Practice final exam (solutions)

ANU-SDU: ODE, 2022

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- This exam has 5 questions, for a total of 39 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- If you need more space, use the page at the back and indicate this clearly.

Name: ____

ID number: ____

1. Consider the equation

$$y(2x^2y + 1)\ dx + x(4x^2y - 1)\ dy = 0$$

   (a) (1 point) Is this equation exact?

Solution: No. If $P = y(2x^2y + 1)$ and $Q = x(4x^2y - 1)$ then

$$\partial_y P = 4x^2y + 1 \neq \partial_x Q = 12x^2y - 1$$

   (b) (3 points) Show that $\frac{1}{x^2}$ is an integrating factor.

Solution: We can look for an integrating factor only involving x.

$$\frac{1}{Q}\left(\partial_y P - \partial_x Q\right) = \frac{-8x^2y + 2}{x(4x^2y - 1)} = -\frac{2}{x}$$

so we have an integrating factor of $\frac{1}{x^2}$.

   (c) (3 points) Use the integrating factor to find an implicit solution to the equation if $y(1) = 1$ (you do not need to solve for y).

Solution: Multiplying through by the integrating factor, the new equation is

$$(2y^2 + \frac{y}{x^2})\ dx + (4xy - \frac{1}{x})\ dy$$

We are looking for a solution where $u(t, y) = 0$ where

$$u = \int 4xy - \frac{1}{x}\ dy + k(x) = 2xy^2 - \frac{y}{x} + k(x)$$

Noting that $\partial_x u = 2y^2 + \frac{y}{x^2}$ and so $k'(x) = 0$ i.e. $k(x) = c$, a constant. Thus

$$2xy^2 - \frac{y}{x} + c = 0$$

Subbing in $y(1) = 1$ we see that $2 - 1 + c = 0$ so $c = -1$ and the implicit solution is

$$2xy^2 - \frac{y}{x} - 1 = 0$$

2. (4 points) **MATLAB code:** Here is some MATLAB code. Describe what this code does indicating what equation it solves (including initial condition) and what it prints to the screen (One sentance is fine).

   In a file `ydot.m`:

```
function out = ydot(t,y)
  out = t*y+1;
end
```

   In a file `main.m`:

```
y0 = 1;
ts = [0:0.01:3];
[t,y] = ode45(@ydot,ts,y0);
plot(t,y);
```

   Solution: The code sets up a differential equation $y; = ty+1$ and solves it over the interval $0 \leq t \leq 3$ with the initial condition $y(0) = 1$. The code then prints out a plot of this solution.

3. **Spring system:** A mass of 2 kg is attached to the bottom of a spring. This stretches the spring by 2m.

   (a) (1 point) What is the spring constant k of the spring? You should assume that the accelleration due to gravity is $g = 10$.

   Solution: We use the equation $kl = mg$ i.e. $2k = 10 \cdot 2$ so $k = 10$

   (b) (4 points) The spring and mass system is placed in water, which acts on the mass with a damping force equal to 4 times it's velocity. Let $y(t)$ be the position of the mass **below** it's equillibrium point at time t (so y increases as we go down). Write down a differential equation that governs the motion of the mass.

   Solution: We use Newton's second law and Hooke's law

$$my'' = mg - k(y + l) - 4y'$$

   Noting that $mg = kl$, the equation becomes

$$y'' + 2y' + 5y = 0$$

(c) (4 points) If the mass is initially pulled down 1 metre (so $y(0) = 1$) and then let go (zero initial velocity), solve this differential equation.

> **Solution:** The ODE has characteristic polynomial $\lambda^2 + 2\lambda + 5 = 0$ which has two complex conjugate solutions
>
> $$\lambda_1 = -1 + 2i \text{ and } \lambda_2 = -1 - 2i$$
>
> So $\alpha = -1$ and $\omega = 2$. Thus the general solution is
>
> $$y(t) = e^{-t}\left(A\sin(2t) + B(\cos(2t))\right)$$
>
> We use the initial conditions and get the equations
>
> $$1 = B$$
> $$0 = 2A - B \text{ so } A = \frac{1}{2}$$
>
> and the solution is
>
> $$y(t) = e^{-t}(\frac{1}{2}\sin(2t) - \cos(t))$$

(d) (3 points) Is the spring system under-damped, over-damped or critically damped? How strong should the damping force be in order for the system to be critically damped?

> **Solution:** Under-damped. In order to be critically damped we would need the characteristic polynomial to have a single real root. Suppose the damping force is $-\alpha y'$. Then the equation is
>
> $$y'' + \frac{1}{2}\alpha y' + 5y = 0$$
>
> so the characteristic polynomial is $\lambda^2 + \frac{1}{2}\alpha\lambda + 5$ which has discriminant
>
> $$\Delta = \frac{1}{4}\alpha^2 - 20$$
>
> If we set this equal to zero we get $\alpha = 4\sqrt{5}$.

4. **Half-life:** A solution of 4 mg/L of a radioactive substance, Fluorine-18, is being pumped into a tank at a rate of 3 L/hour. It is known that the Fluorine-18 has a half-life of $4\ln 2$ hours.

   (a) (4 points) Let $y(t)$ be the total amount of Fluorine-18 in the tank at time t. Write a differential equation modelling $y(t)$. *Hint: the change in y is equal to the rate being pumped in, minus the rate leaving due to the half life.*

   Solution: Let $y(t)$ be the amount of Fluorine-18 at time t. There is 12 mg/hour entering the tank. Due to half-life there is $y/4$ mg/hour leaving the tank. Thus

$$\frac{dy}{dt} = 12 - \frac{1}{4}y.$$

   (b) (4 points) If, initially, the tank contains only water with no Fluorine-18 solve the differential equation. Use the method of exact equations and integrating factors. *Do **not** use separation of variables*

   Solution: The equation is rearranged to

$$(y - 48)\, dt + 4\, dy = 0$$

   Which is not exact, but we can find an integrating factor depending only on t

$$\frac{1}{Q}\left(\partial_y P = \partial_t Q\right) = \frac{1}{4}$$

   So the integrating factor is $e^{\frac{1}{4}t}$ and the equation becomes

$$(y - 48)e^{\frac{1}{4}t}\, dt + 4e^{\frac{1}{4}t}\, dy = 0$$

   and we can integrate one term to get

$$u(t, y) = \int (y - 48)e^{\frac{1}{4}t}\, dt + k(y) = 4(y - 48)e^{\frac{1}{4}t} + k(y)$$

   we see that $4e^{\frac{1}{4}t} + k'(y) = 4e^{\frac{1}{4}t}$ so $k(y) = c$ is a constant. Thus $4(y - 48)e^{\frac{1}{4}t} + c = 0$ and rearranging we get

$$y(t) = 48 - Ce^{-\frac{1}{4}t}$$

   and since $y(0) = 0$ we have $C = 12$. So

$$y(t) = 48\left(1 - e^{-t/4}\right).$$

   (c) (2 points) How many milligrams of the Fluorine-18 is in the tank after 24 hours? You may leave your answer in terms of e.

   Solution: When $t = 24$ there are

$$y(24) = 48\left(1 - e^{-6}\right) \text{ milligrams.}$$

5. **Challenge:** Consider the equation

$$(t+1)^2 y'' + (t+1)y' - y = 0$$

(a) (2 points) If you assume that $y'' = 0$, does this equation have a solution?

Solution: If we assume that $y'' = 0$ then the equation is separable, and we instead solve

$$\begin{aligned} \frac{dy}{y} &= \frac{dt}{t+1} \\ \int \frac{dy}{y} &= \int \frac{dt}{t+1} \\ \ln y &= \ln(t+1) + C \\ y &= C'(t+1) \end{aligned}$$

So for example, $y = t+1$ is a solution. This indeed does have $y'' = 0$ so it is in fact a solution.

(b) (4 points) Find the solution if $y(0) = 1$ and $y'(0) = 3$.

Solution: We can use the solution $t+1$ for a reduction of order argument. We assume that the general solution has the form

$$y(t) = v(t)(t+1)$$

Putting this into the equation gives

$$\begin{aligned} 0 &= (t+1)^2(v''(t+1) + 2v') + (t+1)(v'(t+1) + v) - v(t+1) \\ &= (t+1)^3 v'' + 3(t+1)^2 v' \end{aligned}$$

Making the substitution $w = v'$ gives the equation

$$(t+1)w' + 3w = 0$$

which we can think about as $(t+1)dw + 3wdt = 0$. This is not exact, but we can quickly find that $(t+1)^2$ is an integrating factor. Thus the solution is given by $u(t, w) = 0$ where

$$u = \int (t+1)^3 dw + k(t) = (t+1)^3 w + k(t)$$

Now noting that $\partial_t u = 3(t+1)^2 w$ we see that $k'(t) = 0$ so $k(t)$ is a constant. Thus

$$w = \frac{c}{(t+1)^3} \text{ and therefore } v = \frac{c_1}{(t+1)^2} + c_2$$

and so finally we have the general solution

$$y(t) = (t+1)v(t) = A(t+1) + \frac{B}{t+1}$$

We just need to apply the initial conditions which show that $A + B = 1$ and $A - B = 3$ which gives $A = 2$ and $B = -1$ so

$$y(t) = 2(t+1) - \frac{1}{t+1} = \frac{2t^2 + 4t + 1}{t+1}.$$

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