THE k-HESSIAN EQUATION

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Abstract. The k-Hessian is the k-trace, or the kth elementary symmetric polynomial of eigenvalues of the Hessian matrix. When $k \geq 2$, the k-Hessian equation is a fully nonlinear partial differential equations. It is elliptic when restricted to k-admissible functions. In this paper we establish the existence and regularity of k-admissible solutions to the Dirichlet problem of the k-Hessian equation. By a gradient flow method we prove a Sobolev type inequality for k-admissible functions vanishing on the boundary, and study the corresponding variational problems. We also extend the definition of k-admissibility to non-smooth functions and prove a weak continuity of the k-Hessian operator. The weak continuity enables us to deduce a Wolff potential estimate. As an application we prove the Hölder continuity of weak solutions to the k-Hessian equation. These results are mainly from the papers [CNS2, W2, CW1, TW2, Ld] in the references of the paper.

Contents

2000 *Mathematics Subject Classification.* 35J60, 35J20, 35A15, 28A33.

Key words and phrases. Hessian equation, a priori estimates, Sobolev inequality, variational problem, Hessian measure, potential estimate.

This work was supported by the Australian Research Council.

1. Introduction

Let Ω be a bounded, smooth domain in the Euclidean space $\mathbb{R}^n$. In this note we study the *k-Hessian equation*

$$S_k[u] = f \quad \text{in} \quad \Omega, \tag{1.1}$$

where $1 \le k \le n$, $S_k[u] = \sigma_k(\lambda)$, $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $(D^2 u)$, and

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \tag{1.2}$$

is the *k-th elementary symmetric polynomial*. The k-Hessian equation includes the Poisson equation ($k = 1$)

$$-\Delta u = f, \tag{1.3}$$

and the Monge-Ampère equation ($k = n$)

$$\det D^2 u = f, \tag{1.4}$$

as special examples.

We say a second order partial differential equation

$$F(D^2u, Du, u, x) = 0 \tag{1.5}$$

is *fully nonlinear* if $F(r, p, z, x)$ is nonlinear in r. The k-Hessian equation is fully nonlinear when $k \ge 2$. We say F is *elliptic* (or *degenerate elliptic*) with respect to a solution u if the matrix $\{F_{ij}\}$ is positive definite (or positive semi-definite) at $(r, p, z, x) = (D^2u(x), Du(x), u(x), x)$, where $F_{ij} = \{\frac{\partial F}{\partial r_{ij}}\}$. We say F is *uniformly elliptic* if there exist positive constants Λ and λ such that

$$\lambda I \le \{F^{ij}\} \le \Lambda I, \tag{1.6}$$

where I is the unit matrix. We also say F is elliptic if $-F$ is.

The Monge-Ampère equation (1.4) is elliptic if and only if the function u is uniformly convex or concave. For the k-Hessian equation, it is elliptic when u is k-admissible [CNS2], namely the eigenvalues $\lambda(D^2u)$ lie in the convex cone Γ_k, which will be introduced in Section 2 below. Fully nonlinear equations of mixed type are very difficult. In this note we restrict ourself to k-admissible solutions to the k-Hessian equation.

There are many other important fully nonlinear equations, see §11 below for examples. But the k-Hessian equation (1.1) is variational, and when restricted to k-admissible solutions, it enjoys many nice properties which are similar to those of the Poisson equation. In this paper we discuss the regularity, variational properties, and local behaviors of solutions to the k-Hessian equation.

We divide this note into a number of sections.

In §2 we introduce the notion of k-admissible functions, and show that the k-Hessian equation is elliptic at k-admissible functions. We also collect some inequalities related to the polynomial σ_k.

In §3 we establish the global a priori estimates and prove the existence of solutions to the Dirichlet problem.

In §4 we establish the interior gradient and second derivative estimates. From the interior gradient estimate we also deduce a Harnack inequality.

In §5 we use gradient flow to prove Sobolev type inequalities for k-admissible functions which vanish on the boundary. That is

$$\|u\|_{L^p(\Omega)} \leq C\left[\int_\Omega (-u)S_k[u]\right]^{1/(k+1)}, \tag{1.7}$$

where C depends only on n, k, Ω; $p = \frac{n(k+1)}{n-2k}$ if $k < \frac{n}{2}$, $p < \infty$ if $k = \frac{n}{2}$; and $p = \infty$ if $k > \frac{n}{2}$. Moreover, the corresponding embedding of k-admissible functions into L^p space is compact when p is below the critical exponent. As an application we give an L^∞ estimate for solutions to the k-Hessian equation (1.1) when $f \in L^p(\Omega)$ with $p > \frac{n}{2k}$ if $k \leq \frac{n}{2}$, or $p = 1$ if $k > \frac{n}{2}$.

In §6 we use the Sobolev type inequality (1.7) to study variational problems of the k-Hessian equation. We prove the existence of a min-max solution to the Hessian equation in the sub-critical and critical growth cases.

In §7 we present some local integral estimates. In particular we show that a k-admissible function belongs to $W^{1,p}_{loc}(\Omega)$ for any $p < \frac{nk}{n-k}$.

In §8 we extend the notion of k-admissible functions to nonsmooth functions; and prove that for any k-admissible function u, we can assign a measure $\mu_k[u]$ to u such that if a sequence of k-admissible functions $\{u_j\}$ converges to u almost everywhere, then $\mu_k[u_j]$ converges to $\mu_k[u]$ weakly as measures. As an application we prove the existence of weak solutions to the k-Hessian equation.

This weak continuity has many other applications as well, in particular it enables us to establish various potential theoretical results for k-admissible functions. In §9 we prove a Wolff potential estimate, and deduce a necessary and sufficient condition for a weak solution to be Hölder continuous.

In §10, we include some a priori estimates for the parabolic Hessian equations used in previous sections.

In the last Section 11, we give more examples of fully nonlinear elliptic equations.

Main references for this note are [CNS2, W2, CW1, TW2, Ld]. There are many other works on the k-Hessian equations. The materials in §2 and §3 are mostly taken from [CNS2], but for the key double normal derivative estimate we adapt the approach from [T1]. See also [I] for the k-Hessian equation for some k. The interior derivative estimates in §4 are from [CW1], but for the Monge-Ampère equation they were first established by Pogorelov [P]. The Sobolev type inequalities in §5 were proved by K.S. Chou for convex functions, and in [W2] for general k-admissible functions by a gradient flow method. The existence of min-max solutions in §6 was first obtained by K.S. Chou [Ch1] for the Monge-Ampère equation and later in [CW1] for $2 \le k \le \frac{n}{2}$. See also [W1] for the Monge-Ampère equation by a degree theory method, which also applies to the case $\frac{n}{2} < k < n$ by the embedding in Theorem 5.1. The local integral estimates in §7 and weak continuity in §8 can be found in [TW2]. The Wolff potential estimate and Hölder continuity of k-admissible solutions in §9 were proved in [Ld].

The result in §6.4 on the variational problem in the critical growth case was not published before, it was included in the preprint [CW2]. The proof of the weak continuity in §8, which uses ideas from [TW1,TW5], is different from that in [TW2]. As the reader will see below, most results in the note are generalization of the counterparts for the Poisson equation. But the study of fully nonlinear equations requires new techniques and is usually more complicated, in particular for estimates near the boundary. These results and techniques can also be used in other problems. See e.g., [FZ, KT, STW].

2. Admissible functions

2.1. **Admissible functions.** We say a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is k-*admissible* if

$$\lambda(D^2u) \in \overline{\Gamma}_k, \tag{2.1}$$

where Γ_k is an open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0 \ \forall\ j = 1, \cdots, k\}. \tag{2.2}$$

Clearly $\sigma_k(\lambda) = 0$ for $\lambda \in \partial\Gamma_k$,

$$\Gamma_n \subset \cdots \subset \Gamma_k \subset \cdots \subset \Gamma_1,$$

Γ_n is the positive cone,

$$\Gamma_n = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 > 0, \cdots, \lambda_n > 0\},$$

and Γ_1 is the half space $\{\lambda \in \mathbb{R}^n \mid \Sigma\lambda_i > 0\}$. A function is 1-admissible if and only if it is sub-harmonic, and an n-admissible function must be convex. For any $2 \le k \le n$, a k-admissible function is sub-harmonic, and the set of all k-admissible functions is a convex cone in $C^2(\Omega)$.

The cone Γ_k may also be equivalently defined as the component $\{\lambda \in \mathbb{R}^N \mid \sigma_k(\lambda) > 0\}$ containing the vector $(1, \cdots, 1)$, and characterized as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \bigm| 0 < \sigma_k(\lambda) \le \sigma_k(\lambda + \eta) \ \text{ for all } \eta_i \ge 0, \ \in \mathbb{R}\}. \tag{2.3}$$

We note that the k-Hessian operator S_k is also elliptic or degenerate elliptic if $\lambda(D^2u) \in -\overline{\Gamma}_k$. But by making the change $u \to -u$ it suffices to consider functions with eigenvalues $\lambda \in \Gamma_k$. In this note we consider functions with eigenvalues in Γ_k only.

2.2. **Admissible solution is elliptic.** We show that if u is k-admissible, the matrix

$$\{S_k^{ij}(A)\} = \{\frac{\partial}{\partial a_{ij}}\sigma_k(\lambda(A))\} \geq 0 \tag{2.4}$$

is positive semi-definite at $A = D^2u$ and so the k-Hessian operator is (degenerate) elliptic. To prove (2.4), note that the k-Hessian operator can also be written in the form

$$S_k[u] = [D^2u]_k, \tag{2.5}$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k^{th} principal minors. Therefore

$$S_k^{nn}[u] = [D^2u]'_{k-1}, \tag{2.6}$$

where $[D^2u]' = \{u_{x_ix_j}\}_{1\leq i,j\leq n-1}$. Denote

$$\bar{D}^2u = \begin{pmatrix} [D^2u]', & 0 \\ 0, & u_{nn} \end{pmatrix}.$$

One easily verifies that

$$[\bar{D}^2u]_m \geq [D^2u]_m \quad \forall \quad 1 \leq m \leq k.$$

Hence by (2.2), $\lambda(\bar{D}^2u) \in \overline{\Gamma}_k$. By (2.3) it follows that

$$S_k^{nn}[u] = [D^2u]'_{k-1} = \frac{\partial}{\partial\lambda_n}\sigma_k(\lambda) \geq 0 \quad (\lambda = \lambda(\bar{D}^2u)). \tag{2.7}$$

Note that (2.7) also holds after a rotation of coordinates, so the k-Hessian equation is (degenerate) elliptic if u is k-admissible.

When u is k-admissible, $S_k[u]$ is nonnegative. Therefore in our investigation of the k-Hessian equation, we always assume that f is nonnegative. If f is positive and $u \in C^2(\Omega)$, $S_k[u]$ is elliptic. Note that we allow that the eigenvalues $\lambda(D^2u)$ lie on the boundary of Γ_k, and in such case the k-Hessian equation may become degenerate elliptic.

2.3. **Concavity.** When u is k-admissible,

$$S_k^{1/k}[u] = \left[\sigma_k(\lambda(D^2u))\right]^{1/k},$$

is concave when regarded as a function of $r = D^2u$. In other words,

$$\sum a_{ij}a_{st}\,\partial^2_{u_{ij}u_{st}}S_k^{1/k}[u] \leq 0 \tag{2.8}$$

for any symmetric matrix $\{a_{ij}\}$. This property follows from the concavity of $\sigma_k^{1/k}(\lambda)$ in Γ_k (see (xii) in §2.5 below). Indeed, when u_{ij} is diagonal, one can verify (2.8) directly by the expression (2.5). When u_{ij} is not diagonal, by a rotation of coordinates $y_\alpha = c_{\alpha i}x_i$ such that $u_{\alpha\beta}$ is diagonal, one has

$$\sum a_{ij}a_{st}\,\partial^2_{u_{ij}u_{st}}S_k^{1/k}[u] = \sum a^*_{\alpha\beta}a^*_{\gamma\delta}\,\partial^2_{u_{\alpha\beta}u_{\gamma\delta}}S_k^{1/k}[u] \leq 0,$$

where $a^*_{\alpha\beta} = a_{ij}c_{\alpha i}c_{\beta j}$, subscripts i, j, s, t mean derivatives in x and subscripts $\alpha, \beta, \gamma, \delta$ mean derivatives in y. The concavity is needed in establishing the regularity of fully nonlinear elliptic equations.

2.4. **A geometric assumption on the boundary.** In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a geometric condition, that is

$$\sigma_{k-1}(\kappa) \geq c_0 > 0 \quad \text{on} \quad \partial\Omega \tag{2.9}$$

for some positive constant c_0, where $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. Indeed, let $u \in C^2(\overline{\Omega})$ be a k-admissible function which vanishes on $\partial\Omega$. For any fixed point $x_0 \in \partial\Omega$, by a translation and rotation of coordinates, we may assume that x_0 is the origin and locally $\partial\Omega$ is given by $x_n = \rho(x')$ such that $e_n = (0, \cdots, 0, 1)$ is the inner normal of $\partial\Omega$ at x_0, where $x' = (x_1, \cdots, x_{n-1})$. Differentiating the boundary condition $u(x', \rho(x')) = 0$, we get

$$u_{ij}(0) + u_n\rho_{ij}(0) = 0. \tag{2.10}$$

By our choice of coordinates, the principal curvatures of $\partial\Omega$ at x_0 are the eigenvalues of $\{\rho_{ij}(0)\}_{1\leq i,j\leq n-1}$. When u is k-admissible, it is subharmonic and so $u_n(x_0) < 0$. We obtain

$$S_k^{nn}[u] = |u_n|^{k-1}\sigma_{k-1}(\kappa). \tag{2.11}$$

Hence (2.9) follows from (2.4) provided $\lambda(D^2u) \in \Gamma_k$.

In this note we call a domain whose boundary satisfies (2.9) $(k-1)$-*convex*. When $k = n$, it is equivalent to the usual convexity. In the following we always assume that Ω is $(k-1)$-convex.

If Ω is $(k-1)$-convex, then for any smooth function φ on $\partial\Omega$, there is a function $\underline{u}$, which is k-admissible in a neighborhood of $\partial\Omega$ and satisfies $\underline{u} = \varphi$ on $\partial\Omega$. Indeed, if $\varphi = 0$, let $\underline{u}(x) = -d_x + td_x^2$, where $x \in \Omega$ and d_x is distance from x to $\partial\Omega$. Then $\underline{u}$ is k-admissible near $\partial\Omega$ provided t is sufficiently large. We refer the reader to [GT] for the computation of the second derivatives of the distance function. For a general boundary value φ, extend φ to Ω such that it is harmonic in Ω. Then $\varphi + \sigma\underline{u}$ is k-admissible near $\partial\Omega$ for large σ, and $S_k[\varphi + \sigma\underline{u}]$ can be as large as we want provided σ is sufficiently large.

Note that the function $\underline{u}$ is defined only in a neighborhood of $\partial\Omega$. But it suffices for the a priori estimates in §3. By the existence of solutions to the Dirichlet problem (Theorem 3.4), there is a k-admissible function u defined in the whole domain Ω such that $u = \varphi$ on $\partial\Omega$.

2.5. **Some algebraic inequalities.** We collect some inequalities related to the polynomial $\sigma_k(\lambda)$, which are needed in our investigation of the k-Hessian equation.

Denote $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$. Assume $\lambda \in \Gamma_k$. Arrange $\lambda = (\lambda_1, \cdots, \lambda_n)$ in descending order, namely $\lambda_1 \geq \cdots \geq \lambda_n$. Denote $\sigma_{k;i} =$

$\sigma_k(\lambda)_{|\lambda_i=0}$, so that $\frac{\partial}{\partial\lambda_i}\sigma_k(\lambda) = \sigma_{k-1,i}(\lambda)$. The following ones are easy to verify

$$\begin{aligned}
&(i) \quad \sigma_k(\lambda) = \sigma_{k;i}(\lambda) + \lambda_i\sigma_{k-1;i}(\lambda),\\
&(ii) \quad \sum\nolimits_{i=1}^{n} \sigma_{k;i}(\lambda) = (n-k)\sigma_k(\lambda),\\
&(iii) \quad \sigma_{k-1,n}(\lambda) \geq \cdots \geq \sigma_{k-1,1}(\lambda) > 0,\\
&(iv) \quad \lambda_k \geq 0 \quad \text{and} \quad \sigma_k(\lambda) \leq C_{n,k}\lambda_1\cdots\lambda_k.
\end{aligned}$$

We also have

$$\begin{aligned}
&(v) \quad \sigma_k(\lambda)\sigma_{k-2}(\lambda) \leq C_{n,k}[\sigma_{k-1}(\lambda)]^2,\\
&(vi) \quad \sigma_k(\lambda) \leq C_{n,k}[\sigma_l(\lambda)]^{k/l}, \quad 1 \leq l < k.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&(vii) \quad \lambda_1\sigma_{k-1,1}(\lambda) \geq C_{n,k}\sigma_k(\lambda).\\
&(viii) \quad \sigma_{k-1;k}(\lambda) \geq C_{n,k}\sum\nolimits_{i=1}^{n}\sigma_{k-1;i}(\lambda),\\
&(ix) \quad \sigma_{k-1;k}(\lambda) \geq C_{n,k}\sigma_{k-1}(\lambda),\\
&(x) \quad \prod\nolimits_{i=1}^{n}\sigma_{k;i}(\lambda) \geq C_{n,k}[\sigma_k(\lambda)]^{n(k-1)/k}.
\end{aligned}$$

In the above the constant $C_{n,k}$ may change from line to line. There are more inequalities useful in the study of the k-Hessian equation. For example, we have

$$\begin{aligned}
&(xi) \quad \sum \mu_i\sigma_{k-1,i} \geq k[\sigma_k(\mu)]^{1/k}[\sigma_k(\lambda)]^{1-1/k} \quad \forall\ \lambda, \mu \in \Gamma_k,\\
&(xii) \quad \{\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\sigma_k(\lambda)\} \leq 0 \quad \forall\ \lambda \in \Gamma_k.
\end{aligned}$$

the last inequality means that $\sigma_k^{1/k}(\lambda)$ is concave in Γ_k. We refer the reader to [CNS2, LT, Lg] for these and more inequalities related to σ_k.

3. The Dirichlet problem

In this section we study the existence and regularity of solutions to the Dirichlet problem of the k-Hessian equation,

$$\begin{aligned}
S_k[u] &= f(x) \quad \text{in } \Omega,\\
u &= \varphi \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.1}$$

where Ω is a bounded, $(k-1)$-convex domain in $\mathbb{R}^n$ with $C^{3,1}$ boundary, $\varphi \in C^{3,1}(\partial\Omega)$, $f \geq 0$, $f \in C^{1,1}(\overline{\Omega})$.

3.1. **A priori estimates.** First we establish the global estimate for the second derivatives.

Theorem 3.1 (CNS2, T1)**.** *Let $u \in C^{3,1}(\overline{\Omega})$ be a k-admissible solution to the Dirichlet problem (3.1). Assume that Ω is $(k-1)$-convex, $\partial\Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\partial\Omega)$, $f \geq f_0 > 0$, and $f^{1/k} \in C^{1,1}(\overline{\Omega})$. Then we have the a priori estimate*

$$\|u\|_{C^{1,1}(\overline{\Omega})} \leq C, \tag{3.2}$$

where C depends only on n, k, Ω, f_0, $\|\varphi\|_{C^{3,1}(\partial\Omega)}$ and $\|f\|_{C^{1,1}(\overline{\Omega})}$.

Proof. First consider the L^∞ estimate. Let $w = \frac{1}{2}a|x|^2 - b$, where the constants a, b are chosen large such that $S_k[w] > f$ in Ω and $w \leq \varphi$ on $\partial\Omega$. Then $w - u$ satisfies the elliptic equation $\sum a_{ij}(w-u)_{ij} > 0$ in Ω and $w - u \leq 0$ on $\partial\Omega$, where $a_{ij} = \int_0^1 S_k^{ij}[u + t(w-u)]dt$. It follows that $w \leq u$ in Ω. Extend φ to Ω such that it is harmonic. By the comparison principle we have $w \leq u \leq \varphi$ in Ω.

Next consider the gradient estimate. Denote $F[u] = S_k^{1/k}[u]$, $\hat{f} = f^{1/k}$. Differentiating the equation

$$F[u] = \hat{f} \tag{3.3}$$

in direction x_l, one obtains

$$L[u_l] = \hat{f}_l,$$

where $L = F_{ij}\partial_{ij}$ is the linearized equation of F, $F_{ij} = F_{u_{ij}}$. So $|L[u_l]| \leq C$. Let $w = \frac{1}{2}a|x|^2$. By (ii) and (vi) above, $L[w] \geq c_1 a > 0$ for some positive constant $c_1 > 0$ depends only on n, k. Hence $L[w \pm u_l] \geq 0$, provided a is chosen suitably large. It follows that $w \pm u_l$ attains its maximum on the boundary $\partial\Omega$. Hence

$$\sup_{x\in\Omega} |Du(x)| \leq C(1 + \sup_{x\in\partial\Omega} |Du(x)|). \tag{3.4}$$

Next let $\hat{w} = \varphi + \sigma\underline{u}$ be the function in §2.4. Denote $\mathcal{N} = \{x \in \Omega \mid \hat{w}(x) > w(x)\}$. Then when σ is sufficiently large, $\mathcal{N}$ is a neighborhood of $\partial\Omega$, and $S_k[\hat{w}] > f$ in $\mathcal{N}$. Therefore by the comparison principle, $\hat{w} \leq u \leq \varphi$ in $\mathcal{N}$. Hence by the boundary condition $\hat{w} = u = \varphi$ on $\partial\Omega$, we infer that $\partial_\gamma\varphi \leq \partial_\gamma u \leq \partial_\gamma\hat{w}$, where γ is the unit outer normal to $\partial\Omega$. Hence Du is bounded on $\partial\Omega$.

Finally consider the second derivative estimate. Since u is sub-harmonic, it suffices to prove that $u_{\xi\xi} \leq C$ for any unit vector ξ. Differentiating equation (3.3) twice in direction ξ, we obtain, by the concavity of F,

$$L[u_{\xi\xi}] \geq \hat{f}_{\xi\xi}.$$

Hence $L[Cw + u_{\xi\xi}] \geq 0$ for a suitably large constant C and so

$$\sup_{\Omega} u_{\xi\xi} \leq C + \sup_{\partial\Omega} u_{\xi\xi}. \tag{3.5}$$

Therefore we reduce the estimate to the boundary.

For any given boundary point $x_0 \in \partial\Omega$, by a translation and a rotation of the coordinates we assume that x_0 is the origin and locally $\partial\Omega$ is given by

$$x_n = \rho(x') \tag{3.6}$$

such that $D\rho(0) = 0$, where $x' = (x_1, \cdots, x_{n-1})$. Differentiating the boundary condition $u = \varphi$ on $\partial\Omega$ twice, we have, for $1 \le i, j \le n-1$,

$$u_{ij}(0) + u_n(0)\rho_{ij}(0) = \varphi_{ij}(0) + \varphi_n(0)\rho_{ij}(0). \tag{3.7}$$

Hence

$$|D_{ij}u(0)| \le C \quad i, j \le n-1. \tag{3.8}$$

Next we establish

$$|u_{in}(0)| \le C \quad i < n. \tag{3.9}$$

By a rotation of the $x_1, \cdots, x_{n-1}$ axes, we assume that $x_1, \cdots, x_{n-1}$ are the principal directions of $\partial\Omega$ at the origin. Let $T = \partial_i + \kappa_i(0)(x_i\partial_n - x_n\partial_i)$, where κ_i is the principal curvature of $\partial\Omega$ in direction x_i, $1 \le i \le n-1$. One can verify that

$$|T(u-\varphi)| \le C|x'|^2|\partial_\gamma(u-\varphi)| \le C|x'|^2| \ \text{ on } \partial\Omega.$$

Next observing that S_k is invariant under rotation of coordinates and $(x_i\partial_n - x_n\partial_i)$ is an infinitesimal generator of a rotation, we have $TF[u] = L[T(u)]$. Hence

$$|L(T(u-\varphi))| \le C(1 + \Sigma_i F_{ii}).$$

Let

$$w = \rho(x') - x_n - \delta|x'|^2 + Kx_n^2, \tag{3.10}$$

where $K > 1$ large and $\delta > 0$ small are constants. By the assumption that Ω is $(k-1)$-convex, the function w is k-admissible in $B_\varepsilon(0) \cap \Omega$ for small $\varepsilon > 0$. By the concavity of F,

$$\begin{aligned} L[w] &\ge F[u+w] - F[u] \ge F[w] - F[u] \\ &\ge c_1 K^{1/k} - C \ge \frac{1}{2} c_1 K^{1/k} \end{aligned}$$

for some constants c_1 depending on n, k, and δ, provided K is sufficiently large. Choose a K' large such that $L[K'w \pm T(u-\varphi)] \ge 0$. It follows that the maximum of $K'w \pm T(u-\varphi)$ in $B_\varepsilon \cap \Omega$ is attained on the boundary $\partial(B_\varepsilon \cap \Omega)$. But on the boundary $\partial(B_\varepsilon \cap \Omega)$, it is easy to see that

$$\begin{aligned} &w \le -\frac{1}{2}\delta|x'|^2 \ \text{ on } \ \partial\Omega \cap B_\varepsilon(0), \\ &w < 0 \ \text{ on } \ \Omega \cap \partial B_\varepsilon(0). \end{aligned}$$

Hence $K'w \pm T(u-\varphi) \le 0$ provided K' is chosen large enough. Hence $K'w \pm T(u-\varphi)$ attains its maximum 0 at the origin and we obtain

$$|\partial_n(T(u-\varphi))| \le K'|\partial_n w| \le C,$$

from which (3.9) follows.

Finally we consider the double normal derivative estimate

$$u_{nn}(0) \le C. \tag{3.11}$$

If $\varphi = 0$, by (3.7) we have $u_{ij}(0) = (-u_n)\rho_{ij}$. By the geometric assumption (2.9), we have

$$S_k^{nn}[u] = \sigma_{k-1}[\lambda(D^2u)'] = |u_n|^{k-1}\sigma_{k-1}(\kappa) > 0,$$

where $(D^2u)' = (u_{ij})_{1\le i,j\le n-1}$. Note that

$$S_k[u] = u_{nn}\sigma_{k-1}\{\lambda[(D^2u)']\} + R = f, \tag{3.12}$$

where R is the rest terms which do not involve u_{nn}, and so is bounded by (3.8) and (3.9). Hence $u_{nn}(0)$ must be bounded.

For general boundary function φ, we adapt the approach from [T1]. By (3.12) it suffices to prove $\sigma_{k-1}\{\lambda[(D^2u)']\} > 0$ on $\partial\Omega$. For any boundary point $x \in \partial\Omega$, let $\xi^{(1)}, \cdots, \xi^{(n-1)}$ be an orthogonal vector field on $\partial\Omega$. Denote $\nabla_i = \xi^{(i)}_m D_m u$,

$$\nabla_{ij}u = \xi^{(i)}_m \xi^{(j)}_l D_{ml}u, \quad \mathcal{C}_{ij} = \xi^{(i)}_m \xi^{(j)}_l D_m \gamma_l,$$

and $\nabla^2 u = \{\nabla_{ij}u\}$, $\mathcal{C} = \{\mathcal{C}_{ij}\}$, where γ is the unit inner normal of $\partial\Omega$ at x. Then we have

$$\lambda[(D^2u)'] = \lambda[\nabla^2 u](x).$$

Similar to (3.7) we have

$$\nabla^2 u = D_\gamma(u-\varphi)\mathcal{C} + \nabla^2\varphi. \tag{3.13}$$

For any $(n-1)\times(n-1)$-matrix r with eigenvalues $(\lambda_1, \cdots, \lambda_{n-1})$, denote

$$G(r) = [\sigma_{k-1}(\lambda)]^{1/(k-1)}.$$

and $G^{ij} = \frac{\partial G}{\partial r_{ij}}$. Assume that $\inf_{x\in\partial\Omega} G(\nabla^2 u)$ is attained at x_0. Then by (3.13) and the concavity of G,

$$G_0^{ij}[D_\gamma(u-\varphi)\mathcal{C}_{ij}(x) + \nabla_{ij}\varphi(x)] \ge G_0^{ij}[D_\gamma(u-\varphi)\mathcal{C}_{ij}(x_0) + \nabla_{ij}\varphi(x_0)]$$

for any $x \in \partial\Omega$, where $G_0^{ij} = G^{ij}(\nabla^2 u(x_0))$. We can also write the above formula in the form

$$\begin{aligned} &G_0^{ij}\mathcal{C}_{ij}(x_0)[D_\gamma(u-\varphi)(x) - D_\gamma(u-\varphi)(x_0)] \\ &\quad \ge G_0^{ij}\{[D_\gamma(u-\varphi)(x) - D_\gamma(u-\varphi)(x_0)][\mathcal{C}_{ij}(x_0) - \mathcal{C}_{ij}(x)] \\ &\quad +D_\gamma(u-\varphi)(x_0)][\mathcal{C}_{ij}(x_0) - \mathcal{C}_{ij}(x)] - [\nabla_{ij}\varphi(x) - \nabla_{ij}\varphi(x_0)]\} \end{aligned}$$

Assume that near x_0, $\partial\Omega$ is given by (3.6) with

$$\rho(x') = \frac{1}{2}\sum_{i=1}^{n-1}\kappa_i x_i^2 + O(|x'|^3).$$

Then we have $\mathcal{C}_{ij}(x_0) = \partial_i \gamma_j = \kappa_i \delta_{ij}$. Recall that Ω is $(k-1)$-convex. The eigenvalues of $\{\mathcal{C}_{ij} - c_1\delta_{ij}\}$ (as a vector in $\mathbb{R}^{n-1}$) lies in Γ_{k-1}, provided c_1 is sufficiently small. Hence $G_0^{ij}(\mathcal{C}_{ij} - c_1\delta_{ij}) \ge 0$ at x_0, and so

$$G_0^{ij}\mathcal{C}_{ij}(x_0) \ge c_1 \sum G_0^{ii} \ge \delta_0 > 0.$$

Therefore we obtain

$$D_n(u-\varphi)(x) - D_n(u-\varphi)(x_0) \le \ell(x') + C|x'|^2,$$

where ℓ is a linear function of x' with $\ell(0) = 0$. Denote

$$v(x) = D_n(u-\varphi)(x) - D_n(u-\varphi)(x_0) - \ell(x').$$

We have

$$v(x) \le C|x'|^2 \quad \forall\, x \in \partial\Omega. \tag{3.14}$$

Differentiating equation (3.3) we have

$$|L(v)| \le C(1 + \sum F^{ii}), \tag{3.15}$$

where $L = \sum F^{ij}\partial_{ij}$ is the linearized operator of F.

Let w be the function given in (3.10). Then by (3.15) we can choose K' sufficiently large such that $L(K'w) \ge \pm L(v)$ in $B_\varepsilon \cap \Omega$. By (3.14), we can also choose K' large such that $K'w + v \le 0$ on $\partial(B_\varepsilon \cap \Omega)$. By the comparison principle it follows that $K'w + v \le 0$ in $B_\varepsilon \cap \Omega$. Hence $K'w + v$ attains its maximum at x_0. We obtain $\partial_n(K'w + v) \le 0$ at x_0, namely $u_{nn}(x_0) \le C$.

To complete the proof, one observes that in (3.12),

$$R = -\sum u_{1i}^2 \frac{\partial^2}{\partial u_{11}\partial u_{ii}} S_k[u] \le 0.$$

Hence

$$\sigma_{k-1}\{\lambda[(D^2u)']\}(x_0) \ge \frac{f}{u_{nn}}(x_0) \ge \frac{f_0}{u_{nn}(x_0)}. \tag{3.16}$$

Recall that $\sigma_{k-1}\{\lambda[(D^2u)']\}$ attains its minimum at x_0. Hence by (3.12) we obtain $u_{\gamma\gamma}(x) < C$ at any boundary point $x \in \partial\Omega$. □

By the a priori estimate (3.2), equation (3.1) becomes uniformly elliptic if f is strictly positive. The uniform ellipticity follows from inequality (iii) in §2.5. To get the higher order derivative estimates, we employ the regularity theory of fully nonlinear, uniformly elliptic equations.

3.2. **Regularity for fully nonlinear, uniformly elliptic equation.** We say a fully nonlinear elliptic operator F is *concave* if F, as a function of $r = D^2u$, is a concave function. From §2.3, the k-Hessian equation is concave when u is k-admissible and the equation is written in the form (3.3).

The regularity theory of fully nonlinear elliptic equations was established by Evans and Krylov independently. Their proof is based on Krylov-Safonov's Hölder estimates for linear, uniformly elliptic equation of non-divergent form.

Theorem 3.2. *Consider the fully nonlinear, uniformly elliptic equation*

$$\begin{aligned} F(D^2u) &= f(x) \quad \text{in } \Omega. \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{3.17}$$

Suppose F is concave, $F \in C^{1,1}$, $f \in C^{1,1}(\Omega)$, and $u \in W^{4,n}(\Omega)$ is a solution of (3.17). Then there exists $\alpha \in (0,1)$ depending only on n, λ, Λ (the constants in (1.6)) such that for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C, \tag{3.18}$$

where C depends only on $n, \lambda, \Lambda, \alpha, \Omega$, $dist(\Omega', \partial\Omega)$, $\|f\|_{C^{1,1}(\Omega)}$, and $\sup_\Omega |u|$.

If furthermore $\varphi \in C^{3,1}(\overline{\Omega})$, $\partial\Omega \in C^{3,1}$, and $f \in C^{1,1}(\overline{\Omega})$, then

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le C, \tag{3.19}$$

where C depends only on $n, \lambda, \Lambda, \alpha, \partial\Omega$, $\|f\|_{C^{1,1}(\Omega)}$, $\|\varphi\|_{C^{3,1}(\Omega)}$ and $\sup_\Omega |u|$.

From (3.19) one also obtains $C^{3,\alpha}$ estimates by differentiating the equation (3.17) and apply the Schauder theory for linear, uniformly elliptic equations. Theorem 3.2 also extends to more general equations of the form (1.1) provided F satisfies certain structural conditions. We refer the readers to [E, K1, GT] for details.

As a corollary of Theorem 3.2, we obtain the higher order derivative estimate for the k-Hessian equation.

Theorem 3.3. *Let $u \in C^{3,1}(\overline{\Omega})$ be a k-admissible solution of (3.1). Assume that Ω is $(k-1)$-convex, $f \in C^{1,1}(\overline{\Omega})$, and $f \ge f_0 > 0$ in Ω. Then we have*

$$\|u\|_{C^{3,\alpha}(\overline{\Omega})} \le C, \tag{3.20}$$

where $\alpha \in (0,1)$, C depends only on n, k, α, f_0, Ω, $\|\varphi\|_{C^{3,1}(\partial\Omega)}$, and $\|f\|_{C^{1,1}(\overline{\Omega})}$.

3.3. **Existence of smooth solutions.** By Theorem 3.3 and the continuity method, we obtain the existence of smooth solutions to the Dirichlet problem (3.1).

Theorem 3.4. *Assume that Ω is $(k-1)$-convex, $\partial\Omega \in C^{3,1}$, $f \in C^{1,1}(\overline{\Omega})$, and $f \ge f_0 > 0$. Then there is a unique k-admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ to the Dirichlet problem (3.1).*

Proof. We apply the continuity method to the Dirichlet problem

$$\begin{aligned} S_k[u_t] &= f_t \ \text{ in } \ \Omega, \\ u_t &= \varphi_t \ \text{ on } \ \partial\Omega, \end{aligned}$$

where $t \in [0,1]$, $f_t = C_n^k(1-t) + tf$, $\varphi_t = \frac{1-t}{2}|x|^2 + t\varphi$. Then when $t = 0$, $u_0 = \frac{1}{2}|x|^2$ is the solution to the above Dirichlet problem at $t = 0$. To apply the continuity method, we consider solution $u = v + \varphi_t$ so that $v \in C^{3+\alpha}(\overline{\Omega})$ with $v = 0$ on $\partial\Omega$. Note that the uniqueness of k-admissible solutions follows from the comparison principle. □

3.4. **Remarks.** .
(i) In the proof of Theorem 3.1, the assumption $f \geq f_0$ was used only once in (3.16). Therefore this assumption can be relaxed to $f \geq 0$ for the zero boundary value problem. By approximation and Theorems 3.1 and 3.4, it follows that there is a k-admissible solution $u \in C^{1,1}(\overline{\Omega})$ to the k-Hessian equation (2.1) which vanishes on $\partial\Omega$, provided Ω is $(k-1)$-convex and $f^{1/k} \in C^{1,1}(\overline{\Omega})$, $f \geq 0$.

The above results are also true for a general boundary function $\varphi \in C^{3,1}(\partial\Omega)$. Indeed Krylov [K2] established the a priori estimate (3.2), not only for solutions to the k-Hessian equation, but also for solutions to the Dirichlet problem (3.17) for general functions $\varphi \in C^{3,1}(\overline{\Omega})$, provided $f \geq 0$ and $f \in C^{1,1}(\partial\Omega)$. The main difficulty is again the estimation on the boundary. For the k-Hessian equation, Krylov's proof was simplified in [ITW].

We also note that the geometric assumption (2.6) can be replaced by the existence of a subsolution $\underline{u}$ to (3.1) with $\underline{u} = \varphi$ [G].
(ii) The estimate (3.2) also extends to the Hessian quotient equation [T1]

$$S_{k,l}[u] = \frac{S_k[u]}{S_l[u]} = f, \tag{3.21}$$

where $0 \leq l < k \leq n$ and we define $S_l[u] = 1$ when $l = 0$.
(iii) For the second boundary value problem of the k-Hessian equation, and some other boundary value problems, we refer the reader to [J,S,U3]
(iv) Much more can be said about the regularity of the Monge-Ampère equation. The interior regularity was established by Calabi and Pogorelov [GT, P]. The global regularity for the Dirichlet problem was obtained independently by Caffarelli, Nirenberg and Spruck [CNS1], and by Krylov [K1], assuming all data are smooth enough. Caffarelli [Ca] established the interior $C^{2,\alpha}$ and $W^{2,p}$ estimates for strictly convex solutions, assuming that $f \in C^\alpha$ and $f \in C^0$, respectively. The continuity of f is also necessary for the $W^{2,p}$ estimate [W3].

The boundary $C^{2,\alpha}$ estimate for the Dirichlet problem was established in [TW6], assuming that $f > 0, \in C^\alpha(\overline{\Omega})$, the boundary $\partial\Omega$ is uniformly convex and C^3 smooth, and boundary function $\varphi \in C^3$. If either $\partial\Omega$ or φ is only $C^{2,1}$, the solution may not belong to $W^{2,p}(\Omega)$ for large p, even f is a positive constant.

4. Interior a priori estimates

In this section we establish interior gradient and second derivative estimates for the k-Hessian equation

$$S_k[u] = f(x, u). \tag{4.1}$$

These estimates were previously proved in [CW1]. The interior gradient estimate was also established in [T2]. From the interior gradient estimate, we also deduce a Harnack inequality. Estimates in this section will be repeatedly used in subsequent sections.

4.1. **Interior gradient estimate.**

Theorem 4.1. *Let $u \in C^3(B_r(0))$ be a k-admissible solution of (4.1). Suppose that $f \geq 0$ and f is Lipschitz continuous. Then*

$$|Du(0)| \leq C_1 + C_2 \frac{M}{r}, \tag{4.2}$$

where $M = 4 \sup |u|$, C_2 is a constant depending only on n, k; C_1 depends on n, k, M, r and $\|f\|_{C^{0,1}}$. Moreover, if f is a constant, then $C_1 = 0$.

Proof. Introduce an auxiliary function

$$G(x, \xi) = u_\xi(x)\varphi(u)\rho(x),$$

where $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$, $\varphi(u) = 1/(M - u)^{1/2}$, and $M = 4 \sup |u|$. Suppose G attains its maximum at $x = x_0$ and $\xi = e_1$, the unit vector in the x_1 axis. Then at x_0, $G_i = 0$ and $\{G_{ij}\} \leq 0$. That is

$$u_{1i} = -\frac{u_1}{\varphi\rho}(u_i\varphi'\rho + \varphi\rho_i), \tag{4.3}$$

$$\begin{aligned} 0 \geq S_k^{ij} G_{ij} &= \varphi\rho\partial_1 f + k u_1 f \varphi'\rho + u_1\varphi''\rho S_k^{ij} u_i u_j + u_1 \varphi S_k^{ij} \rho_{ij} \\ &\quad + u_1\varphi' S_k^{ij}(u_i\rho_j + u_j\rho_i) + 2 S_k^{ij} u_{1i}(u_j\varphi'\rho + \varphi\rho_j) \\ &= \varphi\rho\partial_1 f + k u_1 f\varphi'\rho + u_1\rho(\varphi'' - \frac{2\varphi'^2}{\varphi}) S_k^{ij} u_i u_j + u_1\varphi S_k^{ij}\rho_{ij} \\ &\quad - u_1\varphi' S_k^{ij}(u_i\rho_j + u_j\rho_i) - \frac{2u_1\varphi}{\rho} S_k^{ij}\rho_i\rho_j, \end{aligned} \tag{4.4}$$

where we used the relations $S_k^{ij} u_{ij} = kf$ and $S_k^{ij} u_{ij1} = \partial_1 f$, which follows by differentiating equation (4.1).

By our choice of φ, $\varphi'' - \frac{2\varphi'^2}{\varphi} \geq \frac{1}{16}M^{-5/2}$. Denote $\mathcal{S} = \Sigma_i S_k^{ii}$. Note that the term $ku_1 f\varphi'\rho$ is nonnegative. From (4.4) we obtain

$$0 \geq -16M^{5/2}\varphi\rho|\partial_1 f| + \rho S_k^{11}u_1^3 - C\mathcal{S}(\frac{M^2}{\rho r^2}u_1 + \frac{M}{r}u_1^2), \tag{4.5}$$

where C is independent of r, M. To prove (4.2), we assume that $|Du(0)| > CM/r$, otherwise we are through. Then by $G(x_0) \geq G(0)$, we have $u_1\rho(x_0) > CM/r$. Hence by (4.3) we have

$$u_{11} \leq -\frac{\varphi'}{2\varphi}u_1^2 \quad \text{at } x_0. \tag{4.6}$$

Hence by (ix) above, $S_k^{11} \geq C\mathcal{S}$.

To control $\partial_1 f$ by $\mathcal{S}$, by a rotation of the coordinates, we assume that D^2u is diagonal in the new coordinates y, and $u_{y_1y_1} \geq \cdots \geq u_{y_ny_n}$. Then at the point x_0 where G reaches its maximum,

$$u_{y_ny_n} \leq u_{x_1x_1} \leq -\frac{\varphi'}{2\varphi}u_{x_1}^2 \leq -\frac{1}{4M}u_{x_1}^2$$

by (4.6). From equation (4.1),

$$f = u_{y_ny_n}\sigma_{k-1;n}(\lambda) + \sigma_{k;n}(\lambda), \quad \lambda = \lambda(D^2u).$$

By §2 (vi), we obtain

$$0 \leq u_{y_ny_n}\sigma_{k-1;n}(\lambda) + C[\sigma_{k-1;n}(\lambda)]^{k/(k-1)}.$$

Hence

$$\sigma_{k-1;n}(\lambda) \geq C|u_{y_ny_n}|^{k-1} \geq Cu_{x_1}^{2k-2}.$$

We obtain

$$\mathcal{S} \geq Cu_1^{2k-2} \geq C\frac{u_{x_1}^{2k-2}}{M^{k-1}} \quad \text{at } x_0.$$

Recall that in (4.6), we assumed that $u_{x_1} \geq CM/r$. Hence $\mathcal{S} \geq CM^{k-1}/r^{2k-2}$ and $\mathcal{S}^{-1}|\partial_1 f|$ is bounded. Multiplying (4.5) by $\rho^2/\mathcal{S}$, we obtain (4.2). □

4.2. **Harnack Inequality.** From the interior gradient estimate, we obtain a Harnack inequality for the k-Hessian equation. First we prove a lemma, which also follows from the interpolation inequality (2.12) in [TW2].

Lemma 4.1. *Suppose $u \in C^1(B_R(0))$ is a function which satisfies for any $B_r(x) \subset B_R(0)$,*

$$|Du(x)| \leq \frac{C_1}{r}\sup_{B_r(x)}|u|. \tag{4.7}$$

Then

$$|u(0)| \leq \frac{C_2}{|B_R|}\int_{B_R}|u|, \tag{4.8}$$

where C_2 depends only on C_1 and n.

Proof. There is no loss of generality in assuming that $R = 1$, $\int_{B_1} |u| = 1$, and u is a C^1 function defined in $B_{1+\varepsilon}(0)$ for some small $\varepsilon > 0$. Let K be the largest constant such that $|u(x)| \geq K(1-|x|)^{-n}$ for some $x \in B_1(0)$, namely $K = \sup(1-|x|)^n|u(x)|$. Choose $y \in B_1(0)$ such that $|u(y)| = K(1-|y|)^{-n}$ and $|y| = \sup\{|x| \in B_1(0) \mid |u(x)| = K(1-|x|)^{-n}\}$. Then we have $|u| \leq 2^n|u(y)| = Kr^{-n}$ in $B_r(y)$, where $r = \frac{1}{2}(1-|y|)$. Therefore by applying the interior gradient estimate to u in $B_r(y)$, we get $|Du(x)| \leq CKr^{-n-1}$. Hence $|u(x)| > \frac{1}{2}Kr^{-n}$ whenever $|x-y| \leq r/2C$. It follows that $\int_{B_r(y)} |u| \geq K/C$. But by assumption, $\int_{B_R} |u| \leq 1$, we obtain an upper bound for K and Lemma 4.1 follows. □

Theorem 4.2. *Let u be a non-positive, k-admissible solution to*

$$S_k[u] = c \quad in \quad B_R(0), \tag{4.9}$$

where $c \geq 0$ is a constant. Then we have

$$\sup_{B_{R/2}(0)} (-u) \leq C \inf_{B_{R/2}(0)} (-u), \tag{4.10}$$

where C depends only on n, k.

Proof. By Lemma 4.1,

$$\sup_{B_{R/2}} (-u) \leq C \int_{3R/4} (-u).$$

Since u is subharmonic, we have [GT]

$$\int_{3R/4} (-u) \leq C \inf_{B_{R/2}} (-u).$$

From the above two inequalities we obtain (4.10). □

The interior gradient estimate also implies the following Liouville Theorem.

Corollary 4.1. *Let $u \in C^3(\mathbb{R}^n)$ be an entire solution to $S_k[u] = 0$. If $u(x) = o(|x|)$ for large x, then $u \equiv$constant.*

4.3. **Interior second derivative estimate.**

Theorem 4.3. *Let $u \in C^4(\Omega)$ be a k-admissible solution of (4.1). Suppose $f \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$ and $f \geq f_0 > 0$. Suppose there is a k-admissible function w such that*

$$w > u \quad in \quad \Omega, \quad and \quad w = u \quad on \quad \partial\Omega. \tag{4.11}$$

Then

$$(w-u)^4(x)|D^2u(x)| \leq C, \tag{4.12}$$

where C depends only on n, k, f_0, $\sup_\Omega(|Dw| + |Du|)$, and $\|f\|_{C^{1,1}(\overline{\Omega})}$.

Proof. Writing equation (4.1) in the form

$$F[u] = \hat{f},$$

where $\hat{f} = f^{1/k}(x,u)$, and differentiating twice, we get

$$F_{ii}u_{ii\mathfrak{g}\mathfrak{g}} + (F_{ij})_{rs}u_{ij\mathfrak{g}}u_{rs\mathfrak{g}} = \hat{f}_{\mathfrak{g}\mathfrak{g}}.$$

Suppose (D^2u) is diagonal. Then

$$(F_{ij})_{rs} = \begin{cases} \mu'\sigma_{k-2;ir}(\lambda) + \mu''\sigma_{k-1;i}\sigma_{k-1;r} & \text{if } i=j, r=s, \\ -\mu'\sigma_{k-2;ij}(\lambda) & \text{if } i \neq j, r=j, \text{and } s=i, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu(t) = t^{1/k}$. Hence

(4.13)

$$F_{ii}u_{ii\mathfrak{g}\mathfrak{g}} = \hat{f}_{\mathfrak{g}\mathfrak{g}} + \sum_{i,j=1}^{n}\mu'\sigma_{k-2;ij}u_{ij\mathfrak{g}}^2 - \sum_{i,j=1}^{n}[\mu''\sigma_{k-1;i}\sigma_{k-1;j} + \mu'\sigma_{k-2;ij}]u_{ii\mathfrak{g}}u_{jj\mathfrak{g}}.$$

Let

$$G(x) = \rho^\beta(x)\varphi(\frac{1}{2}|Du|^2)u_{\xi\xi},$$

where $\rho = w-u$, $\beta = 4$, $\varphi(t) = (1-\frac{t}{M})^{-1/8}$, and $M = 2\sup_{x\in\Omega}|Du|^2$. Suppose G attains its maximum at x_0 and in the direction $\xi = (1,0,\cdots,0)$. By a rotation of axes we assume that D^2u is diagonal at x_0 with $u_{11} \geq \cdots \geq u_{nn}$. Then at x_0,

$$0 = (\log G)_i = \beta\frac{\rho_i}{\rho} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}, \tag{4.14}$$

$$0 \geq F_{ii}(\log G)_{ii} = \beta F_{ii}[\frac{\rho_{ii}}{\rho} - \frac{\rho_i^2}{\rho^2}] + F_{ii}[\frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2}] + F_{ii}[\frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}]. \tag{4.15}$$

Case 1: $u_{kk} \geq \varepsilon u_{11}$ for some $\varepsilon > 0$.

By (4.14) we have

$$\frac{u_{11i}}{u_{11}} = -(\frac{\varphi_i}{\varphi} + \beta\frac{\rho_i}{\rho}). \tag{4.16}$$

Hence by (4.15),

$$0 \geq \beta F_{ii}[\frac{\rho_{ii}}{\rho} - (1+2\beta)\frac{\rho_i^2}{\rho^2}] + F_{ii}[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}] + F_{ii}\frac{u_{11ii}}{u_{11}}. \tag{4.17}$$

By the concavity of F,

$$F_{ii}u_{11ii} \geq \hat{f}_{11} \geq -C(1+u_{11}).$$

We have

$$\begin{aligned} F_{ii}[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}] &= (\frac{\varphi''}{\varphi} - 3\frac{\varphi'^2}{\varphi^2})F_{ii}u_i^2u_{ii}^2 + \frac{\varphi'}{\varphi}u_{\mathfrak{g}}F_{ii}u_{ii\mathfrak{g}} + \frac{\varphi'}{\varphi}F_{ii}u_{ii}^2 \\ &\geq \frac{\varphi'}{\varphi}F_{ii}u_{ii}^2 + \frac{\varphi'}{\varphi}u_{\mathfrak{g}}\hat{f}_{\mathfrak{g}}, \end{aligned}$$

where by §2 (ix)

$$\sum F_{ii}u_{ii}^2 > F_{jj}u_{jj}^2 \geq \theta\mathcal{F}u_{11}^2,$$

$\mathcal{F} = \sum_{i=1}^n F_{ii}$, $\theta = \theta(n,k,\varepsilon)$. Hence

$$F_{ii}[\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2}] \geq \theta\mathcal{F}u_{11}^2 - C.$$

Since $\rho = w - u$ and w is k-admissible, we have

$$F_{ii}\rho_{ii} \geq -F_{ii}u_{ii} = -\mu' S_k^{ii}u_{ii} = -k\mu' f.$$

Inserting the above estimates to (4.17) we obtain

$$0 \geq \sum F_{ii}(\log G)_{ii} \geq \theta\mathcal{F}u_{11}^2 - C\mathcal{F}\frac{\rho_i^2}{\rho^2} - \frac{k\beta\mu' f}{\rho} - C. \tag{4.18}$$

Note that $u_{kk} \geq \varepsilon u_{11}$, we have

$$\mathcal{F} \geq F_{nn} \geq \theta\mu' u_{11}\cdots u_{k-1,k-1} \geq \theta_1 u_{11}^{k-1}.$$

Multiplying (4.18) by $\rho^{2\beta}\varphi^2$, we obtain $G(x_0) \leq C$.

Case 2: $u_{kk} \leq \varepsilon u_{11}$.

Since $(u_{11},\cdots,u_{nn}) \in \overline{\Gamma}_k$, we have $u_{kk} \geq 0$ and so $|u_{kk}| \leq \varepsilon u_{11}$. By the arrangement $u_{11} \geq \cdots \geq u_{nn}$, we have $u_{jj} \leq \varepsilon u_{11}$ for all $j = k,\cdots,n$. Noting that

$$\frac{\partial^{k-1}}{\partial\lambda_1\cdots\partial\lambda_{k-1}}\sigma_k[\lambda] = \sum_{j=k}^n \lambda_j \geq 0$$

we obtain

$$|u_{jj}| \leq C\varepsilon u_{11} \quad \text{for} \quad j = k,\cdots,n.$$

By (4.14),

$$\frac{\rho_i}{\rho} = -\frac{1}{\beta}(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}) \quad i = 2,\cdots,n. \tag{4.19}$$

Applying (4.16) for $i = 1$ and (4.19) for $i = 2,\cdots,n$ to (4.15), we obtain

$$\begin{aligned} 0 \quad \geq \quad & \left\{\sum_{i=1}^n \left[\beta F_{ii}\frac{\rho_{ii}}{\rho} + F_{ii}(\frac{\varphi_{ii}}{\varphi} - 3\frac{\varphi_i^2}{\varphi^2})\right] - \beta(1+2\beta)F_{11}\frac{\rho_1^2}{\rho^2}\right\} \\ & + \left\{\sum_{i=1}^n F_{ii}\frac{u_{11ii}}{u_{11}} - (1+\frac{2}{\beta})\sum_{i=2}^n F_{ii}\frac{u_{11i}^2}{u_{11}^2}\right\} =: I_1 + I_2 \end{aligned} \tag{4.20}$$

As in case I we have

$$\begin{aligned} I_1 \quad & \geq \quad \theta F_{ii}u_{ii}^2 - F_{11}\frac{C}{\rho^2} - \frac{k\beta\mu' f}{\rho} - C \\ & \geq \quad \frac{1}{2}\theta F_{11}u_{11}^2 - \frac{k\beta\mu' f}{\rho} - C \end{aligned}$$

provided $\rho^2 u_{11}^2$ is suitably large. By (vii) in §2 we obtain

$$I_1 \geq \theta_1 \mu' f u_{11} - \frac{k\beta\mu' f}{\rho} - C.$$

We claim

$$I_2 \geq \hat{f}_{11}/u_{11}. \tag{4.21}$$

If (4.21) is true then (4.20) reduces to

$$0 \geq \theta_1 \mu' f u_{11} + \frac{\hat{f}_{11}}{u_{11}} - \frac{k\beta\mu' f}{\rho} - C. \tag{4.22}$$

Multiplying the above inequality by $\rho^\beta \varphi$ we obtain $G(x_0) \leq C$.

To verify (4.21) we first note that by the concavity of F,

$$-\sum_{i,j=1}^n [\mu'' \sigma_{k-1;i}\sigma_{k-1;j} + \mu' \sigma_{k-2;ij}] u_{ii1}u_{jj1} = -\sum \frac{\partial^2}{\partial\lambda_i \partial\lambda_j} \mu(S_k(\lambda)) u_{ii1}u_{jj1} \geq 0.$$

Hence by (4.13),

$$\begin{aligned} u_{11} I_2 &\geq \hat{f}_{11} + \sum_{i,j=1}^n \mu' \sigma_{k-2;ij} u_{ij1}^2 - (1 + \frac{2}{\beta}) \sum_{i=2}^n F_{ii} \frac{u_{11i}^2}{u_{11}} \\ &\geq \hat{f}_{11} + \sum_{i=2}^n \mu' \left(2\sigma_{k-2;1i} - (1+\frac{2}{\beta}) \frac{\sigma_{k-1;i}}{u_{11}} \right) u_{11i}^2. \end{aligned}$$

Since $\beta = 4$, we need only

$$\sigma_{k-2;1i} - \frac{3}{4}\frac{\sigma_{k-1;i}}{u_{11}} \geq 0. \tag{4.23}$$

But (4.23) follows from the following lemma.

Lemma 4.2. *Suppose $\lambda \in \Gamma_k$ and $\lambda_1 \geq \cdots \geq \lambda_n$. Then for any $\delta \in (0,1)$, there exists $\varepsilon > 0$ such that if*

$$S_k(\lambda) \leq \varepsilon \lambda_1^k \quad \text{or} \quad |\lambda_i| \leq \varepsilon \lambda_1 \text{ for } i = k+1, \cdots, n$$

we have

$$\lambda_1 S_{k-1;1} \geq (1-\delta) S_k. \tag{4.24}$$

Proof. To prove (4.24) we first consider the case $S_k(\lambda) \leq \varepsilon \lambda_1^k$. We may suppose $S_k(\lambda) = 1$. If (4.24) is not true,

$$S_{k-1;1} < (1-\delta)\lambda_1^{-1} \leq \varepsilon^{1/k}.$$

Hence

$$S_{k;1} \leq C S_{k-1;1}^{k/(k-1)} \leq C \varepsilon^{1/(k-1)}.$$

Noting that

$$S_k = S_{k-1;1}\lambda_1 + S_{k;1},$$

we obtain (4.24).

Next we consider the case $|\lambda_i| \le \varepsilon\lambda_1$ for $i = k+1, \cdots, n$. Observing that if $\lambda_k << \lambda_1$, we have $S_k(\lambda) << \lambda_1^k$ and so (4.24) holds. Hence we may suppose $|\lambda_i| << \lambda_k$ for $i = k+1, \cdots, n$. In this case both sides in (4.24) $= \lambda_1 \cdots \lambda_k(1 + o(1))$ with $o(1) \to 0$ as $\varepsilon \to 0$. Hence (4.24) holds. $\square$

In Section 6 we will investigate the existence of nonzero solutions to equation (4.1) with zero Dirichlet boundary condition. Assume that $f \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$, $f(x,u) > 0$ when $u < 0$. Then by choosing $w = -\delta$ for small constant δ in (4.12), we obtain a local second derivative estimate. Therefore by the regularity theory for fully nonlinear, uniformly elliptic equations, one also obtains local $C^{3,\alpha}$ estimate for the solution u. That is

Theorem 4.4. *Let $u \in C^4(\Omega) \cap C^0(\overline{\Omega})$ be a k-admissible solution of (4.1). Suppose $u = 0$ on $\partial\Omega$, $f \in C^{1,1}$ and $f > 0$ when $u < 0$. Then u satisfies a priori estimates in $C^3_{loc}(\Omega) \cap C^{0,1}(\overline{\Omega})$, namely for any $\Omega' \subset\subset \Omega$,*

$$\|u\|_{C^3(\Omega')} + \|u\|_{C^{0,1}(\overline{\Omega})} \le C, \tag{4.25}$$

where C depends only on $n, k, f, \sup|u|$, and dist$(\Omega', \partial\Omega)$. If $f^{1/k} \in C^{1,1}(\overline{\Omega})$, $\partial\Omega \in C^{3,1}$ and Ω is uniformly $(k-1)$-convex, then

$$\|u\|_{C^{1,1}(\overline{\Omega})} \le C.$$

Remark. Theorem 4.3 was established in [P] for the Monge-Ampère equation, and in [CW1] for the k-Hessian equations. The condition (4.11) in Theorem 4.3 is necessary when $k \ge 3$ [P, U1], but may be superfluous when $k = 2$ [WY]. Instead of (4.11), Urbas [U2] established the interior second derivative estimate under the assumption $D^2u \in L^p(\Omega)$, $p > \frac{1}{2}k(n-1)$.

5. Sobolev type inequalities

The k-Hessian operator can also be written in the form

$$S_k[u] = [D^2u]_k, \tag{5.1}$$

see (2.5). Hence by direct computation, one has [R]

$$\sum_i \partial_i S_k^{ij}[u] = 0 \quad \forall\ j. \tag{5.2}$$

It follows that the k-Hessian operator is of divergence form

$$\begin{aligned} S_k[u] &= \frac{1}{k}\sum u_{ij}S_k^{ij}[u] \\ &= \frac{1}{k}\sum \partial_{x_i}(u_{x_j}S_k^{ij}[u]), \end{aligned} \tag{5.3}$$

Denote by $\Phi^k(\Omega)$ the set of all k-admissible functions in Ω, and by $\Phi_0^k(\Omega)$ the set of all k-admissible functions vanishing on $\partial\Omega$. Let

$$(5.4)\qquad \begin{aligned} I_k(u) &= \int_\Omega (-u) S_k[u]dx \\ &= \frac{1}{k}\int_\Omega u_i u_j S_k^{ij}[D^2 u]. \end{aligned}$$

By (5.2), we can compute the first variation of I_k,

$$(5.5)\qquad \langle \delta I_k(u), h\rangle = (k+1)\int_\Omega (-h) S_k[u]$$

for any smooth h with compact support. Hence the Hessian equation (3.1) is variational, namely it is the Euler equation of the functional

$$(5.6)\qquad J(u) = \frac{1}{k(k+1)}\int_\Omega u_i u_j S_k^{ij}[u] + \int_\Omega fu.$$

The second variation is also easy to compute. Indeed by (5.2) we have

$$(5.7)\qquad \frac{d^2}{dt^2} I_k(u+t\varphi) = (k+1)\int_\Omega \varphi_i\varphi_j S_k^{ij}[u]$$

for any $u \in C^2(\Omega), \varphi \in C_0^\infty(\Omega)$, or any $u, \varphi \in C^2(\overline{\Omega})$, both vanishing on $\partial\Omega$. In particular if u is k-admissible, then $\frac{d^2}{dt^2} I_k(u+t\varphi) \geq 0$.

Denote

$$(5.8)\qquad \|u\|_{\Phi_0^k} = [I_k(u)]^{1/(k+1)}, \qquad u \in \Phi_0^k.$$

One can easily verify that $\|\cdot\|_{\Phi_0^k}$ is a norm in Φ_0^k [W2]. In this section we prove Sobolev type inequalities for the functional I_k.

5.1. **Sobolev type inequalities.** The following Theorem 5.1 was proved in [W2]. The proof below is also from there. For convex functions, the theorem was first established in [Ch2].

Theorem 5.1. *Let $u \in \Phi_0^k(\Omega)$.*
(i) If $1 \leq k < \frac{n}{2}$, we have

$$(5.9)\qquad \|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{\Phi_0^k} \quad \forall\ \ p+1 \in [1, k^*],$$

where C depends only on n, k, p, and $|\Omega|$,

$$k^* = \frac{n(k+1)}{n-2k}.$$

When $p+1 = k^*$, the best constant C is attained when $\Omega = \mathbb{R}^n$ by the function

$$(5.10)\qquad u(x) = [1+|x|^2]^{(2k-n)/2k}.$$

(ii) If $k = \frac{n}{2}$,

$$(5.11)\qquad \|u\|_{L^p(\Omega)} \leq C\|u\|_{\Phi_0^k}$$

for any $p < \infty$, where C depends only on n, p, and diam(Ω).
(iii) If $\frac{n}{2} < k \le n$,

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{\Phi_0^k}, \tag{5.12}$$

where C depends on n, k, and diam(Ω).

Remark. Our proof of Theorem 5.1 reduces the above inequalities to rotationally symmetric functions. When $k = \frac{n}{2}$, we have accordingly the embedding of $\Phi_0^k(\Omega)$ in the Orlicz space associated with the function $e^{|t|^{(n+2)/n}}$.

Proof. Step 1. When u is radial and $\Omega = B_1(0)$,

$$\|u\|_{\Phi_0^k(\Omega)} = C\bigg(\int_0^1 r^{n-k}|u'|^{k+1}\bigg)^{1/(k+1)}. \tag{5.13}$$

One can verify Theorem 5.1 for k-admissible, radial functions vanishing on $\partial B_1(0)$. For details see [W2].

Step 2. We prove that Theorem 5.1 holds for general k-admissible functions when $\Omega = B_1(0)$. Indeed, let

$$\begin{aligned} T_p &= \inf\{\|u\|_{\Phi_0^k}^{k+1}/\|u\|_{L^{p+1}(\Omega)}^{k+1} \mid u \in \Phi_0^k(\Omega)\}, \\ T_{p,r} &= \inf\{\|u\|_{\Phi_0^k}^{k+1}/\|u\|_{L^{p+1}(\Omega)}^{k+1} \mid u \in \Phi_0^k(\Omega) \text{ is radial}\}. \end{aligned}$$

Suppose to the contrary that $T_p < T_{p,r}$. Choose a constant $\lambda \in (T_p, T_{p,r})$ and consider the functional

$$J(u) = J(u, \Omega) = \int_\Omega \frac{-u}{k+1} S_k[u] - \frac{\lambda}{k+1}\bigg[(p+1)\int_\Omega F(u)\bigg]^{\frac{k+1}{p+1}}, \tag{5.14}$$

where

$$F(u) = \int_0^{|u|} f(t)dt,$$

and f is a smooth, positive function such that

$$f(t) = \begin{cases} \delta^p & |t| < \frac{1}{2}\delta \\ |t|^p & \delta < |t| < M, \\ \varepsilon t^{-2} & |t| > M + \varepsilon, \end{cases}$$

where $M > 0$ is any fixed constant, $\delta, \varepsilon > 0$ are small constants. We also assume that f is monotone increasing when $\frac{1}{2}\delta < |t| < \delta$, and $\varepsilon M^{-2} \le f(t) \le |t|^p$ when $M < |t| < M + \varepsilon$. The introduction of ε, δ is such that f is positive and uniformly bounded, so the global a priori estimates for parabolic Hessian equations (Theorem 10.1) applies. Obviously F is also uniformly bounded and J is bounded from below. The Euler equation of the functional is

$$S_k[u] = \lambda\beta(u)f(u), \tag{5.15}$$

where

$$\beta(u) = \left[(p+1)\int_\Omega F(u)\right]^{\frac{k-p}{p+1}}.$$

Note that for a given u, $\beta(u)$ is a constant. By our choice of the constant λ, we have

$$\begin{aligned} (5.16) \qquad & \inf\{J(u) \mid u \in \Phi_0^k(\Omega)\} < -1 \quad (\text{if } M >> 1), \\ & \inf\{J(u) \mid u \in \Phi_0^k(\Omega), u \text{ is radial}\} \to 0 \quad \text{as} \quad \delta \to 0. \end{aligned}$$

Consider the parabolic Hessian equation

$$(5.17) \qquad \log S_k[u] - u_t = \log\{\lambda\beta(u)f(u)\} \quad (x,t) \in \Omega \times [0,\infty),$$

subject to the boundary condition

$$u(\cdot,t) = 0 \ \text{ on } \partial\Omega \ \ \forall\, t \geq 0.$$

We say a function $u(x,t)$ is k-admissible with respect to the parabolic equation (5.17) if for any $t \in [0,\infty)$, $u(\cdot,t)$ is k-admissible. Equation (5.17) is a descent gradient flow of the functional J. Indeed, let $u(x,t)$ be a k-admissible solution. We have

$$\frac{d}{dt}J(u(\cdot,t)) = -\int_\Omega (S_k[u]-\psi)\log\frac{S_k[u]}{\psi} \leq 0,$$

and equality holds if and only if u is a solution to the elliptic equation (5.15), where $\psi(u) = \lambda\beta(u)f(u)$.

Let $u_0 \in \Phi_0^k(\Omega)$ be such that

$$J(u_0) \leq \inf_{\Phi_0^k(\Omega)} J(u) + \varepsilon' < -1.$$

By a slight modification (see Remark 5.1 below), we may assume that the compatibility condition $S_k[u_0] = \lambda\beta(u_0)f(u_0)$ holds on $\partial\Omega \times \{t=0\}$. In the parabolic equation (5.17), $\beta(u)$ is a function of t. By (5.16) and since $F(u)$ is uniformly bounded, we have

$$C_1 \leq \beta(u) \leq C_2,$$

for some positive constants C_1, C_2 independent of time t. Note that C_1, C_2 may depend on M but are independent of the small constants ε and δ. Therefore by Theorem 10.1, there is a global smooth k-admissible solution u to the parabolic Hessian equations (5.17).

By the global a priori estimates and since (5.17) is a descent gradient flow, $u(\cdot,t)$ sub-converges to a solution u_1 of the elliptic equation (5.15). By the Aleksandrov's moving plane method, see also [D] (p.327) for the Monge-Ampère equation, we infer that u_1 is a radial function. Therefore we have

$$\inf\{J(u) \mid u \in \Phi_0^k(\Omega), u \text{ is radial}\} \leq -1.$$

We reach a contradiction when ε, δ are small.

Step 3. Denote

$$T_p(\Omega) = \inf\{\|u\|_{\Phi_0^k}^{k+1}/\|u\|_{L^{p+1}(\Omega)}^{k+1} \ | \ u \in \Phi_0^k(\Omega)\}.$$

We claim that for any $(k-1)$-convex domains Ω_1, Ω_2 with $\Omega_1 \subset \Omega_2$,

$$T_p(\Omega_1) \geq T_p(\Omega_2). \tag{5.18}$$

Suppose to the contrary that $T_p(\Omega_1) < T_p(\Omega_2)$. Let $\lambda \in (T_p(\Omega_1), T_p(\Omega_2))$ be a constant. Let $J(u, \Omega)$ be the functional given in (5.14). Then we have

$$\begin{aligned} &\inf\{J(u,\Omega_1) \ | \ u \in \Phi_0^k(\Omega_1)\} < -1 \quad (\text{when } M >> 1), \\ &\inf\{J(u,\Omega_2) \ | \ u \in \Phi_0^k(\Omega_2)\} \to 0 \quad \text{as} \quad \delta \to 0. \end{aligned} \tag{5.19}$$

Let $u_1 \in \Phi_0^k(\Omega_1)$ be the solution to (5.15) obtained in Step 2 which satisfies

$$J(u_1, \Omega_1) \leq -1.$$

Let

$$w(x) = -M - \varepsilon - \frac{1}{2}\varepsilon^{1/2k}(R^2 - |x|^2),$$

where R is chosen large such that $\Omega_1 \subset B_R(0)$. Recall that $f(t) = \varepsilon t^{-2}$ when $|t| > M + \varepsilon$, and $C_1 \leq \beta(u_1) \leq C_2$, where C_1, C_2 are independent of ε. By equation (5.15) we have $S_k[u_1] \leq C\varepsilon$. Hence $S_k[w] \geq C\varepsilon^{1/2} \geq S_k[u_1]$ when $u_1 < -M - \varepsilon$. Applying the comparison principle to u_1 and w in $\{u_1 < -M - \varepsilon\}$, we obtain a lower bound for u_1,

$$u_1 \geq -M - R^2\varepsilon^{1/2k}. \tag{5.20}$$

Hence when ε is sufficiently small, $F(u_1) = |u_1|^{p+1} + o(1)$ if ε, δ is small, though $f(u_1)$ may violate strongly. In particular we have

$$\beta(u_1) = (1 + o(1))\left[\int_{\Omega_1} |u_1|^{p+1}\right]^{\frac{k-p}{p+1}} \tag{5.21}$$

with $o(1) \to 0$ as $\varepsilon, \delta \to 0$.

Extending u_1 to Ω_2 such that $u_1 = 0$ in $\Omega_2 - \Omega_1$ (so u_1 is not k-admissible in Ω_2). Let $\psi(x) = S_k[u_1]$ in Ω_1 and $\psi(x) = 0$ in $\Omega_2 - \Omega_1$. Denote

$$E(\varphi) = \int_{\Omega_2} (-\varphi)\psi - \lambda\left[\int_{\Omega_2} |\varphi|^{p+1}\right]^{\frac{k+1}{p+1}}.$$

Then, since $u_1 = 0$ outside Ω_1,

$$\begin{aligned} E(u_1) &= \int_{\Omega_1} (-u_1)S_k[u_1] - \lambda\left[\int_{\Omega_1} |u_1|^{p+1}\right]^{\frac{k+1}{p+1}} \\ &\leq \int_{\Omega_1} (-u_1)S_k[u_1] - \lambda\left[(p+1)\int_{\Omega_1} F(u_1)\right]^{\frac{k+1}{p+1}} \\ &= J(u_1, \Omega_1) \leq -1, \end{aligned}$$

where we have used, by the construction of f, the fact that $F(u) \le \frac{1}{p+1}|u|^{p+1}$. Let $u_2 = u_{2,m} \in \Phi_0^k(\Omega_2)$ be the solution of

$$S_k[u] = f_m \quad \text{in} \;\; \Omega_2,$$

where f_m be a sequence of smooth, positive functions which converges monotone decreasingly to ψ. By the maximum principle we have $\|u_2\|_{L^\infty(\Omega_2)} \le C$ for some $C > 0$ independent of m. By the comparison principle we have $u_2 < u_1 \le 0$ in Ω_1.

By our choice of λ and by approximation and uniform boundedness of u_2, we have

$$\begin{aligned} E(u_2) &= \int_{\Omega_2} (-u_2)\psi - \lambda\Big[\int_{\Omega_2} |u_2|^{p+1}\Big]^{\frac{k+1}{p+1}} \\ &\ge \int_{\Omega_2} (-u_2)S_k[u_2] - \lambda\Big[\int_{\Omega_2} |u_2|^{p+1}\Big]^{\frac{k+1}{p+1}} - \frac{1}{8} \\ &\ge -\frac{1}{8} \end{aligned}$$

provided m is sufficiently large.

Denote $\rho(t) = E[u_1 + t(u_2 - u_1)]$. Then $\rho(0) = E(u_1) \le -1$ and $\rho(1) = E(u_2) \ge -\frac{1}{8}$. We compute

$$\rho'(0) = \int_{\Omega_2} (u_1 - u_2)S_k[u_1] - (k+1)\lambda\Big[\int_{\Omega_2} |\varphi|^{p+1}\Big]^{\frac{k-p}{p+1}} \int_{\Omega_2} |u_1|^p(u_1 - u_2).$$

Since u_1 is a solution of (5.15), by (5.21) we have

$$\begin{aligned} \int_{\Omega_2} (u_1 - u_2)S_k[u_1] &= \lambda\beta(u_1)\int_{\Omega_1} |u_1|^p(u_1 - u_2) \\ &= \lambda(1 + o(1))\Big[\int_{\Omega_1} |u_1|^{p+1}\Big]^{\frac{k-p}{p+1}} \int_{\Omega_2} |u_1|^p(u_1 - u_2) \\ &< (k+1)\lambda\Big[\int_{\Omega_1} |u_1|^{p+1}\Big]^{\frac{k-p}{p+1}} \int_{\Omega_2} |u_1|^p(u_1 - u_2) \end{aligned}$$

We obtain $\rho'(0) < 0$. Note that the functional E is linear in the first integral and convex in the second integral, we have $\rho''(t) \le 0$ for all $t \in (0,1)$. Therefore we must have $\rho(1) < \rho(0)$. We reach a contradiction. Hence (5.18) holds.

Finally we remark that when $k < \frac{n}{2}$ and $p + 1 = k^*$, the best constant in (5.9) is achieved by the function in (5.10). This assertion follows from Step 2 by solving an ode. By the Hölder inequality, one also sees that when $k < \frac{n}{2}$ and $p + 1 < k^*$, the constant in (5.9) depends on the volume $|\Omega|$ but not the diameter of Ω. When $k > \frac{n}{2}$, The above proof implies the embedding $\Phi_0^k(\Omega) \hookrightarrow L^\infty(\Omega)$. Indeed, in Step 2 we have shown that the best constant

T_p is achieved by radial functions, and so the assertion follows from Step 1. $\square$

Remark 5.1. For any initial function $u_0 \in \Phi_0^k(\Omega)$ satisfying $J(u_0) < 0$, we can modify u_0 slightly near $\partial\Omega$ such that it satisfies the compatibility condition $S_k[u_0] = \lambda\beta(u_0)f(u_0)$ on $\partial\Omega \times \{t = 0\}$. Indeed, it suffices to replace u_0 by the solution $\hat{u}_0 \in \Phi_0^k(\Omega)$ of $S_k[u] = g$, where $g(x) = (1 + a)S_k[u_0]$ when $\text{dist}(x, \partial\Omega) > \delta_1$ and $g(x) = \lambda\beta(u_0)f(u_0)$ when $\text{dist}(x, \partial\Omega) < \frac{1}{2}\delta_1$. We choose δ_1 a sufficiently small constant and a also small such that $\beta(\hat{u}_0) = \beta(u)$.

5.2. **Compactness.** In this section we prove the embedding $\Phi_0^k(\Omega) \hookrightarrow L^p(\Omega)$ is compact when $k < \frac{n}{2}$ and $p < k^*$. First we quote a theorem from [TW4]

Theorem 5.2. *Suppose Ω is $(k-1)$-convex. Then*

$$\|u\|_{\Phi_0^l(\Omega)} \le C\|u\|_{\Phi_0^k(\Omega)} \tag{5.22}$$

for any $1 \le l < k \le n$, and any $u \in \Phi_0^k(\Omega)$. The best constant C is achieved by the solution $u \in \Phi_0^k(\Omega)$ to the Hessian quotient equation

$$\frac{S_k[u]}{S_l[u]} = 1 \quad \text{in} \ \ \Omega. \tag{5.23}$$

Proof. The proof is based on the global existence of smooth solutions to initial boundary problem of the parabolic equation [TW4]

$$u_t - \log\frac{S_k[u]}{S_l[u]} = 0 \ \text{ in } \Omega \times [0, \infty), \tag{5.24}$$

subject to the boundary condition $u = 0$ on $\partial\Omega \times (0, \infty)$. As above, a solution to (5.24) is k-admissible if for any t, $u(\cdot, t) \in \Phi^k(\Omega)$. The a priori estimation for the parabolic equation (5.24) is very similar to that for the elliptic equation (3.1).

By constructing appropriate super- and sub-barriers, we also infer that for any initial function $u(\cdot, 0)$ satisfying the compatibility condition on $S_k[u] = S_l[u]$ on $\partial\Omega$, the solution $u(\cdot, t)$ converges to the solution u^* of (5.23). Note that $u(\cdot, t) \in \Phi_0^k(\Omega)$ implies the boundary condition $u = 0$ on $\partial\Omega \times [0, \infty)$.

With the above results for the parabolic equation (5.24), Theorem 5.2 follows immediately. Indeed, let

$$J(u) = \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \frac{1}{l+1}\int_\Omega (-u)S_l[u].$$

For any $u_0 \in \Phi_0^k(\Omega)$, modify u slightly near $\partial\Omega$ such that $S_k[u_0] = S_l[u_0]$ on $\partial\Omega$. Let $u(\cdot, t) \in \Phi_0^k(\Omega)$ be the solution to the parabolic equation (5.24). Then

$$\frac{d}{dt}J(u(\cdot, t)) = -\int_\Omega \{S_k[u] - S_l[u]\}\log\frac{S_k[u]}{S_l[u]} \le 0.$$

It follows that $J(u^*) \le J(u_0)$ for any $u_0 \in \Phi_0^k(\Omega)$. Replacing u_0 by $u_0\|u^*\|_{\Phi_0^k(\Omega)}/\|u_0\|_{\Phi_0^k(\Omega)}$, we obtain Theorem 5.2. □

Theorem 5.3. *The embedding $\Phi_0^k(\Omega) \hookrightarrow L^p(\Omega)$ is compact when $k < \frac{n}{2}$ and $p < k^*$.*

By the Hölder inequality, Theorem 5.3 follows from Theorem 5.2 and the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $p < \frac{2n}{n-2}$.

Next we show that when $k > \frac{n}{2}$, a k-admissible function is Hölder continuous.

Theorem 5.4. *Suppose $u \in \Phi^k(\Omega) \cap L^\infty(\Omega)$ and $k > \frac{n}{2}$. Then $u \in C^\alpha_{loc}(\Omega)$ with $\alpha = 2 - \frac{n}{k}$, and for any $x, y \in \Omega' \subset\subset \Omega$,*

$$|u(x) - u(y)| \le C|x-y|^\alpha, \tag{5.25}$$

where C depends only on n, k, Ω, dist$(\Omega', \partial\Omega)$, and $\|u\|_{L^\infty(\Omega)}$.

Proof. Let

$$w(x) = |x|^{2-n/k}.$$

By direct computation, w is k-admissible and $S_k[w] = 0$ when $x \ne 0$. For any interior point $x_0 \in \Omega$, Applying the comparison principle to u and $\hat{u}(x) = u(x_0) + Cw(x - x_0)$, where C is chosen large such that $\hat{u} > u$ on $\partial\Omega$, we obtain Theorem 5.4. □

5.3. **An L^∞ estimate.** As an application of Theorem 5.1, we prove an L^∞ estimate for solutions to the k-Hessian equation. See Theorem 2.1 in [CW1]. The proof below is essentially the same as that in [CW1].

Theorem 5.5. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a k-admissible solution of*

$$\begin{cases} S_k(D^2u) = f(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.26}$$

Suppose $f \ge 0$, $f \in L^p(\Omega)$, where $p > n/2k$ if $k \le \frac{n}{2}$, or $p = 1$ if $k > \frac{n}{2}$. Then there exists $C > 0$ depending only on n, k, p, Ω such that

$$|\inf_\Omega u| \le |\inf_\Omega \varphi| + C\|f\|^{1/k}_{L^p(\Omega)}. \tag{5.27}$$

Proof. By replacing the boundary function φ by $\inf \varphi$ and by the comparison principle, we need only to prove (5.27) for $\varphi \equiv 0$. Since the k-Hessian equation is homogeneous, we may assume that $\|f\|_{L^p(\Omega)} = 1$.

First we prove (5.27) for $k = \frac{n}{2}$. Multiplying (5.26) by $-u$ and taking integration, we obtain,

$$\begin{aligned} \|u\|^{k+1}_{\Phi_0^k(\Omega)} &= |\int_\Omega f(x)u(x)dx| \le \|f\|_{L^p}\|u\|_{L^q} \\ &\le |\Omega|^{\frac{1}{q}(1-\frac{1}{\beta})}\|u\|_{q\beta} \le C|\Omega|^{\frac{1}{q}(1-\frac{1}{\beta})}\|u\|_{\Phi_0^k(\Omega)}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\beta>1$ will be chosen large. Hence

$$\|u\|_{\Phi_0^k}\le C|\Omega|^{\frac{1}{qk}(1-\frac{1}{\beta})}.$$

By the Sobolev type inequality (5.11),

$$\|u\|_{L^1(\Omega)}\le|\Omega|^{1-\frac{1}{\beta}}\|u\|_{L^\beta(\Omega)}\le C|\Omega|^{1-\frac{1}{\beta}}\|u\|_{\Phi_0^k}\le C|\Omega|^{1+\delta}, \tag{5.28}$$

where $\delta=\frac{1}{qk}-\frac{1}{\beta}(1+\frac{1}{qk})>0$ provided β is sufficiently large. Hence

$$|\{u(x)<-K\}|\le\frac{C}{K}|\Omega|^{1+\delta}. \tag{5.29}$$

From Sard's theorem, the level set $\{u(x)<t\}$ has smooth boundary for almost all t. Therefore we may assume all the level sets involved in the proof below have smooth boundary.

Denote $u_1=u+K$, $\Omega_1=\{u_1<0\}$. When K is large enough, we have $|\Omega_1|\le\frac{1}{2}|\Omega|$. For $j>1$ we define inductively u_j and Ω_j by $u_j=u_{j-1}+2^{-\delta j}$ and $\Omega_j=\{u_j<0\}$. Then similarly to (5.28) we have

$$\|u_j\|_{L^1(\Omega_j)}\le C|\Omega_j|^{1+\delta}$$

for some C independent of j. Therefore

$$|\Omega_{j+1}|\le C2^{\delta(j+1)}|\Omega_j|^{1+\delta},$$

where $\Omega_{j+1}=\{u_j(x)<-2^{-\delta(j+1)}\}$.

Assume by induction that $|\Omega_i|\le\frac{1}{2}|\Omega_{i-1}|$ for all $i=1,2,\cdots,j$, then by (5.29),

$$|\Omega_j|^\delta\le 2^{-\delta(j-1)}|\Omega_1|\le\frac{2^{-\delta(j-1)}C}{K}|\Omega|^{1+\delta}.$$

When K is large, we obtain $|\Omega_{j+1}|\le\frac{1}{2}|\Omega_j|$. Therefore the set $\{x\in\Omega\mid u(x)<-K-\sum_{j=1}^{\infty}2^{-\delta j}\}$ has measure zero. In other words, we have

$$|\inf u|\le K+\sum\nolimits_{j=1}^{\infty}2^{-\delta j}.$$

Therefore (5.27) is established for $k=\frac{n}{2}$.

When $k<\frac{n}{2}$, let $w\in\Phi_0^{n/2}$ be the solution to

$$S_{n/2}[w]=f^{n/2k}\ \text{ in }\ \Omega.$$

By inequality (vi) in §2, $S_k[w]\ge C_{n,k}S_{n/2}^{2k/n}[w]\ge C_{n,k}f$. Hence by the comparison principle we also obtain (5.27).

When $k>\frac{n}{2}$, multiplying (5.26) by $-u$ and taking integration, we have

$$\|u\|_{\Phi_0^k}^{k+1}=|\int_\Omega f(x)u(x)dx|\le\|f\|_{L^1}\|u\|_{\Phi_0^k}$$

and (5.27) follows from (5.12). This completes the proof. □

We will prove in Section 9 that the solution in Theorem 5.5 is Hölder continuous.

Theorem 5.5 was extended by Kuo and Trudinger to more general elliptic equations [KT]. In their paper [KT], Kuo and Trudinger considered the linear elliptic inequality

$$
\begin{aligned}
L[u] &= \sum a_{ij}(x)u_{x_ix_j} \le f \ \text{ in } \ \Omega. \\
u &\le 0 \ \text{ on } \ \partial\Omega
\end{aligned}
\tag{5.30}
$$

Assume that the eigenvalues $\lambda(\mathcal{A}) \in \Gamma_k^*$, where $\mathcal{A} = -\{a_{ij}(x)\}$, Γ_k^* is the dual cone of Γ_k, given by

$$\Gamma_k^* = \{\lambda \in \mathbb{R}^n \mid \lambda \cdot \mu \ge 0 \ \ \forall \ \mu \in \Gamma_k\}$$

Denote

$$\rho_k^*(\mathcal{A}) = \inf\{\lambda \cdot \mu \mid \mu \in \Gamma_k, \sigma_k(\mu) \ge 1\}.$$

They proved the following maximum principle

Theorem 5.6. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (5.30). Assume that $\lambda(\mathcal{A}) \in \Gamma_k^*$ and $\rho_k^*(\mathcal{A}) > 0$. Then we have the estimate*

$$\sup_\Omega u \le C \|\frac{f}{\rho_k^*(\mathcal{A})}\|_{L^q(\Omega)},$$

where $q = k$ if $k > \frac{n}{2}$, and $q > \frac{n}{2}$ if $k \le \frac{n}{2}$, where C depends only on n, k, q, and Ω.

Theorem 5.6 extended the well-known Aleksandrov maximum principle.

6. Variational problems

Consider the Dirichlet problem

$$
\begin{cases}
S_k(D^2u) = f(x,u) & \text{in } \ \Omega, \\
u = 0 & \text{on } \ \partial\Omega,
\end{cases}
\tag{6.1}
$$

where $f(x,u) \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$ is a nonnegative function in $\overline{\Omega} \times \mathbb{R}$. There has been a huge amount of works on the existence of positive solutions to semilinear elliptic equations, namely equation (6.1) with $k = 1$. In this section we show that there are similar existence results for the k-Hessian equation. Materials in this section are taken from in [CW1, CW2], except the eigenvalue problem in §6.2, which was previously treated in [W2]. We note that the published paper [CW1] is a part of the preprint [CW2]. The preprint [CW2] also contains the existence of solutions in the critical growth case, presented in §6.4 below.

As shown in §5, a solution of (6.1) is a critical point of the Hessian functional

$$J(u) = \frac{-1}{k+1}\int_\Omega uS_k[u] - \int_\Omega F(x,u), \tag{6.2}$$

where $F(x,u)=\int_u^0 f(x,t)dt$. The functional J is defined on the convex cone $\Phi_0^k(\Omega)$. We don't know the behavior of the functional near the boundary of $\Phi_0^k(\Omega)$, and so we cannot use the variational theory directly. To find a critical point of J, we employ a descent gradient flow of the functional, which was previously used by Chou [Ch1] for the Monge-Ampère equation. That is a parabolic Hessian equation of the form

$$\mu(S_k[u]) - u_t = \mu(f(u)). \tag{6.3}$$

We assume that μ is a smooth function defined on $(0,\infty)$, satisfying $\mu'(t)>0$, $\mu''(t)<0$ for all $t>0$,

$$\begin{aligned}&\mu(t)\to-\infty \quad \text{as } t\to 0,\\ &\mu(t)\to+\infty \quad \text{as } t\to+\infty,\end{aligned}$$

and such that $\mu(S_k[u])$ is concave in D^2u. As we consider solutions in Φ_0^k, the boundary condition for (6.3) is

$$u=0 \;\text{ on }\; \partial\Omega\times[0,\infty).$$

Let $u\in\Phi_0^k(\Omega\times\mathbb{R}^+)$ be a k-admissible solution to (6.3). Then

$$\begin{aligned}\frac{d}{dt}J(u(\cdot,t)) &= -\int_\Omega (S_k(\lambda)-f)u_t \\ &= -\int_\Omega (S_k(\lambda)-f)(\mu(S_k(\lambda))-\mu(f))\le 0.\end{aligned} \tag{6.4}$$

As before, we say a solution u is k-admissible with respect to (6.3) if for any $t\ge 0$, $u(\cdot,t)$ is k-admissible. To simplify the notation, we will denote $u\in\Phi_0^k(\Omega\times\mathbb{R}^+)$ if $u(\cdot,t)\in\Phi_0^k(\Omega)$ for all $t\in\mathbb{R}^+=[0,\infty)$.

A typical example of μ is $\mu(t)=\log t$ [Ch1]. But for the k-Hessian equation we have to choose a different μ in our treatment below. For the a priori estimates for the parabolic equation (6.3), we always need to assume that f is strictly positive. But in studying the variational problem (6.1), typically f vanishes when $u=0$. To avoid such situation, we add a small positive constant to f, or modify f slightly near $u=0$.

To study the variational problem associated with the k-Hessian equation, similar to the Laplace equation, we divide the problem into three cases, namely the sublinear case, the eigenvalue problem, and the superlinear case.

6.1. **The sublinear growth case.** We say $f(x,u)$ is sublinear with respect to the k-Hessian operator if

$$\lim_{u\to-\infty}|u|^{-k}f(x,u)\to 0 \tag{6.5}$$

uniformly for $x\in\Omega$. Note that the power k is due to that the k-Hessian operator is homogeneous of degree k.

Theorem 6.1. *Let Ω be $(k-1)$-convex with $C^{3,1}$-boundary. Suppose $f(x,u)\in C^{1,1}(\overline{\Omega}\times\mathbb{R})$, $f(x,u)>0$ when $u<0$, f satisfies (6.5), and $\inf_{\Phi_0^k(\Omega)}J(u)<0$.*

Then there is a nonzero solution $u \in C^{0,1}(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ to (6.1), which is the minimizer of the functional J.

Proof. We sketch the proof, as it was essentially included in the proof of Theorem 5.1.

Replace f by $f + \delta$ for some small $\delta > 0$, so that f is strictly positive. Observe that in the sublinear growth case, by the Sobolev type inequality (Theorem 5.1), we have $J(tu) \to +\infty$ as $t \to \infty$ for any $u \in \Phi_0^k(\Omega)$, $u \neq 0$, and J is bounded from below. As the infimum of J is negative, one can choose an initial function $u_0 \in \Phi_0^k(\Omega)$ such that $J(u_0) < \inf_{u \in \Phi_0^k(\Omega)} J(u) + \delta$. By Remark 5.1, we may assume the compatibility condition $S_k[u_0] = f$ on $\partial\Omega$ at $t = 0$ is satisfied. In the sublinear growth case, one can construct a sub-barrier $\underline{u}$ to the parabolic equation (6.3) such that $\underline{u} \leq u_0$. By the comparison principle one gets a global uniform estimate, and also derivative estimates up to the third order, for solutions to (6.3). Therefore there is a global smooth solution to (6.3). By (6.4), the solution sub-converges to a nonpositive solution $u = u_\delta$ of (6.1).

We claim that all the solutions u_δ are uniformly bounded for $\delta > 0$ small. Indeed, if $m_\delta = -\inf u_\delta \to \infty$, the function $v_\delta = u_\delta / m_\delta$ satisfies the equation

$$S_k[v] = m_\delta^{-k}[f(x, m_\delta v_\delta) + \delta].$$

By (6.5), the right hand side converges to zero uniformly. Hence by the comparison principle, one infers that $\inf v_\delta \to 0$, which contradicts with the fact that $\inf v_\delta = -1$. Next by the assumption that $\inf J < 0$, it is easily seen that u_δ does not converge to zero.

Sending $\delta \to 0$, by the interior a priori estimates in §4, we conclude that u_δ converges to a solution $u \in C^{0,1}(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ of (6.1) which is a minimizer of J. □

A particular case in Theorem 6.1 is when $f \in C^{1,1}(\overline{\Omega})$ is a function of x, independent of u [B]. Then for any initial u_0 satisfying the compatibility condition, the solution $u \in \Phi_0^k(\Omega \times \mathbb{R}^+)$ of (6.3) is uniformly bounded and converges to a solution u^* of (6.1). By the convexity of the functional J (see (5.7)) and the uniqueness of solutions to the Dirichlet problem, u^* is a minimizer of the functional J in $\Phi_0^k(\Omega)$.

6.2. **The Eigenvalue problem.** Similar to the Laplace operator, the k-Hessian operator admits a positive eigenvalue λ_1 such that

$$\begin{aligned} S_k[u] &= \lambda |u|^k \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.6}$$

has a nonpositive k-admissible solution when $\lambda = \lambda_1$. The following theorem was proved in [W2] for $k < n$ and in [Lp] for $k = n$. Here we provide a proof which uses Theorem 6.1.

Theorem 6.2. *Let Ω be $(k-1)$-convex with $C^{3,1}$-boundary. Then there exists $\lambda_1 > 0$ depending only on n, k, Ω, such that*
(i) (6.6) has a nonzero k-admissible solution $\varphi_1 \in C^{1,1}(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ when $\lambda = \lambda_1$.
(ii) If $(\lambda^*, \varphi^*) \in [0, \infty) \times (C^{1,1}(\overline{\Omega}) \cap C^{3,\alpha}(\Omega))$ is another solution to (6.6), then $\lambda^* = \lambda_1$ and $\varphi^* = c\varphi_1$ for some positive constant c.
(iii) If $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.

Proof. First consider part (i). Let $p \in (k - \frac{1}{2}, k)$ and let $c_0 > 0$ be a large constant. By Theorem 6.1, there is an admissible solution $u_p \in \Phi_0^k(\Omega)$ to the problem

$$S_k[u] = c_0 |u|^p,$$

which is a minimizer of the associated functional. Namely $J(u_p) = \inf J(u)$, where

$$J(u) = \frac{1}{k+1} \int_\Omega (-u) S_k[u] - \frac{c_0}{p+1} \int_\Omega |u|^{p+1}.$$

Let $v_p = u_p / m_p$, where $m_p = \sup |u_p|$. Then v_p satisfies

$$S_k[v] = c_0 m_p^{p-k} |v|^p.$$

If $m_p^{p-k} \to 0$ as $p \to k$, the right hand side converges to zero uniformly, which contradicts with the fact that $\inf v_p = -1$. If $m_p^{p-k} \to \infty$, then $m_p \to 0$ uniformly, which implies $J(u_p) = \inf J(u) \to 0$. But if we choose $c_0 > 0$ large, $J(u_p) = \inf J(u) \to -\infty$ as $p \to k$. The contradiction implies that m_p is uniformly bounded. Hence by the a priori estimates in §4, we see that $(c_0 m_p^{p-k}, v_p)$ sub-converges to (λ_1, φ_1), and (λ_1, φ_1) is a solution of (6.6). By Theorem 4.4, $\varphi_1 \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$.

Next we consider (ii). If (λ^*, φ^*) is also a solution of (6.6), we may assume that $\lambda^* > \lambda_1$ and $\varphi^* < \varphi_1$ by multiplying a constant to φ^*. Denote $a_{ij} = \frac{\partial}{\partial u_{ij}} \big[S_k[u]\big]^{1/k}$ at $u = \varphi_1$. Then λ_1 and φ_1 are respectively the eigenvalue and eigenfunction of the elliptic operator $L = \sum a_{ij} \partial_{ij}^2$. By the concavity of $S_k^{1/k}[u]$, and noting that ψ^* and ψ_1 are negative in Ω, we deduce that

$$L(\varphi^* - \varphi_1) \geq S_k^{1/k}[\varphi^*] - S_k^{1/k}[\varphi_1] = -(\lambda^*)^{1/k} \varphi^* + \lambda_1^{1/k} \varphi_1 > \lambda_1^{1/k} (\varphi^* - \varphi_1)$$

in Ω, which contradicts the fact that $\lambda_1^{1/k}$ is the first eigenvalue of L. Hence we have $\lambda^* = \lambda_1$ and $\varphi_1 = \varphi^*$.

Part (iii) was proved in Step 3 of the proof of Theorem 5.1. □

6.3. **Superlinear growth case.** We say f is superlinear with respect to the k-Hessian operator if

$$\lim_{u \to -\infty} |u|^{-k} f(x, u) \to \infty \tag{6.7}$$

uniformly for $x \in \Omega$.

Theorem 6.3. *Suppose that* $f(x,z) > 0$ *for* $z < 0$,

$$\lim_{z\to 0^-} f(x,z)/|z|^k < \lambda_1, \tag{6.8}$$

$$\lim_{z\to -\infty} f(x,z)/|z|^k > \lambda_1, \tag{6.9}$$

where λ_1 *is the eigenvalue of the* k*-Hessian operator. Suppose there exist constants* $\theta > 0$ *and* M *large such that*

$$\int_z^0 f(x,s)\,ds \le \frac{1-\theta}{k+1}|z|f(x,z) \quad \forall\ z < -M. \tag{6.10}$$

When $k \le \frac{n}{2}$, *we also assume that there exists* $p \in (1, k^*-1)$ *such that*

$$\lim_{z\to -\infty} f(x,z)/|z|^p = 0. \tag{6.11}$$

Then (6.1) has a non-zero k*-admissible solution in* $C^{3,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$, $\alpha \in (0,1)$.

When $k = 1$, Theorem 6.3 is a typical result in semilinear elliptic equation. The solution in Theorem 6.3 is a min-max critical point of the functional J. As indicated before, we cannot use the variational theory directly, but by studying a descent gradient flow, we can use the underlying idea in the Mountain Pass Lemma. The main difficulty is to prevent blowup of solutions near the boundary for both the elliptic equation (6.1) and the parabolic equation (6.3), in the case $2 \le k < \frac{n}{2}$. It requires some new techniques.

Proof. For clarity we divide the proof into four steps.

Step 1. Let $f_{\delta,K}$ be a smooth, positive function given by

$$f_{\delta,K}(x,u) = \delta + \eta_{\delta_1}\hat{f}_{\delta,K}(x,u),$$

where $\eta_{\delta_1} \in C_0^\infty(\Omega)$ is a nonnegative function satisfying $\eta_{\delta_1}(x) = 1$ when $\text{dist}(x,\partial\Omega) > 2\delta_1$ and $\eta_{\delta_1}(x) = 0$ when $\text{dist}(x,\partial\Omega) < \delta_1$, and

$$\hat{f}_{\delta,K}(x,u) = \begin{cases} \delta & \text{if } |u| < \frac{1}{2}\delta, \\ f(x,u) & \text{if } \delta < |u| < K, \\ |u|^p & \text{if } |u| > 2K. \end{cases}$$

We will choose the constants $\delta, \delta_1 > 0$ small and $K > 1$ large.

Remark. Before continuing, let us explain why we make these modifications when $k \ge 2$, which are not needed when considering semilinear elliptic equations (the case $k = 1$). The introduction of δ is such that f is positive, so that we can apply the C^3 a priori estimates for the parabolic Hessian equation (6.3). We modify f for large $|u|$ (namely $f = |u|^p$ when $|u| > 2K$) is to use the gradient estimate for the parabolic Hessian equation. The purpose of introducing η_{δ_1} is to prevent the solution to the parabolic Hessian equation blow-up near the boundary. In the following we choose $\delta_1 = \delta$.

Consider the functional

$$(6.12) \qquad J_{\delta,K}(x,u) = \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \int_\Omega F_{\delta,K}(x,u),$$

where $F_{\delta,K}(x,u) = \int_u^0 f_{\delta,K}(x,t)$. When $\delta > 0$ is sufficiently small, by assumption (6.8), there exists a smooth, k-admissible function $u_0 \in \Phi_0^k(\Omega)$ with small L^∞-norm, such that

$$(6.13) \qquad S_k[u_0] > f_{\delta,K}(x,u_0).$$

Consider the parabolic Hessian equation (6.3) with initial condition $u(\cdot,0) = su_0$, where $s > 0$ is a parameter. We choose the function μ in (6.3) such that

$$\begin{aligned} \mu(t) &= \log t \quad \text{if } t < 1/8, \\ \mu(t) &= t^{1/p} \quad \text{if } t > 8, \end{aligned} \qquad (6.14)$$

and

$$(6.15) \qquad (t-s)(\mu(t)-\mu(s)) \ge C(t-s)(t^{1/p} - s^{1/p})$$

for all $t, s > 0$, where C is an absolute constant, independent of s, t. Then equation (6.3) has a unique smooth solution u_s. By (6.11) and (6.14), $\mu(f(u))$ is of linear growth in u. Hence the solution exists for all time t.

Since $S_k[u_0] > 0$, u_0 is a sub-barrier for the solution u_s for small $s > 0$. That is when $s > 0$ is small, one has $0 > u_s(\cdot,t) > u_0$ for all t. Hence $J_{\delta,K}(x,u_s)$ is uniformly bounded,

$$J_{\delta,K}(u_s(\cdot,t)) > -\int_\Omega F_{\delta,K}(u_s(x,t))dx \ge -\frac{1}{2}$$

for all t, provided $\delta > 0$ is small.

On the other hand, when $s > 1$ is large, we have $J_{\delta,K}(su_0) < -1$. Hence $J_{\delta,K}(u_s(\cdot,t)) < -1$ for all $t > 0$ as (6.3) is a descent gradient flow. Let

$$(6.16) \qquad s^* = \inf\{\overline{s} \mid \lim_{t\to\infty} J_{\delta,K}(u_s(\cdot,t)) < -1 \ \ \forall\, s > \overline{s}\}.$$

Then s^* is positive. By the continuous dependence of the solution u_s in s, and the monotonicity (6.4), we see that $J_{\delta,K}(u_{s^*}(\cdot,t)) \ge -1$ for all time t. We also have

$$(6.17) \qquad \sup|u_{s^*}(\cdot,t)| \ge C > 0$$

for some $C > 0$ independent of t. Indeed, if $\sup|u_{s^*}(\cdot,t)|$ is small at some time $\overline{t}$, then $\sup|u_s(\cdot,t)|$ is also small at $\overline{t}$ for $s > s^*$, close to s^*. Hence by (6.13), u_0 is a sub-barrier, and so by the comparison principle, $\sup|u_s(\cdot,t)|$ is small for all $t > \overline{t}$, which contradicts with the definition of (6.16).

Suppose for a moment that

$$(6.18) \qquad |u_{s^*}(\cdot,t)| \le C$$

uniformly for $t \in (0,\infty)$. The constant C is allowed to depend on δ and K. Then by the global regularity of the parabolic Hessian equation, we conclude that $u_{s^*}(\cdot,t)$ sub-converges to a solution $u^*_{\delta,K}$ to the equation

$$S_k[u] = f_{\delta,K}(x,u). \tag{6.19}$$

In Step 4 we show that $u^*_{\delta,K}$ is uniformly bounded in δ and K, and so it sub-converges as $\delta \to 0, K \to \infty$ to a solution $u \in C^{0,1}(\overline{\Omega}) \cap C^3(\Omega)$ in $\Phi^k_0(\Omega)$ of (6.1). From (6.17), $u \neq 0$.

Step 2. In the following we prove (6.18). For brevity we will write u_{s^*} as u, dropping the subscript s^*. Recall that $J_{\delta,K}(u(\cdot,t)) \ge -1$ for all time t. Hence the set

$$K^0 = \{t \in (0,\infty) \mid \frac{d}{dt} J_{\delta,K}(u(\cdot,t)) < -\sigma\} \tag{6.20}$$

has finite measure, where $\sigma > 0$ is a small constant. For any $t \notin K^0$, first we show that

$$\int_\Omega (-u(\cdot,t) S_k[u(\cdot,t)] \le C, \tag{6.21}$$

$$\int_\Omega F_{\delta,K}(x,u(\cdot,t)) \le C, \tag{6.22}$$

where C is a constant independent of t, δ and K. Indeed, if $t \notin K^0$, we have

$$\begin{aligned} &\int_\Omega \{S_k[u] - f_{\delta,K}(x,u)\}\{\mu(S_k[u]) - \mu(f_{\delta,K}(x,u))\} \\ = &\int_\Omega \partial_t u\{S_k[u] - f_{\delta,K}(x,u)\} = -\frac{d}{dt} J_{\delta,K}(u(\cdot,t)) \le \sigma. \end{aligned}$$

Hence by (6.15),

$$\int_\Omega \{S_k[u] - f_{\delta,K}(x,u)\}\{(S_k[u])^{1/p} - (f_{\delta,K}(x,u))^{1/p}\} \le C\sigma.$$

Denote $\alpha = (S_k[u])^{1/p}$, $\beta = (f_{\delta,K}(x,u))^{1/p}$. We obtain

$$\int_\Omega |\alpha-\beta|^{p+1} \le C \int_\Omega (\alpha^p - \beta^p)(\alpha-\beta) \le C\sigma.$$

We have

$$\begin{aligned} \left|\int_\Omega u(\alpha^p - \beta^p)\right| &\le C \int_\Omega |u|\,|\alpha-\beta|\,(\alpha^{p-1} + \beta^{p-1}) dx \\ &\le C\Big[\int_\Omega |\alpha-\beta|^{p+1}\Big]^{\frac{1}{p+1}} \Big[\int_\Omega |u|^{p+1}\Big]^{\frac{1}{p(p+1)}} \Big[\int_\Omega |u||\alpha^p + \beta^p|\Big]^{\frac{p-1}{p}} \\ &\le C\sigma^{1/(p+1)} \|u\|^{1/p}_{L^{p+1}} \Big[\int_\Omega |u||\alpha^p + \beta^p|\Big]^{\frac{p-1}{p}}. \end{aligned}$$

On the other hand,

$$(6.23)\qquad \begin{aligned} J_{\delta,K}(s^*u_0) &= J_{\delta,K}(u(\cdot,0)) \ge J_{\delta,K}(u(\cdot,t)) \\ &= \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \int_\Omega F_{\delta,K}(x,u) \ge -1. \end{aligned}$$

By (6.10),

$$F_{\delta,K}(x,u) \le \delta|u| + \frac{1-\theta}{k+1}|u|f_{\delta,K}(x,u) + C.$$

Hence

$$\begin{aligned} J_{\delta,K}(u(\cdot,t)) &= \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \int_\Omega F_{\delta,K}(x,u) \\ &\ge \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \int_\Omega [\delta|u| + \frac{1-\theta}{k+1}|u|f_{\delta,K}(x,u) + C] \\ &= \frac{1}{k+1}\int_\Omega (-u)(\alpha^p - \beta^p) + \frac{\theta}{k+1}\int_\Omega |u|f_{\delta,K}(x,u) - \int_\Omega (C+\delta|u|). \end{aligned}$$

It follows that, by the Sobolev inequality (Theorem 5.1),

$$\begin{aligned} \int_\Omega |u|f_{\delta,K}(x,u) &\le C\int_\Omega |u|\,|\alpha^p - \beta^p| + J_{\delta,K}(s^*u_0) + \int_\Omega (C+\delta|u|) \\ &\le C\sigma^{1/(p+1)}\|u\|_{L^{p+1}}^{1/p}\left[\int_\Omega |u||\alpha|^p + \int_\Omega |u|\beta^p|\right]^{\frac{p-1}{p}} + \int_\Omega (C+\delta|u|) \\ &\le C\sigma^{1/(p+1)}\|u\|_{\Phi_0^k(\Omega)}^{1/p}\left[\|u\|_{\Phi_0^k(\Omega)}^{k+1} + \int_\Omega |u|\beta^p|\right]^{\frac{p-1}{p}} + \int_\Omega (C+\delta|u|). \end{aligned}$$

By the Sobolev inequality again, $\|u\|_{L^1} \le C\|u\|_{\Phi_0^k}$. We obtain

$$\int_\Omega |u|f_{\delta,K}(x,u) \le C_1\varepsilon_{\sigma,\delta}\int_\Omega |u||\alpha|^p + C_2$$

with $\varepsilon_{\sigma,\delta} \to 0$ as $\sigma,\delta \to 0$. Inserting the estimate into (6.23) we obtain (6.21) and (6.22).

Step 3. Now we use (6.21) and (6.22) to establish (6.18). If $k > \frac{n}{2}$, in view of (5.12), (6.18) follows readily from (6.21). We need only to consider the cases $k \le \frac{n}{2}$. Let $M_t = \sup_{\Omega_\delta}|u(\cdot,t)|$, $\widetilde{M}_t = \sup_\Omega |u(\cdot,t)|$. If M_t is not uniformly bounded, there exists a sequence $t_j \to \infty$ such that $M_{t_j} \to \infty$ and

$$(6.24)\qquad M_t \le M_{t_j} \text{ for all } t < t_j.$$

Let

$$w(x) = \frac{-d_x + Kd_x^2}{-\delta + K\delta^2}M_{t_j},$$

where $d_x = \mathrm{dist}(x,\partial\Omega)$. We choose δ small and $K \in (1,\delta^{-1/2})$ large such that $S_k[w] > \delta$ in $\Omega - \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega \mid d_x > \delta\}$. Then $w = M_{t_j}$ on $\partial\Omega_\delta$. Recall that $f_{\delta,K} = \delta$ in $\Omega - \Omega_\delta$. By the comparison principle we have $u(x,t) \ge w(x)$ for any $x \in \Omega - \Omega_\delta$, $t \in (0,t_j)$. It follows that $\widetilde{M}_t \le M_{t_j}$

for $t \in (0, t_j)$. By (6.11) and (6.14), the right hand side of the parabolic equation (6.3) is of linear growth. Hence we have

$$M_t \geq M_{t_j} e^{C(t_j - t)} \quad \forall\, t < t_j.$$

Hence $M_t \geq C M_{t_j}$ for $t \in (t_j - 2, t_j)$. Since the set K^0 has finite measure, we may assume that $t_j \notin K^0$ for all j and $M_t \leq C M_{t_j}$ for all $t < t_j$.

Suppose the maximum M_{t_j} of $|u(\cdot, t_j)|$ is attained at the point y_j. By the interior gradient estimate (10.10) below, we have

$$u(x, t_j) \leq -\frac{1}{2} M_{t_j} \quad \forall\, x \in B_r(y_j),$$

where $r = c_1 M_{t_j}^{\beta}$ and $c_1 > 0$ is independent of j, and

$$\beta = 1 - \frac{p+k}{2k} = \frac{k-p}{2k}.$$

By (6.21) (where the constant C is independent of δ, K) and the Sobolev inequality (5.9) and (5.11), we have

$$\|u(\cdot, t_j)\|_{L^q(B_r(y_j))} \leq \|u(\cdot, t_j)\|_{L^q(\Omega)} \leq C\|u\|_{\Phi_0^k(\Omega)} \leq C,$$

where $q = k^*$ if $k < \frac{n}{2}$ and $q > p+1$ is any sufficiently large constant if $k = \frac{n}{2}$. On the other hand, we have

$$\|u(\cdot, t_j)\|_{L^q(B_r(y_j)} \geq C r^n M_{t_j}^q \geq C M_{t_j}^{q + b\beta}.$$

Since $p < k^* - 1$, we have $q + b\beta > 0$. Hence when M_{t_j} is sufficiently large, we reach a contradiction. Therefore (6.18) is proved.

Step 4. We have therefore obtained a solution $u_{\delta,M}$ to (6.19) which satisfies (6.21) and (6.22), with the constant C in (6.21) and (6.22) independent of δ, K. If $k > \frac{n}{2}$, by (5.12) we obtain

$$\sup_{\Omega} |u_{\delta,K}| \leq C \tag{6.25}$$

for a different C independent of δ, K. Sending $\delta \to 0$ and $K \to \infty$, we obtain a solution $u \in C^{0,1}(\overline{\Omega}) \cap C^3(\Omega)$ in $\Phi_0^k(\Omega)$ of (6.1).

If $k < \frac{n}{2}$, denote $\psi = f_{\delta,K}(c, u_{\delta,K})$. By (6.21) and (6.11), and the Sobolev inequality, we have $\psi \in L^{\beta}(\Omega)$ for some $\beta > \frac{n}{2k}$. Hence applying Theorem 5.5 to equation $S_k[u] = \psi$ in Ω we obtain again (6.25). In all the cases, we obtain a solution to (6.1). □

Remarks.

(i) In Step 4 above, if $k = \frac{n}{2}$, by the Sobolev embedding (5.11), the right hand side of (6.19) belongs to $L^{\beta}(\Omega)$ for any $\beta > 1$. Write equation (6.19) in the form

$$\sum\nolimits_{i,j} a_{ij} u_{ij} = [S_k[u]]^{1/k} = f_{\delta,K}^{1/k}. \tag{6.26}$$

where $a_{ij} = \frac{1}{k}[S_k[u]]^{1/k-1}S_k^{ij}[u]$. By inequality (x) in Section 2, the determinant $|a_{ij}| \ge C_{n,k} > 0$. Hence by Aleksandrov's maximum principle,

$$\sup_\Omega |u_{\delta,K}| \le C \int \frac{1}{|a_{ij}|} f_{\delta,K}^{n/k}\, dx \le C$$

for some C independent of δ, K. We also obtain (6.25).

(ii) If f is independent of x and the domain Ω is convex, by the method of moving planes, the maximum point of $u_{\delta,K}$ will stay away from the boundary. Hence we can obtain (6.25) by a usual blow-up argument. We don't need to use Theorem 5.5.

(iii) Let u_1 be a k-admissible function with small L^∞ norm, u_2 be a k-admissible function such that $J(u_2) < -1$, where J is the functional in (6.2). Let Γ denote the set of paths in $\Phi_0^k(\Omega)$ connecting u_1 to u_2. Let

$$c_0 = \inf_{\gamma\in\Gamma} \sup_{s\in(0,1)} J(\gamma(s)). \tag{6.27}$$

Then the assumptions (6.8)-(6.11) and the Sobolev inequality (Theorem 5.1) implies that $c_0 > 0$. The above proof implies that there is a solution $u \in \Phi_0^k(\Omega)$ to (6.1) such that $J(u) = c_0$.

6.4. **The critical growth case.** In this section we extend Theorem 6.3 to the critical growth case. Consider the problem

$$\begin{cases} S_k(D^2u) = |u|^{k^*-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.28}$$

where $1 < k < n/2$ and f is a lower order term of $|u|^{k^*-1}$. For simplicity we will consider the case

$$f(x,u) = \lambda |u|^q, \tag{6.29}$$

where $q \in (k, k^*-1), \lambda > 0$. Denote

$$J(u) = \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \frac{1}{k^*}\int_\Omega |u|^{k^*} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}dx.$$

$$c_0 = \inf_{u\in\Phi_0^k(\Omega)} \sup_{s>0} J(su). \tag{6.30}$$

By the Sobolev inequality (5.9), we have $c_0 > 0$.

We also denote

$$J^*(u) = \frac{1}{k+1}\int_\Omega (-u)S_k[u] - \frac{1}{k^*}\int_\Omega |u|^{k^*}.$$

$$c^* = \inf_{\Phi_0^k(\Omega)} \sup_{t>0} J^*(tu). \tag{6.31}$$

The following theorem extends the existence of positive solutions to semilinear elliptic equations in [BN] to the k-Hessian equation. Our proof is completely different, due to the lack of a gradient estimate near the boundary for equation (6.40).

Theorem 6.4. *Suppose*

$$c_0 < c^*. \tag{6.32}$$

Then (6.28) has a non-zero k-admissible solution.

Proof. For any $p \in (q, k^* - 1)$, by Theorem 6.3, there exists a solution $u_p \in C^3(\Omega) \cap C^{0,1}(\overline{\Omega})$ of

$$\begin{cases} S_k(D^2 u) = \psi_p(x, u) =: |u|^p + \lambda |u|^q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.33}$$

with

$$J_p(u_p) = c_p,$$

where

$$J_p(u) = \frac{-1}{k+1} \int_\Omega u S_k(D^2 u) dx - \int_\Omega \Psi_p(x, u) dx,$$

$\Psi_p(x, u) = \int_u^0 \psi_p(x, t) dt$, and

$$c_p = \inf_{u \in \Phi_0^k(\Omega)} \sup_{s>0} J_p(su) > 0.$$

From equation (6.33) we have

$$\int_\Omega u_p S_k(D^2 u_p) dx = \int_\Omega u_p \psi_p(x, u_p) dx,$$

which, together with $J_p(u_p) = c_p$, implies that

$$\|u_p\|_{\Phi_0^k} \le C \quad \text{and} \quad \|u_p\|_{L^{p+1}(\Omega)} \le C. \tag{6.34}$$

We want to prove that

$$M_p = \sup_{x \in \Omega} |u_p(x)|$$

is uniformly bounded for $p < k^* - 1$ and close to $k^* - 1$. If this is true then by the regularity in §4, there exists a subsequence of $u_p(x)$ which converges to a solution u_0 of (6.28). Moreover, one can prove

$$J(u_0) = \lim_{p \to k^*-1} J_p(u_p) = \lim_{p \to k^*-1} c_p > 0.$$

Hence $u_0 < 0$ in Ω.

Suppose to the contrary that there is a subsequence p_j so that $M_j =: M_{p_j} \to \infty$ as $p_j \to k^* - 1$. Suppose the supremum M_j is attained at x_j. Let

$$v_j(y) = M_j^{-1} u(R_j^{-1} y + x_j), \quad y \in \Omega_j,$$

where $R_j = M_j^{(p_j - k)/2k}$, $\Omega_j = \{y \mid R_j^{-1} y + x_j \in \Omega\}$. Then $v_j(0) = -1$, $-1 \le v_j \le 0$ for $y \in \Omega_j$, and v_j satisfies

$$S_k(D_y^2 v) = \widetilde{\psi}_j(y), \tag{6.35}$$

where

$$\widetilde{\psi}_j(y) = |v_j|^{p_j} + \lambda M_j^{q - p_j} |v_j|^q.$$

Moreover,

$$\int_\Omega |u_j|^{p_j+1}dx = M_j^{\delta_j}\int_{\Omega_j}|v_j|^{p_j+1}dy, \tag{6.36}$$

$$\int_\Omega |u_j|S_k(D^2u_j)dx = M_j^{\delta_j}\int_{\Omega_j}|v_j|S_k(D^2v_j)dy,$$

where $\delta_j = p_j + 1 - \frac{n}{2k}(p_j - k) \geq 0$. Hence $\|v_j\|_{L^{p_j+1}}$ and $\|v_j\|_{\Phi_0^k(\Omega_j)}$ are uniformly bounded.

By passing to a subsequence we assume that $x_j \to x_\infty \in \overline{\Omega}$. Denote $d_j = \mathrm{dist}(x_j, \partial\Omega)$. If

$$d_jR_j \to \infty, \tag{6.37}$$

then for any $R > 0$, $B_R(0) \subset \Omega_j$ provided j is large enough. By the interior gradient estimate (Theorem 4.1) and the interior second derivative estimate (Theorem 4.2), we may suppose, by passing to a subsequence if necessary,

$$v_j(y) \to v_\infty \quad \text{in} \quad C^2_{loc}(\mathbb{R}^n),$$

and v_∞ satisfies the equation

$$S_k(D^2v) = |v|^{k^*-1} \quad \text{in} \quad \mathbb{R}^n.$$

Note that to apply Theorem 4.2, we may choose the function w in (4.11) as $w = \frac{\varepsilon}{R_1^2}(|x|^2 - R_1^2)$ for large R_1. Hence

$$-\int_{\mathbb{R}^n} v_\infty S_k(D^2v_\infty)dx = \int_{\mathbb{R}^n}|v_\infty|^{k^*}dx,$$

and so

$$J^*(v_\infty) = \sup_{t>0} J^*(tv_\infty) \geq c^*. \tag{6.38}$$

On the other hand, let $\widetilde{\Psi}_j(y,u) = \int_u^0 \widetilde{\psi}_j(y,t)dt$. We have

$$\widetilde{\Psi}_j(y,u) \to \frac{1}{k^*}|v_\infty(y)|^{k^*} \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^n)$$

as $j \to \infty$. Note that by equation (6.35),

$$\frac{-1}{k+1}v_jS_k(D^2v_j) - \widetilde{\Psi}_j(y,v_j) \geq 0.$$

By Fatou's lemma we obtain

$$\begin{aligned} J^*(v_\infty) &\leq \underline{\lim}_{j\to\infty}\int_{\Omega_j}\big[\frac{-1}{k+1}v_jS_k(D^2v_j) - \widetilde{\Psi}_j(y,v_j)\big]dx \\ &= \underline{\lim}_{j\to\infty}M_j^{-\delta_j}\int_\Omega\big[\frac{-1}{k+1}u_{p_j}S_k^2(D^2u_{p_j}) - \Psi_{p_j}(x,u_{p_j})\big]dx \\ &\leq \underline{\lim}_{j\to\infty}c_{p_j} \leq c_0 < c^*, \end{aligned}$$

which contradicts with (6.38). Hence M_p is uniformly bounded.

Next we consider the case that $d_j R_j$ is uniformly bounded. We may suppose

$$d_j R_j \to \alpha \geq 0. \tag{6.39}$$

By the interior gradient estimate (Theorem 4.1), v_j converges locally uniformly to a function v_∞, and v_∞ satisfies

$$S_k(D^2 v_\infty) = |v_\infty|^{k^*-1} \quad \text{in } \ \Omega_\infty \tag{6.40}$$

in a weak or the viscosity sense (see §9 for definition of the weak solution), where by a rotation of axes, $\Omega_\infty = \{y_n > -\alpha\}$. Moreover we have $-1 \leq v_\infty \leq 0$ in Ω_∞.

We note that the argument in Case 1 doesn't work at the current situation, as we don't have uniform gradient estimate near the boundary $\partial\Omega_\infty$, we don't know whether $v_\infty = 0$ on $\partial\Omega_\infty$.

Let $D_j = \{y \in \mathbb{R}^n \mid v_j(y) \leq -\frac{1}{2}\}$. By (6.34) and (6.36) we have $\|v_j\|_{L^{p_j}} \leq C$ and so $\text{mes}(D_j) \leq C$. Applying Theorem 5.5 to the equation (6.35) and noticing that $\inf v_j = -1$, we have

$$\text{mes}(D_j) \geq C_1 \tag{6.41}$$

for some $C_1 > 0$ independent of j. Let v_j^* be the (usual) rearrangement of v_j. Namely v_j^* is a radially symmetric, monotone increasing function, satisfying $|\{v_j^* < a\}| = |\{v_j < a\}|$ for any $a \in \mathbb{R}$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. Let $v^* = \lim_{j\to\infty} v_j^*$. Then $v^* \not\equiv 0$ because of (6.41). By Fatou's lemma,

$$\begin{aligned} \int_{\mathbb{R}^n} |v^*|^{k^*} dy &\leq \underline{\lim}_{j\to\infty} \int_{\mathbb{R}^n} |v_j^*|^{p_j+1} dy \\ &= \underline{\lim}_{j\to\infty} \int_{\mathbb{R}^n} |v_j|^{p_j+1} dy \leq C. \end{aligned}$$

Therefore $v^*(r) = o(\frac{1}{r})$ as $r \to \infty$. It follows that for any given $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$, with $\delta \to 0$ at $\varepsilon \to 0$, such that

$$\delta \cdot \text{mes}\{y \in \mathbb{R}^n \mid |v^*(y)|^{k^*} > \delta\} < \varepsilon.$$

Hence

$$\delta \cdot \text{mes}\{y \mid |v_j(y)|^{k^*} > \delta\} = \delta \cdot \text{mes}\{y \mid |v_j^*(y)|^{k^*} > \delta\} < \varepsilon \tag{6.42}$$

for sufficiently large j.

Denote

$$\begin{aligned} &\widetilde{v}_j(y) = v_j(y) + \delta^{1/k^*}, \\ &\Omega_{j,\delta} = \{y \in \Omega_j \mid \widetilde{v}_j(y) < 0\}. \end{aligned}$$

We have

$$\begin{aligned}\sup_{t>0} J^*(tv_j \mid \Omega_{j,\delta}) &= \sup_{t>0} \int_{\Omega_{j,\delta}} \left[\frac{-t^{k+1}}{k+1} v_j S_k(D^2 v_j) - \frac{t^{k^*}}{k^*}|v_j|^{k^*}\right] dy \\ &\geq \sup_{t>0} \int_{\Omega_{j,\delta}} \left[\frac{-t^{k+1}}{k+1} \widetilde{v}_j S_k(D^2 \widetilde{v}_j) - \frac{t^{k^*}}{k^*}|v_j|^{k^*}\right] dy.\end{aligned}$$

By (6.42),

$$\begin{aligned}\int_{\Omega_{j,\delta}} |v_j|^{k^*} &= \int_{\Omega_{j,\delta}} |\widetilde{v}_j + \delta^{1/k^*}|^{k^*} \\ &\leq \int_{\Omega_{j,\delta}} [|\widetilde{v}_j|^{k^*} + C\delta^{1/k^*} |\widetilde{v}_j|^{k^*-1} + C\delta] \\ &\leq \int_{\Omega_{j,\delta}} [|\widetilde{v}_j|^{k^*} + (\delta|\Omega_{j,\delta}|)^{1/k^*} \left(\int_{\Omega_{j,\delta}} |\widetilde{v}_j|^{k^*}\right)^{\frac{k^*-1}{k^*}} + C\delta|\Omega_{j,\delta}| \\ &\leq (1 + C\varepsilon^{1/k^*}) \int_{\Omega_{j,\delta}} |\widetilde{v}_j|^{k^*},\end{aligned}$$

where we have used the fact that

$$\|\widetilde{v}_j\|_{L^{k^*}} \geq C > 0,$$

which follows from (6.41). We obtain

$$\sup_{t>0} J^*(tv_j \mid \Omega_{j,\delta}) \geq \sup_{t>0} \int_{\Omega_{j,\delta}} \left[\frac{-t^{k+1}}{k+1} \widetilde{v}_j S_k(D^2 \widetilde{v}_j) - \frac{t^{k^*}}{k^*}(1 + C\varepsilon^{1/k^*})|\widetilde{v}_j|^{k^*}\right] dy.$$

Hence we obtain

$$\sup_{t>0} J^*(tv_j \mid \Omega_{j,\delta}) \geq (1 - C\varepsilon^{1/k^*})c^*.$$

On the other hand, since $f(x, u) = \lambda|u|^q$ is a lower order term of $|u|^{k^*-1}$ and $\text{mes}(\Omega_{j,\delta})$ are uniformly bounded for fixed δ, we see that when M_j is large enough,

$$\sup_{t>0} J_j(tv_j, \Omega_{j,\delta}) \geq \sup_{t>0} J^*(tv_j, \Omega_{j,\delta}) - \varepsilon_j \geq (1 - C\varepsilon^{1/k^*})c^* - \varepsilon_j$$

with $\varepsilon_j \to 0$ as $M_j \to \infty$, where

$$J_j(u, \Omega_{j,\delta}) = J_{p_j}(u, \Omega_{j,\delta}) = \int_{\Omega_{j,\delta}} \left[\frac{-1}{k+1} u S_k(D^2 u) - \widetilde{\Psi}_j(x, u)\right] dx.$$

For any subdomain $D \subset \Omega_j$, since

$$\int_D v_j S_k(v_j) dy = \int_D v_j (|v_j|^{k^*-1} + \lambda M_j^{q-p_j} |v_j|^{q+1}) dy,$$

we have

$$J_j(v_j, D) = \sup_{t>0} J_j(tv_j, D) \geq 0.$$

Hence

$$J_j(v_j,\Omega_j)\ge J_j(v_j,\Omega_{j,\delta})\ge(1-C\varepsilon^{1/k^*})c^*-\varepsilon_j.$$

We reach a contradiction with (6.32) when ε and ε_j are sufficiently small. This completes the proof. □

The technique in the treatment of the case (6.39) is new. Moreover, it also applies to the case (6.37). By carefully examining the argument, one sees that the function f in (6.29) can be replaced by a more general $f(x,u)$, provided

$$\lim_{t\to 0} f(x,t)|t|^{-k}=0,$$
$$\lim_{t\to 0} f(x,t)|t|^{-k^*+1}=0.$$

In the case the constant c_0 in (6.30) should be replaced by

$$c_0=\inf_{\gamma\in\Gamma}\sup_{s\in[0,1]} J(\gamma(s)),$$

where Γ denotes the set of all paths in $\Phi_0^k(\Omega)$ connecting $U\equiv 0$ to a function u_0 satisfying $J(u_0)<0$.

The verification of (6.32) can be carried out in a similar way as [BN], and the computation is also similar. Let

$$w_\varepsilon(x)=C_n^k(\frac{n-2k}{k})^k\left(\frac{\varepsilon^{\frac{1}{k+1}}}{\varepsilon+|x|^2}\right)^{\frac{n-2k}{2k}}, \tag{6.43}$$

where C_n^k is the binary coefficient. Then w_ε satisfies the equation

$$S_k(D^2u)=|u|^{k^*-1}\quad\text{in }\ \mathbb{R}^n, \tag{6.44}$$

and the constant c^* in (9.3) is attained by w_ε when $\Omega=\mathbb{R}^n$.

Lemma 6.1. *Suppose there exists a ball $B_r(x_0)\subset\Omega$ and*

$$q>\max\{k,\frac{(k+1)(nk-n+2k)}{k(n-2k)}-1\}$$

so that

$$f(x,u)\ge\lambda|u|^q\quad\textit{for}\ \ x\in B_r(x_0) \tag{6.45}$$

for some $\lambda>0$, then $c_0<c^*$.

Proof. Let

$$B=\int_{R^n} w_\varepsilon^{k^*}dx=-\int_{\mathbb{R}^n} w_\varepsilon S_k(D^2w_\varepsilon)dx.$$

B is independent of ε. By a translation we may suppose x_0 is the origin. Let $\varphi(x)$ be a radial cut-off function so that $\varphi=1$ in $B_{r/2}(0)$ and $\varphi=0$ outside $B_r(0)$. We may choose φ so that $u_\varepsilon=:\varphi w_\varepsilon\in\Phi_0^k$ for $\varepsilon>0$ small. Direct computations show that

$$\int_\Omega u_\varepsilon^{k^*}dx=B+O(\varepsilon^{(n-2k)/2k}),$$

$$\int_\Omega (-u_\varepsilon) S_k(D^2 u_\varepsilon) dx = B + O(\varepsilon^{(n-2k)/2k}),$$

If $f(x,u)$ satisfies (6.45), we have

$$\int_\Omega F(x,u_\varepsilon)dx \ge C \int_\Omega |u_\varepsilon|^{q+1} dx \ge C\varepsilon^{\frac{n}{2} - \frac{n-2k}{2(k+1)}q},$$

where $F(x,u) = \int_u^0 f(x,t)dt$. If $q > \frac{(k+1)(nk-n+2k)}{k(n-2k)} - 1$, we have $\frac{n}{2} - \frac{n-2k}{2(k+1)} q < \frac{n-2k}{2k}$. Hence if ε is small enough we have $c_0 < c^*$. □

We refer the reader to [CGY] for more details on radially symmetric solutions to the k-Hessian equations in the critical growth case.

7. Hessian integral estimates

In this section we establish some local integral estimates for k-admissible functions [TW2]. The main estimates include (7.2) and (7.5) below. Our estimates are based on the divergence structure of the k-Hessian operator. As shown as the beginning of Section 5, the k-Hessian operator can also be written as

$$\begin{aligned} S_k[u] &= [D^2u]_k \\ &= \frac{1}{k}\sum u_{ij} S_k^{ij}[u] \\ &= \frac{1}{k}\sum \partial_{x_i}(u_{x_j} S_k^{ij}[u]). \end{aligned} \tag{7.1}$$

As before we denote by $\Phi^k(\Omega)$ the set of all smooth k-admissible functions in Ω, and by $\Phi_0^k(\Omega)$ the set of all smooth k-admissible functions vanishing on $\partial\Omega$.

7.1. A basic estimate.

Here we establish the following basic estimate.

Theorem 7.1. *Let $u \in \Phi^k(\Omega)$. Suppose $u \le 0$ in Ω. Then for any subdomain $\Omega' \subset\subset \Omega$,*

$$\int_{\Omega'} S_k[u] \le C\Big(\int_\Omega |u|\Big)^k, \tag{7.2}$$

where C is a constant depending on n, k, Ω and Ω'.

Proof. It suffices to consider the case $\Omega = B_R(y)$, $\Omega' = B_r(y)$, for some $y \in \mathbb{R}^n$ and $r < R$. Let $\eta \in C^\infty(\overline{\Omega})$ satisfy $0 \le \eta \le 1$, $\eta = 1$ in $B_r(y)$, and $\eta = 0$ when $|x-y| \ge (R+2r)/3$. Let $\widetilde{u} \in C^\infty(\overline{\Omega})$ be the unique k-convex solution of the Dirichlet problem

$$\begin{aligned} S_k[\widetilde{u}] &= \eta S_k[u] + \delta^n \ \text{ in } \ \Omega, \\ \widetilde{u} &= 0 \ \text{ on } \ \partial\Omega. \end{aligned}$$

Then $S_k[u + \frac{\delta}{2}|x-y|^2] \geq S_k[\widetilde{u}]$. By the comparison principle, we have $\frac{\delta}{2}|x-y|^2 + u \leq \widetilde{u} \leq 0$ in Ω, so that

$$\int_\Omega |\widetilde{u}| \leq \int_\Omega |u| + C\delta.$$

Let $\zeta \in C_0^\infty(\Omega)$ be a cut-off function. Then, by integration by parts,

$$\begin{aligned}\int_\Omega \zeta S_k[\widetilde{u}] &= \frac{1}{k}\int \zeta S_k^{ij}[\widetilde{u}]D_{ij}\widetilde{u} = \frac{1}{k}\int \widetilde{u}S_k^{ij}[\widetilde{u}]D_{ij}\zeta \\ &\leq \frac{1}{k}\max\left(|D^2\zeta|\,|\widetilde{u}|\right)\int_{\mathrm{supp}D^2\zeta} S^{ii}[\widetilde{u}] \\ &= \frac{n-k+1}{k}\max\left(|D^2\zeta|\,|\widetilde{u}|\right)\int_{\mathrm{supp}D^2\zeta} S_{k-1}[\widetilde{u}].\end{aligned}$$

Choose $\zeta = 1$ in $B_{(R+r)/2}(y)$, $\zeta = 0$ for $|x-y| \geq (5R+r)/6$, $|D^2\zeta| \leq C(R-r)^{-2}$. By the Harnack inequality (4.10), we have

$$|\widetilde{u}(x)| \leq \int_\Omega |\widetilde{u}| \qquad \forall\ |x-y| \geq \frac{1}{2}(R+r).$$

Hence by sending $\delta \to 0$, we obtain

$$\int_{\Omega'} S_k[u] \leq C\int_\Omega S_{k-1}[u]\int_\Omega (-u)$$

for some constant C depending on n, k, r, R. By iteration we obtain (7.2). □

7.2. **Local integral gradient estimates.** First we prove

Theorem 7.2. *Let $u \in \Phi^k(\Omega)$, $k = 1, \cdots, n$, satisfy $u \leq 0$ in Ω. Then for any sub-domain $\Omega' \subset\subset \Omega$, we have the estimates*

$$\int_{\Omega'} |Du|^q S_l[u] \leq C\left(\int_\Omega |u|\right)^{q+l} \tag{7.3}$$

for all $l = 0, \cdots, k-1$, $0 \leq q < \frac{n(k-l)}{n-k}$, where C is a constant depending on Ω, Ω', n, k, l and q.

Corollary 7.1. *Let $u \in \Phi^k(\Omega)$. Then $\forall\ \ \Omega' \subset\subset \Omega$,*

$$\|Du\|_{L^q(\Omega')} \leq C\int_\Omega |u| \tag{7.4}$$

for $q < \frac{nk}{n-k}$, where C depends on n, k, q, Ω, and Ω'.

Inequality (7.4) follows from (7.3) by taking $l = 0$. When $k = 1$, (7.4) can be found in [H]. Corollary 7.1 asserts that a k-admissible function is in the local Sobolev space $W^{1,q}_{\mathrm{loc}}(\Omega)$. When $k > n/2$, we have $q > n$, and by the Sobolev imbedding theorem, $u \in C^\alpha_{loc}(\Omega)$ with $\alpha \in (0, 2-\frac{n}{k})$. But recall that in Theorem 5.4 we have shown that $u \in C^\alpha_{loc}(\Omega)$ with $\alpha = 2 - \frac{n}{k}$.

To prove Theorem 7.2, let us denote, for a real $n \times n$ matrix $\mathcal{A} = [a_{ij}]$, not necessarily symmetric,

$$\begin{aligned} S_k(\mathcal{A}) &= [\mathcal{A}]_k, \\ S_k^{ij}(\mathcal{A}) &= \frac{\partial}{\partial a_{ij}}[\mathcal{A}]_k. \end{aligned} \tag{7.5}$$

Then for any vector field $g = (g_1, \cdots, g_n)$, $g_i \in C^1(\Omega)$, $i = 1, \cdots, n$, it follows that

$$\begin{aligned} D_i S_k^{ij}(Dg) &= 0, \quad j = 1, \cdots, n, \\ S_k^{ij}(Dg) D_i g_j &= k S_k(Dg). \end{aligned} \tag{7.6}$$

Now we introduce a broader class of operators, namely, the p-k-Hessian operators, given for $k = 1, \cdots, n$, $p \geq 2$, $u \in C^2(\Omega)$, by

$$S_{k,p}[u] = \left[D(|Du|^{p-2}Du)\right]_k. \tag{7.7}$$

where

$$D(|Du|^{p-2}Du) = |Du|^{p-2}\left(I + (p-2)\frac{Du \otimes Du}{|Du|^2}\right)D^2u.$$

When $k = 1$, it is the well-known p-Laplacian operator,

$$S_{1,p}[u] = \mathrm{div}\ (|Du|^{p-2}Du). \tag{7.8}$$

One can verify by direct computation that the p-k-Hessian operator is invariant under rotation of coordinates.

Let us call a function $u \in C^2(\Omega)$, *p-k-admissible* in Ω if $S_{l,p}[u] \geq 0$ for all $l = 1, \cdots, k$. We then have the following relation between k-admissibility and p-k-admissibility.

Lemma 7.1. *If u is k-admissible, then u is p-l-admissible for $l = 1, \cdots, k-1$ and $p - 2 \leq \frac{n(k-l)}{l(n-k)}$.*

Proof. At a point $y \in \Omega$, where $Du(y) \neq 0$, we fix a coordinate system so that the x_1 axis is directed along the vector $Du(y)$ and the remaining axes are chosen so that the reduced Hessian $[D_{ij}u]_{i,j=2,\cdots,n}$ is diagonal. It follows then that the p-Hessian is given by

$$D_i\big(|Du|^{p-2}D_ju\big) = |Du|^{p-2}\begin{cases} (p-1)D_{i1}u & \text{if } j = 1, i \geq 1, \\ D_{1j}u & \text{if } i = 1, j > 1, \\ D_{ii}u & \text{if } j = i > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

Hence by calculation, we obtain for $l = 1, \cdots, k-1$ at the point y, setting $\widetilde{\lambda}_i = D_{ii}u(y)$, $i = 1, \cdots, n$,

(7.10)

$$|Du|^{l(2-p)}S_{l,p}[u] = (p-1)\widetilde{\lambda}_1\sigma_{l-1;1}(\widetilde{\lambda}) + \sigma_{l;1}(\widetilde{\lambda}) - (p-1)\sum_{i=2}^{n}\sigma_{l-2;1i}(\widetilde{\lambda})(D_{i1}u)^2.$$

From the k-admissibility of u, we have

$$S_k[u] = \widetilde{\lambda}_1 \sigma_{k-1;1}(\widetilde{\lambda}) + \sigma_{k;1}(\widetilde{\lambda}) - \sum_{i=2}^{n} \sigma_{k-2;1i}(\widetilde{\lambda})(D_{i1}u)^2 \geq 0$$

so that using Newton's inequality, in the form

$$\frac{\sigma_{k;1}}{\sigma_{k-1;1}} \leq \frac{l(n-k)}{k(n-l)} \frac{\sigma_{l;1}}{\sigma_{l-1;1}}, \tag{7.11}$$

we have, for $p - 1 \leq \frac{k(n-l)}{l(n-k)}$, the inequality

$$\begin{aligned}
\frac{1}{p-1}|Du|^{l(2-p)} S_{l,p}[u] &\geq \widetilde{\lambda}_1 \sigma_{l-1;1}(\widetilde{\lambda}) + \frac{\sigma_{k;1}}{\sigma_{k-1;1}} \sigma_{l-1;1}(\widetilde{\lambda}) - \sum_{i=2}^{n} \sigma_{l-2;1i}(\widetilde{\lambda})(D_{i1}u)^2 \\
&\geq \frac{\sigma_{l-1;1}}{\sigma_{k-1;1}} \sum_{i=2}^{n} \sigma_{k-2;1i}(D_{i1}u)^2 - \sum_{i=2}^{n} \sigma_{l-2;1i}(D_{i1}u)^2 \\
&= \frac{1}{\sigma_{k-1;1}} \sum_{i=2}^{n} \Big(\sigma_{l-1;1}\sigma_{k-2;1i} - \sigma_{k-1;1}\sigma_{l-2;1i} \Big) (D_{i1}u)^2 \\
&= \frac{1}{\sigma_{k-1;1}} \sum_{i=2}^{n} \Big(\sigma_{l-1;1i}\sigma_{k-2;1i} - \sigma_{k-1;1i}\sigma_{l-2;1i} \Big) (D_{i1}u)^2 \\
&\geq 0.
\end{aligned}$$

Note that when applying Newton's inequality to the last inequality, the coefficient in $(\sigma_{l-1;1i}\sigma_{k-2;1i} - \sigma_{k-1;1i}\sigma_{l-2;1i})$ is better than we need. □

Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$. Suppose $\lambda_1 \geq \cdots \geq \lambda_n$. Let $u = \frac{1}{2}\lambda_i x_i^2$. By Lemma 7.1, we have $\Delta_p u \geq 0$ for $p \leq 2 + \frac{n(k-1)}{n-k}$. Hence

$$\sum_i \lambda_i + \frac{n(k-1)}{n-k}\lambda_n \geq 0. \tag{7.12}$$

Proof of Theorem 7.2. Setting

$$p^* = 1 + \frac{k(n-l)}{l(n-k)}, \qquad k < n,\ l < k,$$

we obtain from Lemma 7.1 and the formula (7.10), for $2 < p < p^*$ and $u \in \Phi^k(\Omega)$,

$$\begin{aligned}
|Du|^{l(2-p)} S_{l,p}[u] &= \frac{p^*-p}{p^*-2} S_l[u] + \frac{p-2}{p^*-2}|Du|^{l(2-p^*)} S_{l,p^*}[u] \\
&\geq \frac{p^*-p}{p^*-2} S_l[u],
\end{aligned}$$

and hence, for $q = (p-2)l < \frac{n(k-l)}{n-k}$, we have the estimate

$$|Du|^q S_l[u] \leq \frac{p^*-2}{p^*-p} S_{l,p}[u]. \tag{7.13}$$

Theorem 7.2 will follow by estimation of $S_{l,p}[u]$ in $L^1_{\mathrm{loc}}(\Omega)$. For any non-negative cut-off function $\eta \in C^2_0(\Omega)$, we obtain

$$(7.14)\qquad \begin{aligned}\int_\Omega \eta S_{l,p}[u] &= \int_\Omega \eta S_l(D(|Du|^{p-2}Du)) \\ &= \frac{1}{l}\int_\Omega \eta S_l^{ij} D_i(|Du|^{p-2}D_ju) \\ &= -\frac{1}{l}\int_\Omega |Du|^{p-2}S_l^{ij}D_i\eta D_ju.\end{aligned}$$

From (7.9), we have

$$\begin{aligned}S_l^{ij}D_ju &= |Du|^{(l-1)(p-2)}S_l^{ij}(D^2u)D_ju \\ &= |Du|^{(l-1)(p-2)}S_l^{ij}[u]D_ju,\end{aligned}$$

so that, by substituting in (7.14), we obtain

$$\begin{aligned}\int_\Omega \eta S_{l,p}[u] &= -\frac{1}{l}\int_\Omega |Du|^{l(p-2)}S_l^{ij}D_i\eta D_ju \\ &\le \frac{1}{l}\int_\Omega |Du|^{q+1}|D\eta|S_{l-1}[u],\end{aligned}$$

and hence, replacing η by η^l and using (7.13), we obtain

$$\int_\Omega |Du|^q\eta^l S_l[u] \le C\max|D\eta|\int_\Omega |Du|^{q+1}\eta^{l-1}S_{l-1}[u].$$

Consequently,

$$(7.15)\qquad \int_\Omega |Du|^q\eta^l S_l[u] \le \big(C\max|D\eta|\big)^l\int_\Omega |Du|^{q+l},$$

so that the estimate (7.3) is reduced to the case $l = 0$. To handle this case, we take $l = 1$ in (7.15) with

$$q = q(1) < \frac{n(k-1)}{n-k}.$$

If u is k-admissible for $k \ge 2$, we have

$$S_2[u] = \frac{1}{2}\big((\Delta u)^2 - |D^2u|^2\big) \ge 0$$

and hence

$$(7.16)\qquad |D^2u| \le \Delta u.$$

Therefore we obtain from (7.15)

$$\int_\Omega \eta|Du|^q|D^2u| \le C\max|D\eta|\int_\Omega |Du|^{1+q}$$

so that

$$\int_\Omega \eta D\big(|Du|^{1+q}\big) \le C\max|D\eta|\int_\Omega |Du|^{1+q}.$$

Thus by an appropriate choice of η, we obtain for any subdomain $\Omega' \subset\subset \Omega$,

$$\| |Du|^{1+q} \|_{L^{n/(n-1)}(\Omega')} \le C d_{\Omega'}^{-1} \int_\Omega |Du|^{1+q}, \tag{7.17}$$

where $d_{\Omega'} = \mathrm{dist}(\Omega', \partial\Omega)$, C is a constant depending on k, q and n. The estimate (7.3) now follows by interpolation. □

From Theorem 7.2 we may derive corresponding estimates for the k-admissible function themselves.

Theorem 7.3. *Let u be a nonpositive k-admissible function in Ω, $k \le n/2$. Then for any subdomain $\Omega' \subset\subset \Omega$, we have*

$$\int_{\Omega'} |u|^q S_l[u] \le C\Big(\int_\Omega |u|\Big)^{l+q} \tag{7.18}$$

for all $l = 0, \cdots, k-1$, $0 \le q < \frac{n(k-l)}{n-2k}$, where C is a constant depending on Ω, Ω', n, k, l and q.

Proof. With $\eta \ge 0, \in C_0^1(\Omega)$, we estimate

$$\begin{aligned}
\int_\Omega \eta^2(-u)^q S_l[u] &= \frac{q}{l}\int_\Omega \eta^2(-u)^{q-1} S_l^{ij} D_i u D_j u - \frac{1}{l}\int_\Omega (-u)^q S_l^{ij} D_i u D_j \eta^2 \\
&\le \frac{q(n-l+1)}{l}\int_\Omega \eta^2(-u)^{q-1} S_{l-1}|Du|^2 \\
&\qquad + \frac{2(n-l+1)}{l}\int_\Omega \eta(-u)^q S_{l-1}|Du||D\eta| \\
&\le \frac{(q+1)(n-l+1)}{l}\int_\Omega \eta^2(-u)^{q-1} S_{l-1}|Du|^2 \\
&\qquad + \frac{n-l+1}{l}\int_\Omega |D\eta|^2(-u)^{q+1} S_{l-1}
\end{aligned}$$

Now, for any $p < \frac{n(k-l+1)}{n-k}$, we have

$$\int_\Omega \eta^2(-u)^{q-1} S_{l-1}|Du|^2 \le \Big(\int_\Omega \eta^2 S_{l-1}|Du|^p\Big)^{2/p}\Big(\int_\Omega \eta^2(-u)^{\frac{p(q-1)}{p-2}} S_{l-1}\Big)^{1-2/p}$$

so that if $q < \frac{n(k-l)}{n-2k}$, we may choose p so that $q^* = \frac{p(q-1)}{p-2} < \frac{n(k-l+1)}{n-2k}$, and the estimate (7.18) follows from Theorem 7.2 by induction on l. □

8. Hessian measures

In this section we extend the notion of k-admissible functions to non-smooth functions. We assign a measure $\mu_k[u]$ to a k-admissible function u and prove the weak continuity of μ_k. The proof of the weak continuity of μ_k in [TW2] involves delicate integral estimates and is based on the estimates in §7. Here we provide a simpler proof, using ideas from [TW1, TW5]. As an application we prove the existence of a weak solution to the Dirichlet problem of the k-Hessian equation.

8.1. **Non-smooth k-admissible functions.** Observe that a C^2 function u is k-admissible if and only if for any matrix $A = \{a_{ij}\}$ with eigenvalues in the cone

$$\Gamma_k^* = \{\lambda^* \in \mathbb{R}^n \mid \lambda^* \cdot \lambda \le 0 \ \forall \ \lambda \in \Gamma_k\},$$

there holds

$$\sum a_{ij} D_{ij}^2 u \le 0. \tag{8.1}$$

Note that a matrix A with eigenvalues in Γ_k^* must be negative definite. From (8.1) we can extend the notion of k-admissibility to non-smooth functions as follows.

Definition 8.1. *A function u in Ω is k-admissible if*
(i) it is upper semi-continuous and the set $\{u = -\infty\}$ has measure zero; and
(ii) for any matrix $A = \{a_{ij}\}$ with eigenvalues in Γ_k^*,

$$\int_\Omega u a_{ij} D_{ij}^2 \varphi \le 0 \quad \forall \ \varphi \in C_0^\infty(\Omega), \varphi \ge 0. \tag{8.2}$$

Note that when $k = 1$, Γ_k^* contains only the vector $-(1, \cdots, 1)$, and (8.2) becomes $\int_\Omega u(-\Delta\varphi) \le 0$ for any $\varphi \ge 0, \in C_0^\infty(\Omega)$. The above definition implies that an upper semi-continuous function u is k-admissible if it is subharmonic with respect to the operator $L = \sum a_{ij} D_{ij}^2$ for any matrix A with eigenvalues in Γ_k^*.

From (8.2) we see that if u is k-admissible, so is its mollification u_ε, given by

$$u_\varepsilon(x) = \int_\Omega u(x - \varepsilon y)\rho(y)\, dy = \int_\Omega \varepsilon^{-n} \rho(\frac{x-y}{\varepsilon}) u(y), \tag{8.3}$$

where ρ is a mollifier, namely ρ is a smooth, nonnegative function with support in the unit ball $B_1(0)$, and $\int_{B_1(0)} \rho = 1$. Observe that if u is k-admissible, it is also subharmonic. Hence its mollification u_ε converges to u monotone decreasingly. Therefore by Corollary 7.1, a k-admissible function is locally in the Sobolev space $W_{loc}^{1,q}(\Omega)$ for any $q < \frac{nk}{n-k}$.

Lemma 8.1. *Let u_j be a sequence of k-admissible functions which converges to u almost everywhere. Suppose u is upper semi-continuous and the set $\{u = -\infty\}$ has measure zero. Then u is k-admissible and u_j converges to u pointwise.*

Proof. The first assertion follows readily from the definition. The second one is due to that u is upper semi-continuous and u_j is subharmonic and so it satisfies the mean value inequality below. □

Recall that a k-admissible function u is subharmonic, it satisfies the mean value inequality

$$u(y) \le \frac{1}{|B_r(y)|}\int_{B_r(y)} u \quad \forall\ B_r(y) \subset \Omega. \tag{8.4}$$

Therefore if u_j and u are k-admissible and $\{u_j\}$ converges to u almost everywhere, u_j is locally uniformly bounded in $L^1(\Omega)$. Conversely, if a sequence of k-admissible functions $\{u_j\}$ is uniformly bounded in $L^1_{loc}(\Omega)$, then by Corollary 7.1, the set $\{u = -\infty\}$ has measure zero.

We also have the following comparison principle for k-admissible functions.

Lemma 8.2. *Suppose u and v are k-admissible and v is smooth in Ω. Suppose $S_k[v] = 0$ in Ω and for any point $y \in \partial\Omega$, $\underline{\lim}_{x\to y}[v(x) - u(x)] \ge 0$. Then $v \ge u$ in Ω.*

Proof. If there is an interior point $x_0 \in \Omega$ such that $v(x_0) < u(x_0)$, by adding a positive constant $\delta = \frac{1}{2}(u(x_0) - v(x_0))$ to v we may suppose that for any $y \in \partial\Omega$, $\lim_{x\to y}[v(x) - u(x)] \ge \delta > 0$, so that $v > u$ in a neighborhood of the boundary $\partial\Omega$. Therefore for $\varepsilon > 0$ small, we have $v > u_\varepsilon$ near $\partial\Omega$, where u_ε is the mollification of u. By the comparison principle for smooth k-admissible functions, we conclude that $v \ge u_\varepsilon$, which is in contradiction with $v(x_0) < u(x_0) \le u_\varepsilon(x_0)$. □

Lemma 8.3. *Suppose u, v are two k-admissible functions. Then $w = \max(u, v)$ is also k-admissible.*

Proof. Let $u_\varepsilon, v_\varepsilon$ be the mollification of u and v, respectively. Then it suffices to show that $w_\varepsilon = \max(u_\varepsilon, v_\varepsilon)$ is k-admissible. For brevity we drop the subscript ε. Since the function w is semi-convex (i.e., $w + C|x|^2$ is convex for sufficiently large constant C), w is twice differentiable almost everywhere and the eigenvalues of D^2w lies in Γ_k. Let $w_{\varepsilon'}$ be the mollification of w. By integration by parts in (8.3), we have

$$D^2 w_{\varepsilon'}(x) \ge \int_\Omega D^2 w(x - \varepsilon' y)\rho(y)\, dy.$$

Hence for any matrix A with eigenvalues in Γ_k^*,

$$a_{ij}D^2_{ij}w_{\varepsilon'}(x) \le \int_\Omega a_{ij}D^2_{ij}w(x - \varepsilon' y)\rho(y)\, dy \le 0.$$

Hence $w_{\varepsilon'}$, and so also w, is k-admissible. □

8.2. **Perron lifting.** Let u be a k-admissible function in Ω and $\omega \Subset \Omega$ be a subdomain of Ω. The Perron lifting of u in ω, u^ω, is the upper semicontinuous regularization of the function $\hat{u}$,

$$u^\omega(x) = \lim_{t\to 0} \sup_{B_t(x)} \hat{u}, \tag{8.5}$$

where

$$\hat{u}(x) = \sup\{v(x) \mid \ v \text{ is } k\text{-admissible in } \Omega \text{ and } v \le u \text{ in } \Omega - \omega\}.$$

Obviously we have $u^\omega \ge \hat{u}$, and $\hat{u}$ and u^ω coincide in Ω except possibly on $\partial\omega$.

Lemma 8.4. *Assume $\partial\omega$ is $C^{3,1}$ smooth. Then u^ω is a solution of*

$$\begin{aligned} S_k[w] &= 0 \quad in \ \ \omega, \\ w &= u \quad on \ \ \partial\omega, \end{aligned} \tag{8.6}$$

in the sense that there is a sequence of smooth k-admissible functions w_ε which satisfies $S_k[w_\varepsilon] = 0$ in ω and $w_\varepsilon \to u$ on $\overline{\omega}$ pointwise.

Proof. Let u_ε be a mollification of u, as given in (8.3). Let $u_{\varepsilon,j} = u_\varepsilon + 2^{-j}|x|^2$. Then $S_k[u_{\varepsilon,j}] \ge C2^{-kj}$. That is, $u_{\varepsilon,j}$ is a smooth sub-solution to the Dirichlet problem

$$\begin{aligned} S_k[w] &= C2^{-kj} \ \text{ in } \ \omega, \\ w &= u_{\varepsilon,j} \ \text{ on } \ \partial\omega. \end{aligned}$$

Hence from [G], there is a unique global smooth solution $w_{\varepsilon,j} \in C^3(\overline{\omega})$, monotone in j. By (3.5) we have $\sup_\omega |Dw_{\varepsilon,j}| \le \sup_{\partial\omega} |Dw_{\varepsilon,j}|$. On the boundary $\partial\omega$, we have $u_{\varepsilon,j} \le w_{\varepsilon,j} \le \overline{u}_{\varepsilon,j}$, where $\overline{u}_{\varepsilon,j}$ is the harmonic extension of $u_{\varepsilon,j}$ in ω. Hence

$$\sup_{\partial\omega} |Dw_{\varepsilon,j}| \le \sup_{\partial\omega} |Du_{\varepsilon,j}| \le \sup_{\partial\omega} |Du_\varepsilon| + C2^{-j}.$$

Therefore by passing to a subsequence, $w_{\varepsilon,j}$ converges as $j \to \infty$ to a solution w_ε of (8.6) which satisfies the boundary condition $w_\varepsilon = u_\varepsilon$ on $\partial\omega$.

Let $u^\omega_\varepsilon = w_\varepsilon$ in ω and $u^\omega_\varepsilon = u_\varepsilon$ in $\Omega - \omega$. It is easy to see that u^ω_ε is the Perron lifting of u_ε in ω. The proof of Lemma 8.3 implies that u^ω_ε is k-admissible. Since u_ε is monotone decreasing in ε, so is u^ω_ε. By the comparison principle (Lemma 8.2), we have $u^\omega_\varepsilon \ge u^\omega$ in Ω. Hence $u_0 := \lim_{\varepsilon\to 0} u^\omega_\varepsilon \ge u^\omega$.

On the other hand, let u^ω_0 be the upper semicontinuous regularization of u_0. Then by Lemma 8.1, u^ω_0 is k-admissible in Ω. Obviously $u^\omega_0 = u$ in $\Omega - \overline{\omega}$. Hence by definition of u^ω, $u^\omega_0 \le u^\omega$. We obtain $u_0 \ge u^\omega \ge u^\omega_0 \ge u_0$. Hence

$$\lim_{\varepsilon\to 0} u^\omega_\varepsilon = u^\omega.$$

Lemma 8.1 implies the convergence is pointwise. The interior gradient estimate implies that u^ω is locally uniformly Lipschitz continuous in ω. □

Below we will consider the Perron lifting in an annulus $\omega_t = B_{r+t}(x_0) - B_{r-t}(x_0)$. Let us fix r and let t vary. Then u^{ω_t} is monotone in t, namely

$$\lim_{t\to\delta^-} u^{\omega_t}(x) \le u^{\omega_\delta}(x) \le \lim_{t\to\delta^+} u^{\omega_t}(x) \ \ \forall\, x \in \Omega.$$

It follows that $\|u^{\omega_t}\|_{L^1(\Omega)}$, as a function of t, is monotone and bounded. Hence, $\|u^{\omega_t}\|_{L^1(\Omega)}$ is continuous at almost all t. It follows that for almost all $t > 0$

$$\lim_{s\to t} u^{\omega_s}(x) = u^{\omega_t}(x). \tag{8.7}$$

Lemma 8.5. *Suppose u_j, u are k-admissible and $u_j \to u$ a.e. in Ω. Suppose (8.7) holds at t. Then we have $u_j^{\omega_t} \to u^{\omega_t}$ a.e. in Ω as $j \to \infty$.*

Proof. Since $u_j^{\omega_t}$ and u^{ω_t} are locally uniformly Lipschitz continuous in ω_t, by passing to a subsequence, we may assume that $u_j^{\omega_t}$ is convergent. Let $w' = \lim u_j^{\omega_t}$ and w be the upper semicontinuous regularization of w'. Then by Lemma 8.1, w is k-admissible and $w = u$ in $\Omega - \overline{\omega}_t$. Hence by the definition of the Perron lifting, we have $u^{\omega_t} \ge w$.

Next we prove that for any $\delta > 0$, $w \ge u^{\omega_{t-\delta}}$. Once this is proved, we have $u^{\omega_t} \ge w \ge u^{\omega_{t-\delta}}$. Sending $\delta \to 0$, we obtain $u^{\omega_t} = w$ by (8.7).

To prove $w \ge u^{\omega_{t-\delta}}$, it suffices to prove that for any given $\varepsilon > 0$, $u_j^{\omega_t} \ge u - \varepsilon$ on $\partial\omega_{t-\delta}$ for sufficiently large j. By the interior gradient estimate, $u_j^{\omega_t}$ is locally uniformly Lipschitz continuous in ω_t. If there exists a point $x_0 \in \partial B_{r-\delta/2}$ such that $u(x_0) > u_j^{\omega_r}(x_0) + \varepsilon$ for all large j, by (8.4), there is a Lebesgue point $x_1 \in B_{\delta/4}(x_0)$ of u such that $u(x_1) > u_j^{\omega_r}(x_1) + \frac{1}{2}\varepsilon$ for all large j. It follows that the limit function w' is strictly less than u a.e. near x_1. We reach a contradiction as $w' = \lim_{j\to\infty} u_j^{\omega_r} \ge \lim_{j\to\infty} u_j = u$. □

8.3. **Weak continuity.** Denote $\mu_k[u] = S_k[u]dx$. It is a nonnegative measure if u is a C^2 smooth, k-admissible function. First we prove the following monotonicity formula.

Lemma 8.6. *Let u, v be two smooth k-admissible function in Ω. Suppose $u = v$ on $\partial\Omega$ and $u(x) > v(x)$ for $x \in \Omega$, near $\partial\Omega$. Then*

$$\int_\Omega S_k[u] \le \int_\Omega S_k[v]. \tag{8.8}$$

Proof. We may assume that $\partial\Omega$ is smooth, otherwise it suffices to prove (8.8) in $\{u - \delta > v\}$ and send $\delta \to 0$. We have

$$\begin{aligned} \frac{d}{dt}\int_\Omega S_k[u + t(v-u)] &= \int_\Omega S_k^{ij}[u + t(v-u)](v-u)_{ij} \\ &= \int_{\partial\Omega} (v-u)_i \gamma_j S_k^{ij}[u + t(v-u)]. \end{aligned}$$

It is easy to see that the integrand on the right hand side is nonnegative. □

Lemma 8.7. *Let $u_j \in C^2(\Omega)$ be a sequence of k-admissible functions which converges to a k-admissible function u in Ω almost everywhere. Then $\mu_k[u_j]$ converges to a measure μ weakly, namely for any smooth function φ with compact support in Ω,*

$$\int_\Omega \varphi \, d\mu_k[u_j] \to \int_\Omega \varphi \, d\mu. \tag{8.9}$$

Proof. For any open set $\Omega' \Subset \Omega$, by Theorem 7.1, $\mu_k[u_j](\Omega')$ is uniformly bounded. Hence there is a subsequence of $\mu_k[u_j]$ which converges weakly to a measure μ. We need to prove that μ is independent of the choice of subsequences of $\{u_j\}$.

Let $\{u_j\}, \{v_j\}$ be two sequences of k-admissible functions. Suppose both sequences $\{u_j\}$ and $\{v_j\}$ converge to u almost everywhere in Ω. Suppose that

$$\mu_k[u_j] \to \mu, \qquad \mu_k[v_j] \to \nu \tag{8.10}$$

weakly as measures. To prove that $\mu = \nu$, it suffices to prove that for any ball $B_r(x_0) \Subset \Omega$, $\mu(B_r) = \nu(B_r)$, or equivalently, for any small $t > 0$,

$$\mu(B_{r-2t}) \le \nu(B_{r+2t}), \tag{8.11}$$

$$\nu(B_{r-2t}) \le \mu(B_{r+2t}). \tag{8.12}$$

Let ε_j be a sequence of small positive constants converging to zero. Let $\hat{u}_j = \frac{1}{2}\varepsilon_j |x|^2 + u_j$ and $\hat{v}_j = \frac{1}{2}\varepsilon_j |x|^2 + v_j$. Then

$$S_k[\hat{u}_j] = S_k[u_j] + \sum_{i=1}^{k} C_{k,i} \varepsilon_j^i S_{k-i}[u] \ge C\varepsilon_j^k.$$

By Theorem 7.1, $S_{k-i}[u_j]$ is locally uniformly bounded in $L^1(\Omega)$. Hence by (8.10), $\mu_k[\hat{u}_j] \to \mu$ weakly. Therefore we may assume directly that $S_k[u_j] \ge \varepsilon_j > 0$ and $S_k[v_j] \ge \varepsilon_j > 0$.

We prove (8.11) and (8.12) in two steps. In the first one we assume that $u_j, v_j \in C^2(\Omega)$, $u \in C^0(\Omega)$, and $u_j, v_j \to u$ locally uniformly in Ω.

Let $\hat{v}_j = v_j + \delta_j[|x - x_0|^2 - r^2]$. Since $|u_j - v_j|$ converges to zero uniformly, there exists $\delta_j \to 0$ such that $\hat{v}_j < u_j$ in $B_{r-\frac{1}{2}t}$ and $\hat{v}_j > u_j$ on $\partial B_{r+\frac{1}{2}t}$. Let $A = \{x \in \Omega \mid \hat{v}_j(x) < u_j(x)$. Then $B_{r-\frac{1}{2}t}(x_0) \subset A \subset B_{r+\frac{1}{2}t}(x_0)$. Hence by Lemma 8.6,

$$\begin{aligned}
\int_{B_{r-2t}(x_0)} S_k[u_j] &\le \int_A S_k[u_j] \le \int_A S_k[\hat{v}_j] \\
&\le \int_A S_k[v_j] + O(\delta_j) \le \int_{B_{r+2t}} S_k[v_j] + O(\delta_j).
\end{aligned}$$

Sending $j \to \infty$ we obtain (8.11). Similarly we can prove (8.12).

The second step essentially repeats the first step. From the first step we see that for any continuous k-admissible function u, we can assign a measure $\mu_k[u]$ such that if a sequence of smooth k-admissible functions u_j converges to u uniformly, then $\mu_k[u_j]$ converges to $\mu_k[u]$ weakly as measure. In particular it means $\mu_k[u_j^t]$ and $\mu_k[v_j^t]$ are well defined, where we denote by u_j^t, v_j^t and u^t the Perron lifting of u_j, v_j and u in $\omega_t = B_{r+t}(x_0) - B_{r-t}(x_0)$. By Lemma 8.5, we have

$$u_j^t, v_j^t \to u^t \quad \text{in} \quad \Omega.$$

By the interior gradient estimate, u_j^t, v_j^t are locally uniformly Lipschitz continuous in ω_t. But u_j and v_j may not be C^2 in ω_t. To avoid such situation, we replace u_j^t (and v_j^t) in ω_t by the solution of $S_k[u] = \varepsilon_j'$ in ω_t satisfying the boundary condition $u_j^t = u_j$ (and $v_j^t = v_j$) on $\partial\omega_t$, for sufficiently small ε_j'.

Let $\hat{v}_j^t = v_j^t + \delta_j[|x-x_0|^2 - r^2]$. Since $|u_j^t - v_j^t|$ converges to zero uniformly in $B_{r+\frac{3}{4}t} - B_{r-\frac{3}{4}t}$, there exists $\delta_j \to 0$ such that $\hat{v}_j^t < u_j^t$ on $\partial B_{r-\frac{1}{2}t}$ and $\hat{v}_j^t > u_j^t$ on $\partial B_{r+\frac{1}{2}t}$. Let A' be the component of $\{u_j^t < \hat{v}_j^t\}$ which contains $\partial B_{r-\frac{1}{2}t}$. Let $\partial' A'$ be the boundary of A' in the annulus $B_{r+\frac{3}{4}t} - B_{r+\frac{1}{4}t}$. Let A be the domain enclosed by $\partial' A'$. Then $B_{r-\frac{1}{2}t} \subset A \subset B_{r+\frac{1}{2}t}$. Hence as above,

$$\begin{aligned}
\int_{B_{r-2t}(x_0)} S_k[u_j] &\le \int_A S_k[u_j^t] \le \int_A S_k[\hat{v}_j^t] \\
&\le \int_A S_k[v_j^t] + O(\delta_j) \le \int_{B_{r+2t}} S_k[v_j] + O(\delta_j).
\end{aligned}$$

Sending $j \to \infty$ we obtain (8.11). Similarly we can prove (8.12). □

Therefore for any k-admissible function u, we can assign a measure $\mu_k[u]$ to u, and μ_k is weakly continuous in u.

Theorem 8.1. *For any k-admissible function u, there exists a Radon measure $\mu_k[u]$ such that*
(i) $\mu_k[u] = S_k[u]dx$ if $u \in C^2(\Omega)$; and
(ii) if $\{u_j\}$ is a sequence of k-admissible functions which converges to u a.e., then $\mu_k[u_j] \to \mu_k[u]$ weakly as measure.

As an application, we compute the k-Hessian measure for the function

$$w_k(x) = \begin{cases} |x-y|^{2-n/k} & k > n/2, \\ \log|x-y| & k = n/2, \\ -|x-y|^{2-n/k} & k < n/2. \end{cases}$$

We have

$$\mu_k[w_k] = \begin{cases} \left(2 - \frac{n}{k}\right)\left[\binom{n}{k}\omega_n\right]^{1/k}\delta_y & \text{if } k \ne \frac{n}{2}, \\ \left[\binom{n}{k}\omega_n\right]^{1/k}\delta_y & \text{if } k = \frac{n}{2}, \end{cases}$$

where ω_n is the area of the unit sphere, and δ_y is the Dirac measure at y.

8.4. **The Dirichlet problem.** As another application of Theorem 8.1, we consider the Dirichlet problem

$$(8.13)\qquad \begin{aligned} S_k[u] &= \nu \text{ in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega. \end{aligned}$$

When u is not smooth, the Hessian operator $S_k[u]$ in (8.13) is understood as $\mu_k[u]$, and u is called a *weak solution*. The following theorem was included in [TW2]. Here we give a different proof.

Theorem 8.2. *Let Ω be a $(k-1)$-convex domain with smooth boundary. Let φ be a continuous function on $\partial\Omega$ and ν be a nonnegative Radon measure. Suppose that ν can be decomposed as*

$$(8.14)\qquad \nu = \nu_1 + \nu_2$$

such that ν_1 is a measure with compact support in Ω, and $\nu_2 \in L^p(\Omega)$ for some $p > \frac{n}{2k}$ if $k \le \frac{n}{2}$, or $p = 1$ if $k > \frac{n}{2}$. Then there exists a k-admissible weak solution u to (8.13).

Proof. Let ν_j be a sequence of smooth, positive functions which converges to ν weakly as measure. By the decomposition (8.14), we may assume that ν_j is uniformly bounded in $L^p(N_{\delta_0})$, where $N_\delta = \{x \in \Omega,\ \text{dist}(x, \partial\Omega) < \delta\}$. Let φ_j be a sequence of smooth functions which converges monotone increasingly to φ. Let u_j be the solution of

$$(8.15)\qquad \begin{aligned} S_k[u] &= \nu_j \text{ in } \Omega, \\ u &= \varphi_j \text{ on } \partial\Omega. \end{aligned}$$

If u_j is uniformly bounded in $L^1(\Omega)$, by Corollary 7.1, $\{u_j\}$ contains a convergent subsequence which converges to a k-admissible function u. By Lemma 8.7, u is a weak solution to (8.13). Therefore it suffices to prove that u_j is uniformly bounded in $L^1(\Omega)$ and the limit function u satisfies the boundary condition $u = \varphi$ on $\partial\Omega$.

For $\delta > 0$, let $\eta_\delta \in C_0^\infty(\Omega)$ be a nonnegative function satisfying $\eta_\delta(x) = 1$ when $\text{dist}(x, \partial\Omega) > \delta$ and $\eta_\delta(x) = 0$ when $\text{dist}(x, \partial\Omega) < 3\delta/4$. Let

$$\begin{aligned} \nu_{j,\delta} &= \nu_j \eta_\delta + \delta, \\ \nu'_{j,\delta} &= \nu_j (1 - \eta_\delta) + \delta. \end{aligned}$$

Then both $\nu_{j,\delta}$ and $\nu'_{j,\delta}$ are smooth, positive functions. Let $u_{j,\delta}$ be the solution of the

$$(8.16)\qquad \begin{aligned} S_k[u] &= \nu_{j,\delta} \text{ in } \Omega, \\ u &= \varphi_j \text{ on } \partial\Omega. \end{aligned}$$

Let $u'_{j,\delta}$ be the solution of the

$$\begin{aligned} S_k[u] &= \nu'_{j,\delta} \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{aligned}$$

By Theorem 5.5, we have

$$\|u'_{j,\delta}\|_{L^\infty(\Omega)} \le C\|\nu'_{j,\delta}\|^{1/k}_{L^p(\Omega)} \to 0$$

uniformly in j, as $\delta \to 0$. By the concavity of $S_k^{1/k}[u]$,

$$S_k^{1/k}[u_{j,\delta} + u'_{j,\delta}] \ge S_k^{1/k}[u_{j,\delta}] + S_k^{1/k}[u'_{j,\delta}]$$

Hence $u_{j,\delta} + u'_{j,\delta}$ is a sub-barrier to the Dirichlet problem (8.15). Hence it suffices to prove that for any given $\delta > 0$, $u_{j,\delta}$ is uniformly bounded in $L^1(\Omega)$ and $u_\delta = \lim u_{j,\delta}$ satisfies the boundary condition $u_\delta = \varphi$ on $\partial\Omega$.

For any fixed $\delta > 0$, we claim that $u_j = u_{j,\delta}$ is uniformly bounded in $N_{\delta/2}$ (in the following we drop the subscript δ). Indeed, if this is not true, for a fixed, sufficiently small $\varepsilon \in (0, \frac{1}{4}\delta)$, let $D_\varepsilon = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \varepsilon\}$ be the ε-neighborhood of Ω. Let $\eta = K(\rho_{D_\varepsilon}(x) + K'\rho^2_{D_\varepsilon}(x))$, where $\rho_{D_\varepsilon}(x)$ is the distance from x to ∂D_ε. Then η is k-admissible when K' is large and $d_x < \delta_1$ for some $\delta_1 > 0$ depending only n, k and $\partial\Omega$. If u_j is not uniformly bounded in $N_{\delta/2}$, by the Harnack inequality (Theorem 4.2), $u_j(x) \to -\infty$ uniformly for $x \in \{x \in \Omega \mid \text{dist}(x, \partial\Omega) = 2\varepsilon\}$. Choose K large such that $\eta < u_j$ on $\partial\Omega$ and $u_j < \eta$ on $\partial\Omega_\varepsilon$. Let

$$\partial A_j = \{x \in D_\varepsilon \mid \rho_{D_\varepsilon}(x) \in (0, 2\varepsilon), \eta(x) = u_j(x)\}.$$

Then $\partial A_j \subset N_{\delta/2}$. Let A_j be the domain enclosed by ∂A_j. By Lemma 8.6,

$$\nu_{j,\delta}(\Omega) \ge \nu_{j,\delta}(A_j) = \int_{A_j} S_k[u_j] \ge \int_{A_j} S_k[\eta] \to \infty$$

as $K \to \infty$. But the left hand side is uniformly bounded. The contradiction implies that u_j is uniformly bounded in $N_{\delta/2}$. Since u_j is sub-harmonic, by the mean value inequality (8.4) it follows that u_j is uniformly bounded in $L^1_{loc}(\Omega)$.

To show that $u = \lim u_j$ satisfies the boundary condition $u = \varphi$ on $\partial\Omega$, extend φ to a harmonic function in Ω. Since u is sub-harmonic, by the comparison principle we have $u \le \varphi$ in Ω. Hence for any $y \in \partial\Omega$, $\overline{\lim}_{x\to y} u(x) \le \varphi(y)$. Next we prove that $\underline{\lim}_{x\to y} u(x) \ge \varphi(y)\ \forall\ y \in \partial\Omega$. Let w_j be the solution to the Dirichlet problem

$$\begin{aligned} S_k[w] &= K \ \text{ in } \ \Omega \\ w &= \varphi_j \ \text{ on } \ \partial\Omega. \end{aligned}$$

Since u_j is uniformly bounded in $N_{\delta/2}$, we can fix a sufficiently large K, which may depend on δ but is independent of j, such that the solution

$w_j < -K$ on $\Omega \cap \partial N_{\delta/2}$. Recall that $\nu_{j,\delta} = \delta < 1$ in $N_{\delta/2}$. By the comparison principle we have $w_j \le u_j$ in $N_{\delta/2}$. But when K is fixed, we have $\lim_{j\to\infty} \underline{\lim}_{x\to y} w_j(x) = \varphi(y)$ uniformly. $\square$

The uniqueness is a more complicated issue. It is proved in [TW3] that if $\nu \in L^1$, the solution in Theorem 8.2 is unique.

9. Local behavior of admissible functions

In this section we prove a Wolff potential estimate and give a necessary and sufficient condition such that a weak solution is Hölder continuous. Results in this section are due to D. Labutin [Ld].

9.1. **The Wolff potential estimate.** Given a Radon measure μ on Ω, we denote

$$W_k^\mu(x,r) = \int_0^r \left(\frac{\mu(B_t(x))}{t^{n-2k}}\right)^{\frac{1}{k}} \frac{dt}{t}. \tag{9.1}$$

$W_k^\mu(x,r)$ is called Wolff potential.

Lemma 9.1. *Let $u \le 0$ be k-admissible in $B_R(0)$. Then*

$$\left[\frac{\mu_k[u](B_{9R/10})}{R^{n-2k}}\right]^{\frac{1}{k}} \le C \inf_{\partial B_{R/2}} (-u). \tag{9.2}$$

If furthermore $\mu_k[u] = 0$ in $(B_{5R/8} - B_{3R/8}) \cup (B_{11R/10} - B_{9R/10})$, then

$$\inf_{\partial B_R} u - \inf_{\partial B_{R/2}} u \le C\left(\frac{\mu[u](B_R)}{R^{n-2k}}\right)^{\frac{1}{k}}, \tag{9.3}$$

where C is independent of R and u.

Proof. First we prove (9.2). Let ψ be the solution of $S_k[\psi] = 0$ in $B_R - \overline{B}_{9R/10}$, $\psi = 0$ on ∂B_R, and $\psi = u$ on $\partial B_{9R/10}$. By replacing u by ψ in the annulus $B_R - \overline{B}_{9R/10}$, we may assume that $u = 0$ on ∂B_R and $\mu_k[u] = 0$ in $B_R - \overline{B}_{9R/10}$. By the Harnack type inequality,

$$\sup_{\partial B_{19R/20}} (-u) \le C \inf_{\partial B_{19R/20}} (-u). \tag{9.4}$$

Let φ be a radial k-admissible function satisfying $S_k[\varphi] = 0$ in $B_R - B_{19R/20}$, $\varphi = 0$ on ∂B_R, and $\varphi = \inf_{\partial B_{19R/20}} u$ on $\partial B_{19R/20}$. Then by the comparison principle we have $u \ge \varphi$ in $B_R - \overline{B}_{9R/10}$. By Lemma 8.6, it follows that

$$\mu_k[u](B_R) \le \mu_k[\varphi](B_R) = C_{n,k} R^{n-2k} \big| \inf_{\partial B_{19R/20}} u\big|^k.$$

We obtain by the Harnack inequality (9.4)

$$\left[\frac{\mu_k[u](B_{9R/10})}{R^{n-2k}}\right]^{\frac{1}{k}} \le C \inf_{\partial B_{19R/20}} |u|.$$

Note that u is subharmonic, $\inf_{\partial B_{R/2}} |u| \le \inf_{\partial B_{19R/20}} |u|$. We obtain (9.2).

To prove (9.3), let w be the solution of $S_k[w] = \mu_k[u]$ in B_R and $w = \inf_{\partial B_R} u$ on ∂B_R. The solution w should be obtained as the limit of the solution w_ε to $S_k[w] = \mu_k[u_\varepsilon]$ in B_R and $w_\varepsilon = u_\varepsilon$ on ∂B_R. Note that by assumption, $S_k[u] = 0$ near ∂B_R, so u is Lipschitz continuous near ∂B_R. It follows that $w \le u$ and so

$$\inf_{\partial B_R} u - \inf_{\partial B_{R/2}} u \le \inf_{\partial B_R} w - \inf_{\partial B_{R/2}} w.$$

Therefore to prove (9.3), we may assume that u =constant on ∂B_R. By subtracting we may assume that $u = 0$ on $\partial\Omega$.

Let φ be a radial k-admissible function satisfying $S_k[\varphi] = 0$ in $B_R - B_{R/2}$, $\varphi = 0$ on ∂B_R, and $\varphi = \sup_{\partial B_{R/2}} u$. Then by the comparison principle we have $u \le \varphi$ in $B_R - B_{R/2}$. It follows that

$$\mu_k[u](B_R) \ge \mu_k[\varphi](B_R) = C_{n,k} R^{n-2k} \big| \sup_{\partial B_{R/2}} u \big|^k.$$

By the Harnack inequality, $|\sup_{\partial B_{R/2}} u| \ge C |\inf_{\partial B_{R/2}} u|$. We obtain (9.3). □

Theorem 9.1. *Let $u \le 0$ be a k-admissible function in $B_{2R}(0)$. Then we have*

$$C^{-1} W_k^\mu(0, R) \le -u(0) \le C\{W_k^\mu(0, 2R) + |\sup_{\partial B_R} u|\}, \tag{9.5}$$

where $\mu = \mu[u]$, and C is independent of u and R.

Proof. First we prove the left inequality, namely $|u(0)| \ge C^{-1} W_k^\mu(0, R)$. For any $r \in (0, \frac{1}{2}R)$, let $\omega = B_{9r/8} - B_{3r/4}$, let u^ω be the Perron lifting of u over ω, and let $\widetilde{u}$ be the Perron lifting of u^ω over $B_{7r/8}$. By (9.2) we have

$$\begin{aligned}\left[\frac{\mu_k[\widetilde{u}](B_{9r/10})}{r^{n-2k}}\right]^{1/k} &\le C\big(\sup_{\partial B_{9r/8}} \widetilde{u} - \sup_{\partial B_{7r/8}} \widetilde{u}\big) \\ &\le C\big(\sup_{\partial B_{3r/2}} u - \sup_{\partial B_{3r/4}} u\big).\end{aligned}$$

Observing that

$$\mu_k[u](B_{r/2}) = \mu_k[u^\omega](B_{r/2}) \le \mu_k[u^\omega](B_{9r/10}) = \mu_k[\widetilde{u}](B_{9r/10}),$$

we obtain,

$$\left[\frac{\mu_k[u](B_{r/2})}{r^{n-2k}}\right]^{1/k} \le C\big(\sup_{\partial B_{3r/2}} u - \sup_{\partial B_{3r/4}} u\big). \tag{9.6}$$

For $j = 0, 1, \cdots$, let $R_j = 2^{-j}R$. We have,

$$C^{-1}\sum_{j=0}^{\infty}\left[\frac{\mu_k(B_{R_j})}{R_j^{n-2k}}\right]^{1/k} \le W_k^{\mu}(0,R) \le C\sum_{j=0}^{\infty}\left[\frac{\mu_k(B_{R_j})}{R_j^{n-2k}}\right]^{1/k}. \tag{9.7}$$

Hence letting $r = R_j$ in (9.6) and summing up we obtain the first inequality of (9.5).

To prove the second inequality, we may suppose $\mu_k[u] = 0$ in $B_{2R} - B_R$. Let $R_j = 2^{-j}R$, $\omega = \bigcup_{j=1}^{\infty}(B_{5R_j/4} - B_{3R_j/4})$, and let u^{ω} be the Perron lifting of u over ω. Then $u = u^{\omega}$ in $B_{2R} - \omega$, $\mu_k[u^{\omega}] = 0$ in ω. Since $\mu_k[u]$ depends on u locally, we have, for any $r > 0$,

$$\mu_k[u](B_r) \le \mu_k[u^{\omega}](B_{2r}) \le \mu_k[u](B_{4r}).$$

Hence to prove the second inequality we may suppose directly that $\mu_k[u] = 0$ in ω.

Let $u_j = u^{B_{R_j}}$, the Perron lifting of u over B_{R_j}. Then $u_j \searrow u$ pointwise. In particular $u_j \searrow u$ at the origin. Hence to prove the second inequality it suffices to show that for all $s \ge 1$,

$$|\inf_{\partial B_{R_s}} u| \le C\sum_{j=0}^{s-1}\left[\frac{\mu_k[u](B_{R_j})}{R_j^{n-2k}}\right]^{1/k} + C|\sup_{\partial B_R} u| \tag{9.8}$$

and send $s \to \infty$.

By (9.3) we have

$$|\inf_{\partial B_{R_s}} u| \le |\inf_{\partial B_{R_{s-1}}} u| + C\left[\frac{\mu_k[u](B_{R_{s-1}})}{R_{s-1}^{n-2k})}\right]^{1/k}.$$

Applying (9.3) repeatedly, we obtain, for $0 \le j \le s$,

$$|\inf_{\partial B_{R_s}} u| \le |\inf_{\partial B_{R_j}} u| + C\sum_{i=j}^{s-1}\left[\frac{\mu_k[u](B_{R_j})}{R_j^{n-2k}}\right]^{1/k}.$$

Letting $j = 0$, we obtain (9.8). Since $\mu_k[u] = 0$ in $B_{5R/4} - B_{3R/4}$, we have $|\inf_{\partial B_R} u| \le C|\sup_{\partial B_R} u|$ by the Harnack inequality. This completes the proof. □

9.2. **Hölder continuity of weak solutions.** From Theorem 9.1 we obtain a necessary and sufficient condition for a weak solution to be Hölder continuous.

Theorem 9.2. *A k-admissible function u in Ω is Hölder continuous if and only if there exists a constant $\varepsilon > 0$ such that for any $x \in \Omega$, $r \in (0,1)$, the measure $\mu_k[u]$ satisfies*

$$\mu_k[u](B_r \cap \Omega) \le Cr^{n-2k+\varepsilon}. \tag{9.9}$$

Proof. If u is Hölder continuous with exponent $\alpha \in (0, \frac{1}{2})$, from the first inequality in (9.5) we obtain

$$\int_0^r \left(\frac{\mu(B_t(x))}{t^{n-2k}} \right)^{\frac{1}{k}} \frac{dt}{t} \leq Cr^\alpha.$$

Hence

$$\frac{\mu(B_{r/2}(x))}{t^{n-2k}} \leq Cr^{k\alpha}.$$

We obtain (9.9) with $\varepsilon = k\alpha$.

Next assume that (9.9) holds. Consider the function u in $B_R(0)$. We want to prove that $|u(x) - u(0)| < Cr^\alpha$ for $|x| < r = \frac{1}{2}R^2$. Replacing u by the Perron lifting u^ω, where $\omega = B_R - B_{R/2}$, we may assume that $\mu_k[u] = 0$ in ω. Let w_1 be the solution of $S_k[w] = 0$ in B_R and $w = u$ on ∂B_R. Let w_2 be the solution of $S_k[w] = \mu_k[u]$ in B_R and $w = 0$ on ∂B_R. Then

$$w_1 \geq u \geq w_1 + w_2.$$

Hence

$$u(x) - u(0) \leq w_1(x) - [w_1(0) + w_2(0)] \leq [w_1(x) - w_1(0)] + w_2(0).$$

By (9.9) and the second inequality in (9.5), we have $w_2(0) \leq CR^{\varepsilon/k}$. By the interior gradient estimate, w_1 is Lipschitz continuous. Hence

$$|w_1(x) - w_1(0)| \leq \frac{C}{R}|x| \leq C|x|^{1/2}.$$

We obtain

$$u(x) - u(0) \leq C|x|^{1/2} + CR^{\varepsilon/k}.$$

Similarly we have $u(0) - u(x) \leq C|x|^{1/2} + CR^{\varepsilon/k}$. Hence u is Hölder continuous at the origin with exponent $\varepsilon/2k$. □

From Theorem 9.2, we obtain

Corollary 9.1. *Let u be a k-admissible solution ($k \leq \frac{n}{2}$) to*

$$S_k[u] = f. \tag{9.10}$$

Suppose $f \in L^p(\Omega)$ for some $p > \frac{n}{2k}$. Then u is Hölder continuous.

From Theorem 9.1, one can also prove that a k-admissible function u is continuous at x if and only if $W_k^\mu(x, r) \to 0$ as $r \to 0$. One can also introduce the notion of capacity, and establish various potential theoretical results, such as quasi-continuity of k-admissible functions and the Wiener criterion for the continuity of k-admissible functions at the boundary, just as in the Newton potential theory. We refer the reader to [TW3, Ld] for more details. More applications of the Wolff potential estimate can be found in [PV1, PV2].

10. Parabolic Hessian equations

This section includes the a priori estimates and existence of solutions for the parabolic Hessian equations used before. We refer the reader to [Lg] for more general fully nonlinear parabolic equations of parabolic type.

Consider the initial boundary value problem

$$F[u] - u_t = f(x, t, u) \quad \text{in } \Omega \times [0, \infty) \tag{10.1}$$
$$u(\cdot, 0) = u_0,$$
$$u = 0 \quad \text{on } \partial\Omega \times [0, \infty)$$

where

$$F[u] = \mu(S_k[u]). \tag{10.2}$$

We assume that μ is a smooth function defined on $(0, \infty)$, satisfying $\mu'(t) > 0$, $\mu''(t) < 0$ for all $t > 0$, and

$$\mu(t) \to -\infty \quad \text{as } t \to 0, \tag{10.3}$$

$$\mu(t) \to +\infty \quad \text{as } t \to +\infty. \tag{10.4}$$

Furthermore we assume that $\mu(\sigma_k(\lambda))$ is concave in λ, which implies that $F[u]$ is concave in D^2u. A natural candidate for μ is $\mu(t) = \log t$, such as in (5.17). But we have also used different function μ, such as in (6.3) (6.14).

We say a function $u(x, t)$ is k-admissible with respect to the parabolic equation (10.1) if for any given $t \geq 0$, $u(\cdot, t)$ is k-admissible. Equation (10.1) is parabolic when u is k-admissible. Condition (10.3) is to ensure that $\sigma_k(\lambda) > 0$ so that the admissibility can be kept at all time.

Theorem 10.1. *Assume $u_0 \in C^4(\overline{\Omega})$ is k-admissible, and satisfies the compatibility condition*

$$F[u_0] = f \quad \text{on} \quad \partial\Omega \times \{t = 0\}. \tag{10.5}$$

Assume $\partial\Omega \in C^{3,1}$ and Ω is $(k-1)$-convex, $f \in C^{2,1}_{x,t}(\overline{Q}_T)$, and μ satisfies the conditions above. Let u be an k-admissible solution of (10.1). Then we have the a priori estimate

$$\|u\|_{C^{2,1}_{x,t}(\overline{Q}_T)} \leq C, \tag{10.6}$$

where $Q_T = \Omega \times (0, T]$, C depends on n, k, $\partial\Omega$, $\|u_0\|_{C^4(\overline{\Omega})}$, $\sup_{Q_T} |u|$, and $\|f\|_{C^{1,1}(\overline{Q}_T)}$.

To prove Theorem 10.1, one first establish an upper bound for $\sup_{Q_T} |u_x|$ and $\sup_{Q_T} |u_t|$, then prove $\sup_{Q_T} |u_{xx}|$ is bounded. The estimates for $\sup_{Q_T}(|u_x| + |u_t|)$ will be given in the proof of Theorem 10.2 below. The estimate for $\sup |u_{xx}|$ is similar to that for the elliptic equation (3.1) and is omitted here. We refer the reader to [Ch1, W2] for details. See also [Lg].

Note that when applying Theorem 10.1 to equation (5.17), the a priori bound for $\sup_{Q_T} |u|$ is guaranteed by our truncation of $|u|^p$, namely the function $f(u)$ in (5.15). In equation (5.17), the right hand side involves an integration $\beta(u)$, which satisfies the estimate $C_1 \le \beta(u) \le C_2$ for two absolute positive constants C_1, C_2. This integration does not affect the a priori estimate for $\sup_{Q_T} |u_t|$. See the proof of (10.11) below. Once u_t is bounded, $\beta(u)$ is positive and Lipschitz continuous in t.

Estimate (10.6) implies that equation (10.1) is uniformly parabolic. Therefore by Krylov's regularity theory for uniformly parabolic equation [K1], we obtain higher order derivative estimates. By the a priori estimates, one can then prove the local existence of smooth solutions by the contraction mapping theorem. In particular, if $\sup_{\Omega\times[0,T]} |u| < \infty$ for any $T > 0$, the smooth solution exists at all time $t > 0$.

In Step 3 of the proof of Theorem 6.3, we need a special interior gradient estimate, namely (10.10) below, for solutions to equation (10.1) with μ given in (6.14). We provide a proof for it below. See also [CW1]. Estimate for higher order derivatives and existence of solutions can be obtained similarly as above.

Theorem 10.2. *Let μ be the function in (6.14). Assume $u_0 \in C^4(\overline{\Omega})$ is k-admissible, and satisfies the compatibility condition (10.5). Assume that f is C^2 in x and u, C^1 in t, and satisfies,*

$$f(x,t,u) \le C_0(1+|u|) \quad \forall\ (x,t,u) \in \overline{\Omega}\times\mathbb{R}. \tag{10.7}$$

Suppose $u \in C^{4,2}_{x,t}(\overline{\Omega}\times[0,\infty))$ is a k-admissible solution of (10.1). Then we have, for $0 < t < T$,

$$u(x,t) \ge -e^{C_1 t}\sup_{\Omega}|u_0(x)|, \tag{10.8}$$

$$|\nabla_x u(x,t)| \le C_2\Big(1+\frac{1}{r}M_t^{(p+k)/2k}\Big), \tag{10.9}$$

$$|u_t(x,t)| \le C_3(1+M_t), \tag{10.10}$$

where $M_t = \sup_{Q_t}|u|$, $r = dist(x,\partial\Omega)$. The constant C_1 depends only on n,k,p and C_0; C_2 and C_3 depends additionally on u_0 and the gradient of f.

Proof. Estimate (10.8) is obvious as the right hand side is a lower barrier. To prove (10.9) and (10.10) we assume for simplicity that $M_t \ge 1$. First we prove (10.10). Let

$$G = \frac{u_t}{M-u},$$

where $M = 2M_t$. If G attains its minimum on the parabolic boundary of Q_t, we have $u_t \ge -C$ for some $C > 0$ depending on the initial value u_0. Hence we may suppose G attains its minimum at an interior point in Q_t. At this point we have

$$\begin{aligned} &u_{tt} + (M-u)^{-1}u_t^2 \le 0,\\ &u_{jt} + (M-u)^{-1}u_t u_j = 0, \quad j = 1,2,\cdots,n, \end{aligned}$$

and the matrix

$$\begin{aligned}\{u_{ijt}+(M-u)^{-1} & (u_{it}u_j+u_{jt}u_i+u_tu_{ij})+2(M-u)^{-2}u_iu_ju_t\} \\ = & \ \{u_{ijt}+(M-u)^{-1}u_tu_{ij}\}\geq 0.\end{aligned}$$

Differentiating the equation (10.1) we get

$$\begin{aligned}F_{ij}u_{ijt}-u_{tt}=f_t+f_uu_t,\\ F_{ij}u_{rij}-u_{rt}=f_r+f_uu_r,\end{aligned}$$

where $F_{ij}=\frac{\partial}{\partial u_{ij}}F[u]$. We may suppose $u_t\leq 0$ at this point. From the above formulae we obtain

$$\begin{aligned}(M-u)^{-1}u_t^2 &\leq -F_{ij}u_{ijt}+f_t+f_uu_t\\ &\leq (M-u)^{-1}u_tF_{ij}u_{ij}+f_t+f_uu_t\\ &\leq f_t+f_uu_t.\end{aligned}$$

Hence $u_t\geq -C$ for some C depending on $\sup_{Q_t}f_t$ and $\inf_{Q_t}f_u$.

Similarly let

$$G=\frac{u_t}{M+u}.$$

If G attains its maximum on the parabolic boundary of Q_t, we have $u_t\leq C$. If it attains its maximum at some point in Q_t. At this point we have

$$\begin{aligned}&u_{tt}-(M+u)^{-1}u_t^2\geq 0,\\ &u_{jt}-(M+u)^{-1}u_tu_j=0,\quad j=1,2,\cdots,n,\\ &\{u_{ijt}-(M+u)^{-1}u_tu_{ij}\}\leq 0.\end{aligned}$$

Hence as above we obtain

$$\begin{aligned}(M+u)^{-1}u_t^2 &\leq F_{ij}u_{ijt}-f_t-f_uu_t\\ &\leq (M+u)^{-1}u_tF_{ij}u_{ij}-f_t-f_uu_t\\ &= (M+u)^{-1}ku_t\mu' S_k[u]-f_t-f_uu_t.\end{aligned}$$

If $S_k[u]\leq 10$ at the point, by the equation (10.1) we have

$$u_t=F[u]-f\leq C.$$

Otherwise we have $\mu(t)=t^{1/p}$ and so

$$\mu' S_k[u]=\frac{1}{p}\mu(S_k[u])=\frac{1}{p}(u_t+f).$$

It follows that

$$(M+u)^{-1}u_t^2\leq (M+u)^{-1}ku_t(u_t+f)/p-f_t-f_uu_t.$$

That is

$$\frac{p-k}{p}\frac{u_t^2}{M+u}\leq \frac{kfu_t}{p(M+u)}-f_t-f_uu_t.$$

We obtain $u_t\geq -C$ for some C depending on $\inf_{Q_t}f_t$ and $\inf_{Q_t}f_u$.

Next we prove (10.9). For simplicity let us take $t = T$. The proof below is similar to that of the interior gradient estimate in §4. Assume that $B_r(0) \subset \Omega$. Consider

$$G(x, t, \xi) = \rho(x)\varphi(u)u_\xi,$$

where $\rho(x) = 1 - |x|^2/r^2$, $\varphi(u) = (M - u)^{-1/4}$. Suppose

$$\sup\{G(x, t, \xi) \mid \ x \in B_r(0), t \in (0, T], |\xi| = 1\}$$

is attained at (x_0, t_0) (with $t_0 > 0$) and $\xi_0 = (1, 0, \cdots, 0)$. Then at the point we have

$$\begin{aligned}
0 &= (\log G)_i = \frac{\rho_i}{\rho} + \frac{\varphi_i}{\varphi} + \frac{u_{1i}}{u_1}, \\
0 &\geq F_{ij}(\log G)_{ij} - (\log G)_t \\
&= F_{ij}(\frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2}) - \frac{\rho_t}{\rho} + F_{ij}(\frac{\varphi_{ij}}{\varphi} - \frac{\varphi_i\varphi_j}{\varphi^2}) - \frac{\varphi_t}{\varphi} + F_{ij}(\frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2}) - \frac{u_{1t}}{u_1} \\
&\geq F_{ij}(\frac{\rho_{ij}}{\rho} - 3\frac{\rho_i\rho_j}{\rho^2}) - \frac{\rho_t}{\rho} + F_{ij}(\frac{\varphi_{ij}}{\varphi} - 3\frac{\varphi_i\varphi_j}{\varphi^2}) - \frac{\varphi_t}{\varphi} + \frac{1}{u_1}(F_{ij}u_{1ij} - u_{1t}) \\
&\geq -\frac{C}{\rho^2}\mathcal{F} + (\frac{\varphi''}{\varphi} - 3\frac{\varphi'^2}{\varphi^2})F_{11}u_1^2 + \frac{\varphi'}{\varphi}(F_{ij}u_{ij} - u_t) + \frac{f_1}{u_1},
\end{aligned}$$

where $\mathcal{F} = \sum F_{ii}$. By our choice of φ,

$$\frac{\varphi''}{\varphi} - 3\frac{\varphi'^2}{\varphi} = \frac{1}{8(M-u)^2}$$

Note that $F_{ij}u_{ij} \geq 0$ and $\varphi' \geq 0$. We obtain

$$0 \geq \frac{F_{11}u_1^2}{32M^2} - \frac{C}{\rho^2}\mathcal{F} - \frac{\varphi'}{\varphi}u_t + \frac{f_1}{u_1}.$$

Therefore we have either

$$F_{11}u_1^2 \leq \frac{CM^2}{\rho^2}\mathcal{F}, \tag{10.11}$$

or

$$F_{11}u_1^2 \leq CM^2(\frac{\varphi'}{\varphi}u_t - \frac{f_1}{u_1}) \leq CM^2. \tag{10.12}$$

In (10.12) we have used the estimate (10.10).

Recall that

$$u_{11} = -u_1(\frac{\rho_1}{\rho} + \frac{\varphi'}{\varphi}u_1).$$

We may assume that $u_1 \geq CM/r$, for otherwise (10.9) is readily verified. Hence we have

$$u_{11} \leq -\frac{\varphi'}{2\varphi}u_1^2 \leq -\frac{C}{M}u_1^2.$$

From the proof of Theorem 4.1 we then have

$$S_k^{11}[u] \ge C \sum S_k^{ii}[u],$$
$$S_k^{11}[u] \ge C \frac{u_1^{2k-2}}{M^{k-1}},$$

Therefore in the case (10.11), we obtain $\rho u_1 \le CM$ and (10.9) follows. In the case (10.12), we observe by equation (10.1) and estimate (10.10) that $S_k[u] \le CM^p$ at (x_0, t_0). Hence $\mu'(S_k[u]) \ge CM^{-p+1}$. We therefore obtain

$$F_{11} \ge \frac{Cu_1^{2k-2}}{M^{p+k-2}}.$$

Inserting into (10.12) we obtain $u_1(x_0, t_0) \le CM^{(p+k)/2k}$. Hence at the center $x = 0$ we have

$$|Du(0,t)| \le \frac{\rho\varphi(x_0,t_0)}{\rho\varphi(0,t)} u_1(x_0,t_0) \le CM^{(p+k)/2k}.$$

This completes the proof. □

11. Examples of fully nonlinear elliptic equations

This is the notes for my lectures under the title *Fully nonlinear elliptic equations*, given in a workshop at C.I.M.E., Italy. So it is appropriate to give more examples of fully nonlinear elliptic equations here.

(i) One of the most important fully nonlinear equations is the *Monge-Ampère equation*

$$\det D^2 u = f(x, u, Du). \tag{11.1}$$

The Monge-Ampère equation finds many applications in geometry and applied sciences. A special case of the Monge-Ampère equation is the *prescribing Gauss curvature equation*

$$\frac{\det D^2 u}{(1+|Du|^2)^{(n+2)/2}} = \kappa(x), \tag{11.2}$$

where κ is the Gauss curvature of the graph of u.

(ii) A related equation is the *complex Monge-Ampère equation*

$$\det u_{z_i \bar{z}_j} = f, \tag{11.3}$$

which plays an important role in complex geometry.

(iii) The *k-Hessian equation*

$$S_k[u] = f(x), \tag{11.4}$$

studied in previous sections.

(iv) The *k-curvature equation*

$$H_k[u] = f(x), \tag{11.5}$$

where $1 \le k \le n$ and $H_k[u] = \sigma_k(\kappa)$, is a class of prescribing Weingarten curvature equations, where $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the principal curvatures of the graph of u. The k-curvature equation is just the mean curvature equation when $k = 1$, and the Gauss curvature equation when $k = n$.

Related to the k-Hessian and k-curvature equations are the Hessian quotient and curvature quotient equations, that is

$$\frac{S_k[u]}{S_l[u]} = f(x), \tag{11.6}$$

$$\frac{H_k[u]}{H_l[u]} = f(x), \tag{11.7}$$

where $0 \le l < k \le n$, and S_k and H_k are respectively the k-Hessian and k-curvature operator. A special case of (11.7) is the prescribing harmonic curvature equation, that is when $k = n, l = n - 1$.

(v) The *special Lagrangian equation*

$$\mathrm{arctan}\lambda_1 + \cdots \mathrm{arctan}\lambda_n = c \tag{11.8}$$

is a fully nonlinear equation arising in geometry. If u is a solution, the graph $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$. When $n = 3$ and $c = k\pi$, equation (11.8) can be written as

$$\Delta u = \mathrm{det} D^2 u. \tag{11.9}$$

(vi) In stochastic control theory there arises the *Bellman equation*

$$F[u] = \inf_{\alpha \in V} \{L_\alpha[u] - f_\alpha(x)\}, \tag{11.10}$$

or more generally the Bellman-Isaacs equation

$$F[u] = \sup_{\alpha \in U} \inf_{\beta \in V} \{L_{\alpha,\beta}[u] - f_{\alpha,\beta}(x)\}, \tag{11.11}$$

where α, β are indexes and $L_{\alpha,\beta}$ are linear elliptic operators.

(vii) Another well-known fully nonlinear equation is Pucci's equation [GT], which is a special Bellman equation. For $\alpha \in (0, \frac{1}{n}]$, let $\mathcal{L}_\alpha$ denote the set of linear uniformly elliptic operator of the form $L[u] = a_{ij}(x)\partial_{ij}u$ with bounded measurable coefficients a_{ij} satisfying $a_{ij}\xi_i\xi_j \ge \alpha|\xi|^2$, $\Sigma a_{ii} = 1$ for all $\xi \in \mathbb{R}^n, x \in \Omega$. Pucci's operators are defined by

$$\begin{aligned} P_\alpha^+[u] &= \sup_{L \in \mathcal{L}_\alpha} L[u], \\ P_\alpha^-[u] &= \inf_{L \in \mathcal{L}_\alpha} L[u]. \end{aligned} \tag{11.12}$$

By direct calculation [GT],

$$
\begin{aligned}
(11.13) \qquad P^+_\alpha[u] &= \alpha\Delta u + (1-n\alpha)\lambda_1(D^2u),\\
P^-_\alpha[u] &= \alpha\Delta u + (1-n\alpha)\lambda_n(D^2u),
\end{aligned}
$$

where $\lambda_1(D^2u)$ and $\lambda_n(D^2u)$ denote the maximum and minimum eigenvalues of D^2u.

(viii) Equation (11.1) is the standard Monge-Ampère equation. In many applications one has the Monge-Ampère equation of general form,

$$
(11.14) \qquad \det\{D^2u - A(x,u,Du)\} = f(x,u,Du),
$$

where A is an $n\times n$ matrix. Similarly one has an extension of the k-Hessian equation (11.3), that is

$$
(11.15) \qquad S_k\{\lambda(D^2u - A(x,u,Du)\} = f.
$$

Equation (11.14) arises in applications such as reflector design, optimal transportation, and isometric embedding. Equation (11.15) is related to the so-called k-Yamabe problem in conformal geometry.

Some of the above equations may not be elliptic in general, such as the Monge-Ampère equation (11.1) and the k-Hessian equation (11.4). But they are elliptic when restricted to an appropriate class of functions.

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