $v \neq 0$, Ω_i, $i = 1, 2, 3, 4$, whose closures $\bar{\Omega}_i$ contain O, with $v > 0$ on Ω_1, Ω_3, $v < 0$ on Ω_2, Ω_4. Moreover, by the maximum principle, $\left(\bar{\Omega}_1 - O\right) \cap \left(\bar{\Omega}_3 - O\right), \left(\bar{\Omega}_2 - O\right) \cap \left(\bar{\Omega}_4 - O\right) = \emptyset$ and each component Ω_i is unbounded. Bernstein concluded that at least one of these components must lie in the strip S and be bounded on one side. Indeed if any component Ω_i is bounded on one side, it automatically lies in S, since if Ω_i intersects either of the lines $L^{\pm}$, it must contain the whole line and be unbounded in both directions. Consequently, it is enough to show one of the components is bounded in one direction. Once this is done, we can apply a superlinear growth result, Lemma 2.2, to get a contradiction. Bernstein did not take account of the possibility of infinite oscillations in both directions of the Ω_i so his inference is not correct. Instead we show that the boundedness property at least holds for a small perturbation of the original function. Suppose then that the four components Ω_i, $i = 1, \ldots, 4$, are unbounded in both x directions and set

$$\begin{aligned} a_i &= \sup\{y \mid (1, y) \in \Omega_i, |y| < N^*(r)\}, \\ b_i &= \inf\{y \mid (1, y) \in \Omega_i, |y| < N^*(r)\}, \\ a &= \max a_i, \quad b = \min b_i. \end{aligned} \tag{2.18}$$

We examine first the case when $a = a_{i_1} > a_{i_j}$ for all $i_j \neq i_1$. Then there exists $i_2 \neq i_1$, i_3 such that

$$a = a_{i_1} > a_{i_2} > b_{i_2} > b_{i_3} = b.$$

Fix numbers y_1, y_3 satisfying $(1, y_1) \in \Omega_{i_1}, (1, y_3) \in \Omega_{i_3}$, $a_{i_1} > y_1 > a_{i_2} > b_{i_2} > y_3 > b_{i_3}$. Then there exists a curve $\gamma_1 \subset \Omega_{i_1}$ joining the point $(1, y_1)$ to O and a curve $\gamma_3 \in \Omega_{i_3}$ joining the point $(1, y_3)$ to O. Let $\gamma = \gamma_1 \cup \gamma_2 \cup O$ and suppose γ lies between the lines $x = x_1$ and $x = x_2$. If $\Omega_{i_2} \cap \{x = x_1\}$ and $\Omega_{i_2} \cap \{x = x_2\}$ are both non-empty, there exists a curve in Ω_{i_2} joining points in each of these sets which must cross γ which is a contradiction. Hence if every component Ω_i is unbounded in both directions we must have $a = a_{i_1} = a_{i_2}$ for some $i_1 \neq i_2$, and by a similar argument $b = b_{i_3} = b_{i_4}$ for some $i_3 \neq i_4$, with i_1, i_2, i_3, i_4 all distinct. Without loss of generality we may assume $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 4$. For sufficiently small $\varepsilon > 0$, there exists corresponding components Ω_i^ε of the set where $v + \varepsilon \neq 0$, intersecting at least one of the line segments $\ell^\pm$ and satisfying

$$\begin{aligned} \bar{\Omega}_i \subset \Omega_i^\varepsilon &\quad \text{for } i = 1, 3, \\ \bar{\Omega}_i^\varepsilon \subset \Omega_i &\quad \text{for } i = 2, 4. \end{aligned}$$

Hence $(1, a_1) \in \Omega_1^\varepsilon$, $(1, b_3) \in \Omega_3^\varepsilon = \Omega_1^\varepsilon$ and $b_3 < y < a_1$ for all $(1, y) \in \Omega_2^\varepsilon$. A similar argument, to that in the previous case, then shows that the component Ω_2^ε cannot be unbounded in both directions.

Applying the following lemma, we then conclude that if $\det D^2u \leqslant 0$ in $\mathbb{R}^2$, we must have $\det D^2u \equiv 0$ and the Bernstein-Liouville Theorem follows immediately. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ lie in a wedge of angle $< \pi$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy*

$$\begin{aligned} \det D^2u &\leqslant 0 \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{2.19}$$

Then there exists a constant $C > 0$ such that $N(r) \geqslant Cr$ for r sufficiently large.

Proof. Let us fix coordinates so that the wedge is bisected by the positive x axis, with its vertex at the origin. By the maximum principle, (2.12), the domain Ω is unbounded. We claim that the function of one variable M given by

$$M(x) = \sup_{(x,y) \in \Omega} u(x, y) \tag{2.20}$$

is convex. For if not, there exist numbers $x_0 < x_1 < x_2$ such that

$$M(x_1) > \frac{x_1 - x_0}{x_2 - x_0} M(x_2) + \frac{x_2 - x_1}{x_2 - x_0} M(x_0). \tag{2.21}$$

By subtracting an affine function A from M, we may assume $M(x_0) = M(x_2) = 0$. Defining

$$v(x,y) = u(x,y) - M(x), \tag{2.22}$$

we then have $v \leqslant 0$ on $\Omega \cap \{x = x_0, x_2\}$, so that by the maximum principle, $v \leqslant 0$ in $\Omega \cap \{x_0 < x < x_2\}$, which contradicts (2.21). Since M is convex, we must have $M(x) \geqslant Cx$ for x sufficiently large, for some positive constant C, whence Lemma 2.2 follows. □

Lecture 3

Hölder estimates

In this lecture we outline the derivation of Hölder estimates for solutions of fully nonlinear elliptic equations from the maximum principle of Alexandrov and Bakel'man. Since this is already covered in books [], we shall not include all details but rather concentrate on simplifications achieved through restricting the preparatory result from the linear theory to those really essential for the nonlinear applications.

First we recall that a function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent $\alpha \in (0, 1]$ in Ω if

$$[u]_{0,\alpha;\Omega} := \sup_{\substack{x,\,y\,\in\,\Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} < \infty. \tag{3.1}$$

An important property of Hölder continuity for nonlinear equations is that it is preserved under composition.

For classical existence results for fully nonlinear elliptic equations under appropriate boundary conditions, we need a priori estimates for

1. solutions
2. first derivatives
3. second derivatives
4. Hölder norms of second derivatives

The procedure through the method continuity is described in []. The corresponding results for linear equations, (needed for the openness part of the method of continuity), are embodied in the Schauder estimates.

Alexandrov Bakel'man maximum principle

We begin with a brilliant but simple device of A.D. Alexandrov. Let $u \in C^0(\Omega)$ and for a point $y \in \Omega$, define the (lower) *normal mapping* $\chi = \chi_u$ of u at y by

$$\chi_u(y) := \{p \in \mathbb{R}^n \mid u(x) \geqslant u(y) + (x-y)\cdot p \quad \text{for all } x \in \Omega.\} \tag{3.2}$$

That is, $\chi_u(y)$ is the set of "slopes" of support hyperplanes from below at y. The (lower) *contact set* $\Gamma^- = \Gamma^-_u$ is defined as the subset of Ω where χ_u is not empty, that is

$$\begin{aligned}\Gamma^- = \{y \in \Omega \mid u(x) &\geqslant u(y) + (x-y)\cdot p \\ &\text{for all } x \in \Omega,\ \text{for some } p = p(y) \in \mathbb{R}^n.\}\end{aligned} \tag{3.3}$$

It follows that

(i) if u is differentiable at $y \in \Gamma^-$, then $\chi_y(y) = Du(y)$;

(ii) if u is convex in Ω, then $\Gamma^-_u = \Omega$ and χ_u is the subgradient of u;

(iii) if u is twice differentiable at $y \in \Gamma^-$, then $D^2u(y) \geqslant 0$.

We investigate the relation between χ_u and the minimum value of $u \in C^0(\bar{\Omega})$. Let $p_0 \notin \chi_u(\Omega) := \bigcup_{y\in\Omega} \chi_u(y)$. By vertically translating upwards a hyperplane of slope p_0, lying below the graph of u in $\mathbb{R}^{n+1}$, we see that there must exist a point $y \in \partial\Omega$ such that

$$\begin{aligned}u(x) &\geqslant u(y) + p_0\cdot(x-y) \qquad \text{for all } x \in \Omega,\\ &\geqslant \min_{\partial\Omega} u - |p_0|\, d, \qquad\qquad d = \operatorname{diam}\Omega.\end{aligned} \tag{3.4}$$

Consequently, we have an estimate,

$$u \geqslant \min_{\partial\Omega} u - \left\{\frac{1}{\omega_n}|\chi_u(\Omega)|\right\}^{1/n} d, \tag{3.5}$$

relating u to the volume of the image of χ_u. The same argument embraces oblique boundary conditions. Indeed, if $u \in C^1(\bar{\Omega})$ and β is a vector field on $\partial\Omega$, pointing out of Ω at convex points of $\partial\Omega$ then

$$u \geqslant \min_{\partial\Omega}(\beta\cdot Du + u) - \left\{\frac{1}{\omega_n}|\chi_u(\Omega)|\right\}^{1/n}(|\beta| + d). \tag{3.6}$$

The estimates (3.5), (3.6) are related to the Monge-Ampère operator by the change of variables formula

$$|\chi_u(\Omega)| = |\chi_u(\Gamma^-)| = \int_{\Gamma^-} \det D^2 u \tag{3.7}$$

which holds for $u \in C^2(\Omega)$. Accordingly if the function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies the differential inequality,

$$\det D^2 u \leqslant f \quad \text{in } \Gamma^-, \tag{3.8}$$

we have the estimate

$$\min_{\Omega} u \geqslant \min_{\partial\Omega} u - \left(\frac{1}{\omega_n} \int_{\Gamma^-} f \right)^{1/n} d. \tag{3.9}$$

For the application to Hölder estimates for second derivatives of solutions of fully nonlinear elliptic equations, we need the following simple consequence, due to Alexandrov and Bakel'man.

Lemma 3.1. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy the linear differential inequality,*

$$Lu = a^{ij} D_{ij} u \leqslant g \ \text{in } \Omega, \tag{3.10}$$

where $a^{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \ldots, n$, $\mathcal{A} = [a^{ij}] > 0$ and $g \in L^n(\Omega)$. Then we have the estimate

$$\min_{\Omega} u \geqslant \min_{\partial\Omega} u - \frac{d}{n\, \omega_n^{1/n}} \left|\left| g/(\det \mathcal{A})^{1/n} \right|\right|_{L^n(\Gamma^-)} \tag{3.11}$$

Proof. To derive (3.11), we use the matrix inequality,

$$\det A \ \det B \leqslant \left(\frac{\text{trace}\, AB}{n} \right)^n, \tag{3.12}$$

which holds for any two non-negative, symmetric $n \times n$ matrices. (Note AB is not necessarily symmetric). The estimate (3.11) then follows from (3.9), by taking $B = D^2 u$ in (3.12) on Γ^-. □

By replacing u by $-u$ in Lemma 3.1, we have the corresponding estimate for $\max u$, namely if $Lu \geqslant g$ in Ω, then

$$\max_{\Omega} u \leqslant \max_{\partial\Omega} u + \frac{d}{n\, \omega_n^{1/n}} \left|\left| g/(\det \mathcal{A})^{1/n} \right|\right|_{L^n(\Gamma^+)}, \tag{3.13}$$

where the upper contact set, $\Gamma^+ = \Gamma_u^+ = -\Gamma_{-u}^-$.

Further results follow for more general Monge-Ampère type operators and quasilinear operators. For the application to Hölder estimates the critical feature in (3.11) and (3.13) is that the estimate depends on an L^p norm, $p < \infty$, of the inhomogeneous term, rather than the L^∞ norm given by the classical maximum principle, (2.8).

Local pointwise estimates

We move on now to the famous results of Krylov and Safonov. In particular, they proved Harnack inequalities and local Hölder estimates for *uniformly elliptic* operators L of the form (3.10), that is for coefficient matrices $\mathcal{A}$ satisfying

$$\lambda I \leqslant \mathcal{A} \leqslant \Lambda I, \tag{3.14}$$

where λ, Λ are positive constants. For the application to second derivatives of solutions of fully nonlinear elliptic equations, we need, in the light of (2.7), partial versions for differential inequalities.

Lemma 3.2 (Weak Harnack inequality). *Let $\in C^2(\Omega)$ satisfy (3.10), with $u \geqslant 0$ on some ball $B = B_R(y) \subset \Omega$. Then there exists a constant $p = p(n, \Lambda/\lambda) > 0$ such that for any $\sigma, \tau \in (0,1)$,*

$$\left\{ \frac{1}{|B|} \int_{B_{\tau R}(y)} u^p \right\}^{1/p} \leqslant C \left\{ \inf_{B_{\sigma R}(y)} u + \frac{R}{\lambda} ||g||_{L^n(B_R)} \right\}, \tag{3.15}$$

where $C = C(n, \Lambda/\Lambda, \sigma, \Lambda)$

For applications to Hölder estimates, we do not need as strong a term on the left hand side of (3.15) but the full strength of (3.15) appears automatically from the Krylov-Safonov argument. Prior to their great discovery the closest result known was a simple quantification of the strong maximum principle, namely, for $\tau < \sigma$,

$$\inf_{B_{\tau R}(y)} u \leqslant C \left\{ \inf_{B_{\sigma R}(y)} u + \frac{R}{\lambda} ||g||_{L^n(B_R)} \right\}, \tag{3.16}$$

from which, with $g \equiv 0$, the strong maximum principle is a ready consequence. To prove (3.16), we take an auxiliary or barrier function,

$$w(x) = \left\{ \left(\frac{|x|}{R} \right)^{-k} - 1 \right\} / \left(\tau^{-k} - 1 \right) \tag{3.17}$$

and compute

$$Lw \geqslant 0 \quad \text{for } k + 2 \geqslant \operatorname{trace} \mathcal{A} / \lambda, \ |x| > 0, \tag{3.18}$$

and since

$$w \leqslant u \Big/ \inf_{B_{\tau R}} u \quad \text{on } \partial B_{\tau R} \text{ and } \partial B_R, \tag{3.19}$$

the result follows by the weak maximum principle (Lemma 3.1). The first step in the Krylov-Safonov approach is to use Lemma 3.1 to perturb (3.15) in measure. For simplicity we will take $f = 0$ from now on. The general case may be recovered by replacing u by $u + c||f||$ for an appropriate constant c

Lemma 3.3. *Under the hypotheses of Lemma 3.2, there exists a positive constant $\theta < 1$, $\theta = \theta(n, \Omega/\lambda, \sigma, \Gamma)$ such that if*

$$\big|\{u \geqslant 1\} \cap B_{\tau R}\big| \geqslant \theta |B_{\tau R}| \tag{3.20}$$

then $u \geqslant C^{-1}$ on $B_{\sigma R}$.

Proof. Without loss of generality we may take $y = 0$. Let $\eta(x) = 1 - \frac{|x|^2}{\mathbb{R}^2}$ so that $w = \eta - u \leqslant 0$ on ∂B_R. Since $Lw = L\eta - Lu \geqslant -2g/\mathbb{R}^2$, we have by Lemma 3.1, applied on the set where $w > 0$,

$$\eta - u \leqslant \frac{C\Lambda}{\lambda R} \big|\{\eta > u\}\big|^{1/n} \leqslant \frac{1}{2}\eta \quad \text{on } B_{\tau R} \tag{3.21}$$

if $|\{u < 1\}|/|B_R|$ is sufficiently small. Replacing R by σR and using (3.15) we thus obtain Lemma 3.3. □

Lemma 3.3 embodies all the information we need from the differential inequality (3.10). The rest of the proof of Lemma 3.2 follows a measure theoretic covering argument, that is described for example in [] or [].

The Hölder and Harnack estimates both follow from Lemma 3.2.

Theorem 3.4. *Let $u \in C^2(\Omega)$ by a solution of the linear partial differential equation,*

$$Lu = f \tag{3.22}$$

where L, given by (3.10), satisfies (3.14). Then there exists a positive exponent $\alpha = \alpha(n, \Lambda/\lambda)$ such that for any ball $B = B_R(y) \subset \Omega$, $0 < \sigma < 1$

$$\operatorname*{osc}_{B_{\sigma R}} \leqslant C\left(\sigma^\alpha \operatorname*{osc}_{B} u + \frac{R}{\lambda}||f||_{L^n(B)}\right), \tag{3.23}$$

where $C = C(n, \Lambda/\lambda)$. If $u \geqslant 0$ on B, then

$$\sup_{B_{\sigma R}} u \leqslant C\left(\inf_{B_{\sigma R}} u + \frac{R}{\lambda}||f||_{L^n(B)}\right), \tag{3.24}$$

where $C = C(n, \Lambda/\lambda, \sigma)$.

Remarks

(i) For $n = 2$, these results are due respectively to Morrey [], who proved Hölder estimates for first derivatives and Serrin [], who proved the Harnack inequality; see [].

(ii) The exponent α in Theorem 3.4 is necessarily small for $n > 2$, (Safanov []).

(iii) The exponent p in Lemma 3.2 is also necessarily small; (see [], Problem ??).

(iv) Lemma 3.2 and Theorem 3.4 are the analogues of the celebrated De Giorgi, Nash and Moser results for divergence form operators,

$$Lu = D_i(a^{ij}\, D_j u). \tag{3.25}$$

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