The Square Root Problem of Kato and Related First Order Systems

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Abstract

The first order Cauchy-Riemann equations have long been used in the study of harmonic boundary value problems in the plane. The Dirac operator can sometimes be employed in higher dimensions.

First order systems have provided insight into the solution of the Kato square-root problem for elliptic operators. In recent joint work with Andreas Axelsson and Pascal Auscher, we show how they can also be used to study the solvability of elliptic equations with square integrable boundary conditions. I shall survey this chain of ideas.

Self Adjoint Operators

Let D be a **self adjoint operator** in a **Hilbert space** $\mathcal{H}$. By the spectral theorem, the decomposition of the real line

$$\mathbb{R} = \{0\} \cup \mathbb{R}^+ \cup \mathbb{R}^-$$

leads to a **spectral decomposition**

$$\mathcal{H} = \mathcal{N}(D) \overset{\perp}{\oplus} \mathcal{H}_{D,+} \overset{\perp}{\oplus} \mathcal{H}_{D,-}$$

where

$$\mathcal{N}(D) = \{u \in \mathcal{H}\,;\, Du = 0\} = \text{null space of } D$$
$$\mathcal{H}_{D,+} = \{u \in \overline{\mathcal{R}}(D)\,;\, Du = \sqrt{D^2}u\} = \text{positive eigenspace of } D$$
$$\mathcal{H}_{D,-} = \{u \in \overline{\mathcal{R}}(D)\,;\, Du = -\sqrt{D^2}u\} = \text{negative eigenspace of } D$$
$$\overline{\mathcal{R}}(D) = \text{closure of the range of } D$$

Spectral Decomposition

Associated with the spectral decomposition

$$\begin{aligned}\mathcal{H} &= \mathcal{N}(D) \overset{\perp}{\oplus} \mathcal{H}_{D,+} \overset{\perp}{\oplus} \mathcal{H}_{D,-} \\ I &= P_0 + P_+ + P_- \\ \operatorname{sgn}(D) &= P_+ - P_-\end{aligned}$$

are orthogonal spectral projections P_0 , $P_\pm$
and the signum operator $\operatorname{sgn}(D)$

Given $u \in \mathcal{H}$, consider the **evolution equation E**

$$\frac{\partial U}{\partial t}(t) + DU(t) = 0$$
$$\lim_{t \to 0} U(t) = u \in \mathcal{H}$$

- If $u \in \mathcal{H}_{D,+}$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}^+, \mathcal{H})$:
$$U(t) = e^{-tD}u \, , \quad 0 < t < \infty$$
- If $u \in \mathcal{H}_{D,-}$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}^-, \mathcal{H})$:
$$U(t) = e^{-tD}u \, , \quad -\infty < t < 0$$
- If $u \in \mathcal{N}(D)$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}, \mathcal{H})$:
$$U(t) = u \, , \quad -\infty < t < \infty$$

In each case, $\|U(t)\| \leq \|u\|$ for all t . Note: $C_b^1 = \{U \in C^1 ; u' \in L^\infty\}$

Motivating Example

- $\mathcal{H} = L^2(\mathbb{R})$ with inner product $(u, v) = \int_0^\infty u(x)\overline{v(x)}\,dx$ and norm $\|u\|^2 = (u, u) = \int_0^\infty |u(x)|^2\,dx$
- $D = \frac{1}{i}\,\partial_x = \frac{1}{i}\,\frac{\partial}{\partial x}$ with domain

$$\mathcal{D}(D) = W^{1,2}(\mathbb{R}) = \{u \in L^2(\mathbb{R})\,;\,\partial_x u \in L^2(\mathbb{R})\}$$

- Note that $\mathcal{N}(D) = \{0\}$ and $\overline{\mathcal{R}}(D) = L^2(\mathbb{R})$ so the spectral decomposition

$$\mathcal{H} = \mathcal{N}(D) \overset{\perp}{\oplus} \mathcal{H}_{D,+} \overset{\perp}{\oplus} \mathcal{H}_{D,-}$$

becomes

$$L^2(\mathbb{R}) = L^2_{D,+} \overset{\perp}{\oplus} L^2_{D,-}$$

where

$$L^2_{D,+} = \{u \in L^2(\mathbb{R})\,;\,\tfrac{1}{i}\partial_x u = \sqrt{-\partial_x{}^2}\,u\,\}$$

$$L^2_{D,-} = \{u \in L^2(\mathbb{R})\,;\,\tfrac{1}{i}\partial_x u = -\sqrt{-\partial_x{}^2}\,u\,\}$$

Now

$$L^2(\mathbb{R}) = L^2_{D,+} \overset{\perp}{\oplus} L^2_{D,-}$$

is the usual **Hardy space decomposition** of $L^2(\mathbb{R})$ as

$$\begin{aligned} L^2_{D,+} &= \{u \in L^2(\mathbb{R})\,;\ \tfrac{1}{i}\partial_x u = \sqrt{-\partial_x{}^2}\,u\ \} \\ &= \{u \in L^2(\mathbb{R})\,;\ \xi\hat{u}(\xi) = |\xi|\hat{u}(\xi)\ \} \\ &= \{u \in L^2(\mathbb{R})\,;\ \text{sppt}\,\hat{u}(\xi) \subset [0,\infty)\ \} \\ &= \text{positive Hardy space of } L^2(\mathbb{R}) \end{aligned}$$

with $\hat{u}$ denoting the Fourier transform of u , and

$$\begin{aligned} L^2_{D,-} &= \{u \in L^2(\mathbb{R})\,;\ \tfrac{1}{i}\partial_x u = -\sqrt{-\partial_x{}^2}\,u\ \} \\ &= \{u \in L^2(\mathbb{R})\,;\ \xi\hat{u}(\xi) = -|\xi|\hat{u}(\xi)\ \} \\ &= \{u \in L^2(\mathbb{R})\,;\ \text{sppt}\,\hat{u}(\xi) \subset (-\infty,0]\ \} \\ &= \text{negative Hardy space of } L^2(\mathbb{R}) \end{aligned}$$

Motivating Example

The evolution equation **E** for $D = \frac{1}{i}\partial_x$ acting in $\mathcal{H} = L^2(\mathbb{R})$ is

$$\frac{\partial U}{\partial t}(t) + \frac{1}{i}\partial_x U(t) = 0$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

i.e.

$$\frac{\partial U}{i\,\partial t}(t,x) = \frac{\partial U}{\partial x}(t,x)$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

namely the Cauchy-Riemann equation for U as a function of $x + it$

Motivating Example

So if $u \in L^2_{D,+}$ then $U(t) = e^{-tD}u$, $0 < t < \infty$ is the unique bounded analytic function in the upper half plane with L^2 boundary value u on the real axis.

If $u \in L^2_{D,-}$ then $U(t) = e^{-tD}u$, $-\infty < t < 0$ is the unique bounded analytic function in the lower half plane with L^2 boundary value u on the real axis.

In each case, $\|U(t)\|_2 \leq \|u\|_2$ for all t.

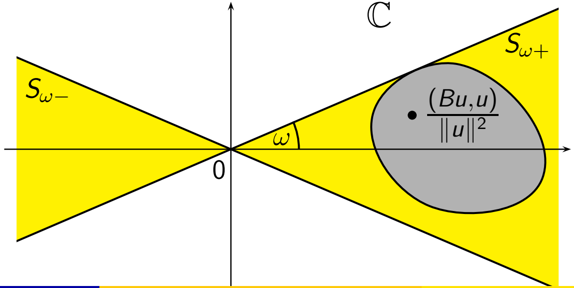
The projections $P_\pm$ on the Hardy spaces are limits of the Cauchy integrals, and $\mathrm{sgn}(D) = P_+ - P_- = H$, the Hilbert transform:

$$P_\pm u(x) = \pm \lim_{\tau \to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x \pm i\tau - y}\, dy$$

$$Hu(x) = \text{p.v. } \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x - y}\, dy$$

Perturbed Self Adjoint Operators

Let D be a **self adjoint operator** in a **Hilbert space** $\mathcal{H}$, and suppose that B is a **bounded strictly accretive operator** in $\mathcal{H}$. Then DB is an ω - **bisectorial operator** for some $\omega < \pi/2$ i.e.
$\sigma(DB) \subset S_\omega = S_{\omega+} \cup S_{\omega-}$ and $\|(DB - \zeta I)^{-1}\| \leq C/\text{dist}(\zeta, S_\omega)$



Spectral Decomposition

Also $\quad \mathcal{H} = \mathcal{N}(DB) \oplus \overline{\mathcal{R}}(D)$ (non-orthogonal)

Main Question: Is there a **spectral decomposition**

$$\mathcal{H} = \mathcal{N}(DB) \oplus \mathcal{H}_{DB,+} \oplus \mathcal{H}_{DB,-}$$
$$I = \quad P_0^B \quad + \quad P_+^B \quad + \quad P_-^B \qquad \text{where}$$

$\mathcal{H}_{DB,+} = \{u \in \overline{\mathcal{R}}(D)\,;\ DBu = \sqrt{(DB)^2}\, u\} =$ positive eigenspace of DB

$\mathcal{H}_{DB,-} = \{u \in \overline{\mathcal{R}}(D)\,;\ DBu = -\sqrt{(DB)^2}\, u\} =$ negative eigenspace of DB

Equivalent Question: Is

$$\|\sqrt{(DB)^2}\, u\| \approx \|DBu\| \qquad ?$$

[meaning $C^{-1}\|\sqrt{(DB)^2}\, u\| \leq \|DBv\| \leq C\|\sqrt{(DB)^2}\, u\| \ \ \forall u$]

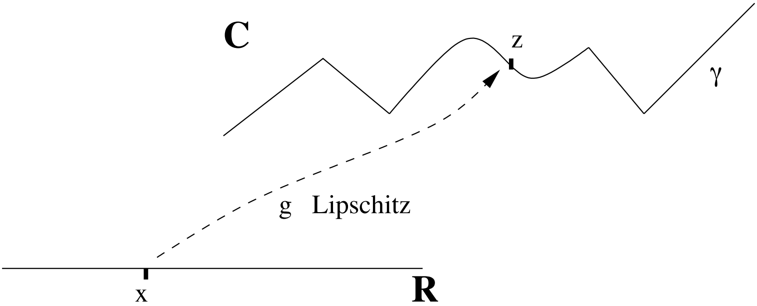
Note: Theory and questions for BD are similar, as $BD = B(DB)B^{-1}$

Main Question: Is there a **spectral decomposition**

$$\mathcal{H} = \mathcal{N}(D) \oplus \mathcal{H}_{BD,+} \oplus \mathcal{H}_{BD,-} \qquad ?$$

or equivalently, is $\|\sqrt{(BD)^2}\,u\| \approx \|BDu\|$?

1. YES when $D = \frac{1}{i}\partial_x$ in $\mathcal{H} = L^2(\mathbb{R})$ and B is multiplication by a bounded function $b \in L^\infty(\mathbb{R})$ with $\mathrm{Re}\, b(x) \geq \kappa > 0$. For then $BD = b(x)\frac{1}{i}\frac{\partial}{\partial x} = \frac{1}{i}\frac{\partial}{\partial z}\big|_\gamma$ where γ is the Lipschitz curve parametrised by $g : \mathbb{R} \to \mathbb{C}$ and $g'(x) = 1/b(x)$



Then $\mathcal{N}(D) = \{0\}$ and

$$L^2(\mathbb{R}) = L^2_{BD,+} \oplus L^2_{BD,-}$$

is equivalent to the Hardy space decomposition

$$L^2(\gamma) = L^2_+(\gamma) \oplus L^2_-(\gamma)$$

A function $u \in L^2_\pm(\gamma)$ is the boundary value of a bounded analytic function in the region $\Omega_\pm$ above or below γ. The spectral projections $P^B_\pm$ are limits of Cauchy integrals, and $\mathrm{sgn}(BD) = P^B_+ + P^B_- = C_\gamma$ is the Cauchy integral on γ

$$P^B_\pm u(z) = \pm \lim_{\tau \to 0} \frac{i}{2\pi} \int_\gamma \frac{u(w)}{z \pm i\tau - w}\, dw$$

$$C_\gamma u(z) = \text{p.v.}\ \frac{i}{\pi} \int_\gamma \frac{u(w)}{z - w}\, dw$$

The boundedness of these operators, and hence the spectral - or Hardy decomposition, is a theorem of harmonic analysis

Historical Interlude I

In the 1960's, Zygmund and Calderón made fundamental contributions to the theory of singular integrals.

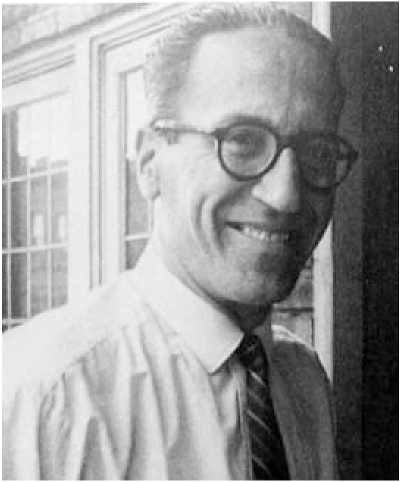
Zygmund asked whether the singular Cauchy integral on a Lipschitz curve,

$$C_\gamma u(z) = \text{p.v.}\ \frac{i}{\pi}\int_\gamma \frac{u(w)}{z-w}\,dw = \lim_{\epsilon\to 0}\frac{i}{\pi}\int_{\{w\in\gamma;|w-z|>\epsilon\}} \frac{u(w)}{z-w}\,dw$$

is defined at almost every point $x \in \gamma$.

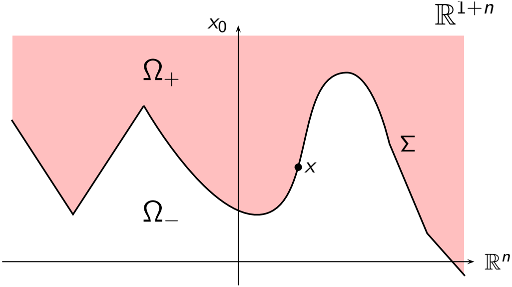
Calderón asked whether $C_\gamma : L^2(\gamma) \to L^2(\gamma)$ is bounded.

Alberto Calderón
1920 – 1998



Historical Interlude I

- First commutator theorem: [Calderón 1965]
- Higher commutator theorems: [Coifman, Meyer 1975, 1978]
- $C_\gamma : L^2(\gamma) \to L^2(\gamma)$ is bounded on curves with small Lipschitz constants: [Calderón, 1977]
- $C_\gamma : L^2(\gamma) \to L^2(\gamma)$ is bounded on all Lipschitz curves: [Coifman-McIntosh-Meyer 1982] In this paper the authors also solve the one-dimensional Kato square root problem
- $T(b)$ Theorem of [David-Journé-Semmes 1986]



The L^2 bounds on Cauchy integrals generalise immediately to give L^2 bounds on singular integrals on Lipschitz surfaces, e.g. the singular double layer potential operator

$$T_\Sigma u(x) = \text{p.v. } \frac{2}{\sigma_n} \int_\Sigma \frac{(x-y).\nu(y)}{|x-y|^{n+1}} u(y)\, dy$$

They do NOT immediately generalise to provide a proof of the higher dimensional Kato problem

Spectral Decomposition

Main Question: Is there a **spectral decomposition**

$$\mathcal{H} = \mathcal{N}(DB) \oplus \mathcal{H}_{DB,+} \oplus \mathcal{H}_{DB,-}$$
$$I = \quad P_0^B \quad + \quad P_+^B \quad + \quad P_-^B$$

2. NO Not always. There are counterexamples.

Operator theory is not enough. Harmonic analysis of functions is needed to prove the quadratic estimates.

A Higher Dimensional Example

From now on $D = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix}$ in $\mathcal{H} = \begin{matrix} L^2(\mathbb{R}^n) \\ \oplus \\ L^2(\mathbb{R}^n, \mathbb{C}^n) \end{matrix}$

i.e. $$Du = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} \mathbf{div}\, \tilde{u} \\ -\nabla u_0 \end{bmatrix}$$

Observe that D is self adjoint,

$\mathcal{N}(D) = \left\{ \begin{bmatrix} 0 \\ \tilde{u} \end{bmatrix} ; \mathbf{div}\, \tilde{u} = 0 \right\}$; $\overline{\mathcal{R}}(D) = \begin{matrix} L^2(\mathbb{R}^n) \\ \oplus \\ \overline{\mathcal{R}}(\nabla) \end{matrix}$ and

$$D^2 = \begin{bmatrix} -\Delta & 0 \\ 0 & -\nabla \mathbf{div} \end{bmatrix}$$

A Higher Dimensional Operator

Notation:

Define $\nabla : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{C}^n)$ by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)$$

for $f \in \mathcal{D}(\nabla) = W^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n)\,;\, \frac{\partial f}{\partial x_j} \in L^2(\mathbb{R}^n) \text{ for each } j\}$

and define **div** $: L^2(\mathbb{R}^n, \mathbb{C}^n) \to L^2(\mathbb{R}^n)$ by

$$\mathbf{div}\,\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \dots + \frac{\partial u_n}{\partial x_n}$$

for $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathcal{D}(\mathbf{div}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}^n)\,;\, \mathbf{div}\,\mathbf{u} \in L^2(\mathbb{R}^n)\}$

Note: $\mathbf{div} = -\nabla^*$ and $\mathbf{div}\,\nabla = \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$

A Higher Dimensional Example

From now on $D = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix}$ in $\mathcal{H} = \begin{matrix} L^2(\mathbb{R}^n) \\ \oplus \\ L^2(\mathbb{R}^n, \mathbb{C}^n) \end{matrix}$

$$\text{i.e.} \qquad Du = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} \mathbf{div}\, \tilde{u} \\ -\nabla u_0 \end{bmatrix}$$

Observe that D is self adjoint,

$$\mathcal{N}(D) = \left\{ \begin{bmatrix} 0 \\ \tilde{u} \end{bmatrix} ; \mathbf{div}\, \tilde{u} = 0 \right\} \quad ; \quad \overline{\mathcal{R}}(D) = \begin{matrix} L^2(\mathbb{R}^n) \\ \oplus \\ \overline{\mathcal{R}}(\nabla) \end{matrix} \qquad \text{and}$$

$$D^2 = \begin{bmatrix} -\Delta & 0 \\ 0 & -\nabla \mathbf{div} \end{bmatrix}$$

Spectral Decomposition

Recall that u belongs to the positive eigenspace of D provided $u \in \overline{\mathcal{R}}(D)$ and $Du = \sqrt{D^2}\, u$. That is $\tilde{u} \in \overline{\mathcal{R}}(\nabla)$ and

$$\begin{bmatrix} \mathbf{div}\, \tilde{u} \\ -\nabla u_0 \end{bmatrix} = \begin{bmatrix} \sqrt{-\Delta} & 0 \\ 0 & \sqrt{-\nabla \mathbf{div}} \end{bmatrix} \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} \sqrt{-\Delta}\, u_0 \\ \sqrt{-\nabla \mathbf{div}}\, \tilde{u} \end{bmatrix}$$

This holds if and only if $\quad \tilde{u} = -\nabla(-\Delta)^{-1/2} u_0$

Similarly u belongs to the negative eigenspace of D if and only if

$$\tilde{u} = \nabla(-\Delta)^{-1/2} u_0$$

Spectral Decomposition

So the spectral decomposition for $D = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix}$ is

$$\mathcal{H} = \mathcal{N}(D) \overset{\perp}{\oplus} \mathcal{H}_{D,+} \overset{\perp}{\oplus} \mathcal{H}_{D,-}$$

where

$$\mathcal{N}(D) = \left\{ \begin{bmatrix} 0 \\ \tilde{u} \end{bmatrix} ; \mathbf{div}\, \tilde{u} = 0 \right\}$$

$$\mathcal{H}_{D,+} = \left\{ \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} ; \tilde{u} = -\nabla(-\Delta)^{-1/2} u_0 \right\}$$

$$\mathcal{H}_{D,-} = \left\{ \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} ; \tilde{u} = \nabla(-\Delta)^{-1/2} u_0 \right\}$$

Evolution Equation

The evolution equation **E** for $D = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix}$ is

$$\frac{\partial U}{\partial t}(t) + DU(t) = 0$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

i.e. $U = \begin{bmatrix} U_0 \\ \tilde{U} \end{bmatrix}$ satisfies the generalized Cauchy-Riemann equations

$$\frac{\partial U_0}{\partial t}(t, \mathbf{x}) + \mathbf{div}\, \tilde{U}(t, \mathbf{x}) = 0$$
$$\frac{\partial \tilde{U}}{\partial t}(t, \mathbf{x}) - \nabla U_0(t, \mathbf{x}) = 0$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

If $u \in \mathcal{H}_{D,+}$ i.e. $\tilde{u} = -\nabla(-\Delta)^{-1/2} u_0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}^+, \mathcal{H})$:

$$U(t) = e^{-tD} u \,, \quad 0 < t < \infty$$

If $u \in \mathcal{H}_{D,-}$ i.e. $\tilde{u} = +\nabla(-\Delta)^{-1/2} u_0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}^-, \mathcal{H})$:

$$U(t) = e^{-tD} u \,, \quad -\infty < t < 0$$

If $u \in \mathcal{N}(D)$ i.e. $\textbf{div}\, \tilde{u} = 0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}, \mathcal{H})$:

$$U(t) = u \,, \quad -\infty < t < \infty$$

In each case, $\|U(t)\| \leq \|u\|$ for all t .

First Perturbed Problem in $\mathcal{H} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, \mathbb{C}^n)$:

$$DB = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{div}\, A \\ -\nabla & 0 \end{bmatrix} \quad \text{where}$$

$A \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^n))$ with $\mathrm{Re}(A(x)) \geq \kappa I > 0$, so that $B = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$ is a bounded strictly accretive operator in $\mathcal{H}$. Note:

$$DBu = \begin{bmatrix} 0 & \mathbf{div}\, A \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} \mathbf{div}\, A\tilde{u} \\ -\nabla u_0 \end{bmatrix} \qquad \text{and}$$

$$(DB)^2 = \begin{bmatrix} L & 0 \\ 0 & -\nabla \mathbf{div}\, A \end{bmatrix} \quad \text{where} \quad L = -\mathbf{div}\, A\nabla$$

i.e. L is the second order divergence form elliptic operator

$L = -\mathbf{div}\, A\nabla = -\sum_{j,k=1}^n \partial_j A_{j,k}(x)\partial_k$

Spectral Decomposition

Main Question: Is there a **spectral decomposition** for

$$DB = \begin{bmatrix} 0 & \textbf{div}\, A \\ -\nabla & 0 \end{bmatrix} : \qquad \mathcal{H} = \mathcal{N}(DB) \oplus \mathcal{H}_{DB,+} \oplus \mathcal{H}_{DB,-} \qquad ?$$

Equivalent Question: Is $\|\sqrt{(DB)^2}\, u\| \approx \|DBu\|$?

Equivalent Question: Is

$$\left\| \begin{bmatrix} \sqrt{L}\, u_0 \\ \sqrt{-\nabla \textbf{div}\, A}\, \tilde{u} \end{bmatrix} \right\| \approx \left\| \begin{bmatrix} \textbf{div}\, A\, \tilde{u} \\ -\nabla u_0 \end{bmatrix} \right\| \qquad ?$$

Equivalent Question: Is $\|\sqrt{L}\, u_0\| \approx \|\nabla u_0\|$?

3. YES This is the Square Root Problem of Kato

Historical Interlude II

Let $L = -\textbf{div}\, A\nabla$ be the second order divergence form elliptic operator

$$Lu(x) = -\sum_{j,k=1}^{n} \tfrac{\partial}{\partial x_j}(A_{jk}\tfrac{\partial u}{\partial x_k})(x)$$

with domain $\mathcal{D}(L) = \{u \in W^{1,2}(\mathbb{R}^n)\,;\, Lu \in L^2(\mathbb{R}^n)\}$
where
$A \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^{1+n}))$, $\text{Re}(A(x)\zeta\,,\,\zeta) \geq \kappa|\zeta|^2$ a.e. $x \in \mathbb{R}^n$, $\forall\zeta \in \mathbb{C}^n$
for some $\kappa > 0$.

Note that the set of all $\frac{(A(x)\zeta\,,\,\zeta)}{|\zeta|^2}$ is contained in a compact subset of the right half plane, and therefore in a sector $S_{\omega+}$ for some $\omega < \pi/2$.

i.e. A is a bounded strictly accretive operator in $L^2(\mathbb{R}^n)$

Historical Interlude II: Properties of the Operator

- $(Lu, u) = (A\nabla u, \nabla u) \in S_{\omega+}$ for all $u \in \mathcal{D}(L) \subset W^{1,2}(\mathbb{R}^n)$
- $\sigma(L) \subset S_{\omega+}$ (Lax-Milgram theorem)
- Together these mean that L is ω -accretive in $L^2(\mathbb{R}^n)$
- L generates a contraction semigroup e^{-tL} in $L^2(\mathbb{R}^n)$
- L has unique fractional powers L^α ($0 < \alpha < 1$) which are $\alpha\omega$ -accretive
- In particular, there exists a unique $\omega/2$ -accretive operator $\sqrt{L} = L^{1/2}$ such that $\sqrt{L}\sqrt{L} = L$
- See Kato's book on "Perturbation Theory for Linear Operators" for details.

Tosio Kato 1917 – 1999

Historical Interlude II: Selfadjoint Operators

When $A = A^*$ (i.e. $\omega = 0$), then $L = L^*$ and $\sqrt{L} = (\sqrt{L})^*$. In this case $\boxed{\mathcal{D}(\sqrt{L}) = W^{1,2}(\mathbb{R}^n)}$ with $\|\sqrt{L}\,u\| \approx \|\nabla u\|$ because

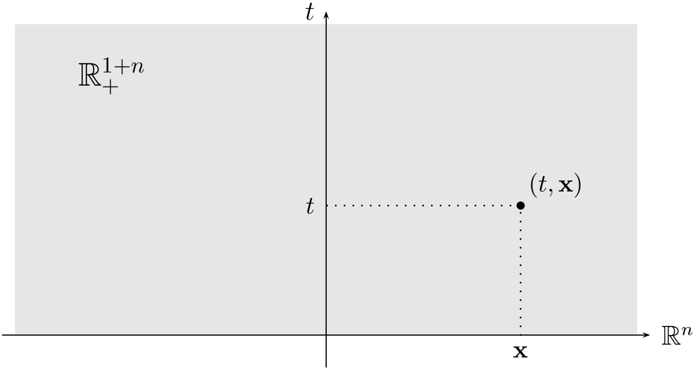
$$\begin{aligned}\|\sqrt{L}\,u\|_2^2 &= (\sqrt{L}\,u, \sqrt{L}\,u) = (Lu, u) = (-\mathbf{div}\, A\nabla u, u)\\ &= (A\nabla u, \nabla u) \approx \|\nabla u\|_2^2\end{aligned}$$

Historical Interlude II: Kato's Question [1961]

When $A \neq A^*$ (i.e. $0 < \omega < \pi/2$), is $\boxed{\mathcal{D}(\sqrt{L}) = W^{1,2}(\mathbb{R}^n)}$ with $\|\sqrt{L}\,u\| \approx \|\nabla u\|$?

- $n = 1$: Yes [Coifman, McIntosh, Meyer 1982]
- $\|A - I\| < \epsilon$: Yes [Coifman, Deng, Meyer 1983], [Fabes, Jerison, Kenig 1984]
- Many partial results. Systemetized in book [Auscher, Tchamitchian 1998]
- $n = 2$: Yes [Hofmann, Mc 2000]
- $n \geq 3$, under Gaussian kernel bounds: Yes [H, Lacey, Mc 2002]
- $n \geq 3$: Yes [A, H, L, Mc, T 2002]

Historical Interlude II: Neumann Problem



Recall $$L = -\mathbf{div}\, A\nabla = -\sum_{j,k=1}^{n} \partial_j A_{j,k}(x)\partial_k$$

where $A \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^n))$ with $\mathrm{Re}(A(x)) \geq \kappa I > 0$,

Neumann Problem for $\frac{\partial^2 F}{\partial t^2} + \mathbf{div}\, A \nabla F = 0$ on $\mathbb{R}^{1+n}_+$

The Neumann problem **N** on the upper half space

$$\mathbb{R}^{1+n}_+ = \{(t, \mathbf{x})\,;\, \mathbf{x} \in \mathbb{R}^n\,,\; t > 0\}$$

with L^2 boundary data is:

$$\frac{\partial^2 F}{\partial t^2}(t, \mathbf{x}) - LF(t, \mathbf{x}) = 0\,, \qquad t > 0$$

$$-\lim_{t \to 0} \frac{\partial F}{\partial t}(t) = w \in L^2(\mathbb{R}^n)$$

Consequence of Kato Estimate [Kenig]: The unique solution F (up to constants) with $\frac{\partial F}{\partial t}\,,\; \frac{\partial F}{\partial x_j} \in C^1_b(\mathbb{R}^+\,,\, L^2(\mathbb{R}^n))$ is

$$F(t) = (\sqrt{L})^{-1} e^{-t\sqrt{L}} w \qquad \text{so that} \qquad \frac{\partial F}{\partial t}(t) = -e^{-t\sqrt{L}} w \qquad \text{with}$$

$$\|\nabla_x F(t)\|_2 \approx \|\sqrt{L} F(t)\|_2 = \left\| e^{-t\sqrt{L}} w \right\|_2 = \left\| \frac{\partial F}{\partial t}(t) \right\|_2 \le \|w\|_2 \quad \forall t > 0$$

Evolution Equation

The evolution equation **E** for $DB = \begin{bmatrix} 0 & \mathbf{div}\, A \\ -\nabla & 0 \end{bmatrix}$ is

$$\frac{\partial U}{\partial t}(t) + DBU(t) = 0$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

i.e. $U = \begin{bmatrix} U_0 \\ \tilde{U} \end{bmatrix}$ satisfies

$$\frac{\partial U_0}{\partial t}(t, \mathbf{x}) + \mathbf{div}\, A\, \tilde{U}(t, \mathbf{x}) = 0$$
$$\frac{\partial \tilde{U}}{\partial t}(t, \mathbf{x}) - \nabla U_0(t, \mathbf{x}) = 0$$
$$\lim_{t \to 0} U(t) = u \in L^2(\mathbb{R})$$

Modification of the material for D gives:

If $u \in \mathcal{H}_{DB,+}$ i.e. $\tilde{u} = -\nabla L^{-1/2} u_0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}^+, \mathcal{H})$:

$$U(t) = e^{-tDB} u \,, \quad 0 < t < \infty$$

If $u \in \mathcal{H}_{DB,-}$ i.e. $\tilde{u} = +\nabla L^{-1/2} u_0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}^-, \mathcal{H})$:

$$U(t) = e^{-tDB} u \,, \quad -\infty < t < 0$$

If $u \in \mathcal{N}(DB)$ i.e. $\mathbf{div}\, A\, \tilde{u} = 0$, then **E** has a unique solution $U \in C_b^1(\mathbb{R}, \mathcal{H})$:

$$U(t) = u \,, \quad -\infty < t < \infty$$

In each case $\|U(t)\| \lesssim \|u\|$ for all t

Neumann Problem and Evolution Equation

Recall the Neumann problem with L^2 boundary data is:

$$\frac{\partial^2 F}{\partial t^2}(t,\mathbf{x}) + \mathbf{div}\, A\nabla F(t,\mathbf{x}) = 0\ , \qquad t>0$$
$$-\lim_{t\to 0}\frac{\partial F}{\partial t}(t) = w \in L^2(\mathbb{R}^n)$$

Let $\quad U = \begin{bmatrix} U_0 \\ \tilde{U} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial t} \\ \nabla F \end{bmatrix}$

Equivalent Claim: There exists a unique function $U \in C_b^1(\mathbb{R}^n, \mathcal{H})$ s. th.

$$\frac{\partial}{\partial t}\begin{bmatrix} U_0 \\ \tilde{U} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{div}\, A \\ -\nabla & 0 \end{bmatrix}\begin{bmatrix} U_0 \\ \tilde{U} \end{bmatrix} = \begin{bmatrix} \frac{\partial U_0}{\partial t} + \mathbf{div}\, A\tilde{U} \\ \frac{\partial \tilde{U}}{\partial t} - \nabla U_0 \end{bmatrix} = 0 \text{ on } \mathbb{R}_+^{1+n}$$
$$\tilde{U}(t) \in \overline{\mathcal{R}}(\nabla)$$
$$\lim_{t\to 0} U_0(t) = -w$$

Neumann Problem and Evolution Equation

Same Claim: There exists a unique function $U \in C_b^1(\mathbb{R}^n, \mathcal{H})$ such that

$$\frac{\partial U}{\partial t} + DBU = 0 \qquad \text{where} \qquad B = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$$

$$U(t) \in \overline{\mathcal{R}}(D) \quad \text{and} \quad \lim_{t \to 0} U_0(t) = -w$$

Equivalent Claim: There exists a unique function

$$u = \begin{bmatrix} u_0 \\ \tilde{u} \end{bmatrix} \in \mathcal{H}_{DB,+}$$

such that $u_0 = -w$. For then $U(t) = e^{-tDB}u$ is the unique solution of the evolution equation **E**

Neumann Problem and Evolution Equation

Solution:

$$\begin{bmatrix} \frac{\partial F}{\partial t}(t) \\ -\nabla F(t) \end{bmatrix} = U(t) = e^{-tDB}u = e^{-tDB}\begin{bmatrix} -w \\ \nabla L^{-1/2}w \end{bmatrix}$$

with bound

$$\|\tfrac{\partial F}{\partial t}(t)\|_2 + \|\nabla F(t)\|_2 \approx \|U(t)\|_2 \lesssim \|u\|_2 \approx \|w\|_2$$

Exercise: Show that this agrees with the previous expression

$$F(t) = (\sqrt{L})^{-1}e^{-t\sqrt{L}}w$$

where $L = -\mathbf{div}\, A\nabla$

Spectral Decomposition

More General Perturbed Problem in $\mathcal{H} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, \mathbb{C}^n)$:

$$DB = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,k} \\ B_{j,0} & B_{j,k} \end{bmatrix}$$

where $B \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C} \oplus \mathbb{C}^n))$ with $\mathrm{Re}(B(x)) \geq \kappa I > 0$, so that B is a bounded strictly accretive operator in $\mathcal{H}$.

Main Question: Is there a **spectral decomposition**

$$\mathcal{H} = \mathcal{N}(DB) \oplus \mathcal{H}_{DB,+} \oplus \mathcal{H}_{DB,-} \qquad ?$$

where

$$\mathcal{H}_{DB,\pm} = \{u \in \overline{\mathcal{R}}(D)\,;\; DBu = \pm\sqrt{(DB)^2}\, u\}$$

4. YES The technology of the solution of the Kato Square Root Problem can be extended to prove this [Axelsson, Keith, Mc 2006]

Given $u \in \mathcal{H}$, consider the **evolution equation E**

$$\frac{\partial U}{\partial t}(t) + DBU(t) = 0$$
$$\lim_{t \to 0} U(t) = u$$

- If $u \in \mathcal{H}_{DB,+}$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}^+, \mathcal{H})$:

$$U(t) = e^{-tDB}u\,, \quad 0 < t < \infty$$

- If $u \in \mathcal{H}_{DB,-}$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}^-, \mathcal{H})$:

$$U(t) = e^{-tDB}u\,, \quad -\infty < t < 0$$

- If $u \in \mathcal{N}(DB)$ then **E** has a unique solution $U \in C_b^1(\mathbb{R}, \mathcal{H})$:

$$U(t) = u\,, \quad -\infty < t < \infty$$

In each case, $\|U(t)\| \lesssim \|u\|$ for all t.

Historical Interlude III: First Order Systems

- First order systems and the Kato problem in one dimension: [Auscher, Nahmod, McIntosh 1997]
- Quadratic estimates and functional calculi of perturbed Dirac operators: [Axelsson, Keith, Mc 2006] These results imply the spectral decomposition for $DB = \begin{bmatrix} 0 & \mathbf{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,k} \\ B_{j,0} & B_{j,k} \end{bmatrix}$
- For a more direct proof, see also [Auscher-Axelsson-Mc 2010a]
- What is needed is

$$\int_0^\infty \left\| \frac{tDB}{I + t^2(DB)^2} u \right\|^2 \frac{dt}{t} \approx \|u\|^2$$

The Quadratic Estimate

- Therefore

$$
\begin{aligned}
\|\sqrt{(DB)^2}\,u\|^2 &\approx \int_0^\infty \left\| \frac{tDB}{I + t^2(DB)^2} \sqrt{(DB)^2}\,u \right\|^2 \frac{dt}{t} \\
&\approx \int_0^\infty \left\| \frac{t\sqrt{(DB)^2}}{I + t^2(DB)^2} DB\,u \right\|^2 \frac{dt}{t} \\
&\approx \|DB\,u\|^2
\end{aligned}
$$

- We need to prove

$$
\int_0^\infty \left\| \frac{tDB}{I + t^2(DB)^2}\,u \right\|^2 \frac{dt}{t} \leq C\|u\|^2
$$

as the reverse direction is a dual result.

The quadratic estimate

$$\int_0^\infty \left\| \frac{tDB}{I + t^2(DB)^2} u \right\|^2 \frac{dt}{t} \leq C\|u\|^2$$

is reduced to proving the Carleson measure bounds like

$$\int_0^{\ell(Q)} \int_Q |\gamma_t(x)|^2 \frac{dxdt}{t} \leq C \operatorname{vol}(Q)$$

for all dyadic cubes $Q \subset \mathbb{R}^n$, where $\gamma_t(x) \in \mathcal{L}(\mathbb{C}^{1+n})$ is defined by

$$[\gamma_t(x)]w = \left(\frac{tDB}{I + t^2(DB)^2} w \right)(x) \qquad \forall w \in \mathbb{C}^{1+n}$$

When $n = 1$ we can do so with a $T(b)$ argument. For $n > 1$ we use a much more complicated local $T(b)$ type proof.

Recent Applications

To conclude, we consider the Neumann problem **N** on the upper half space $\mathbb{R}^{1+n}_+$ with L^2 boundary data:

$$\frac{\partial}{\partial t} B_{0,0}(\mathbf{x}) \frac{\partial F}{\partial t} + \frac{\partial}{\partial t} B_{0,k}(\mathbf{x}) \partial_k F + \partial_j B_{j,0}(\mathbf{x}) \frac{\partial F}{\partial t} + \partial_j B_{j,k}(\mathbf{x}) \partial_k F = 0 \ , \ t > 0$$

$$-\lim_{t \to 0} \left(B_{0,0} \frac{\partial F}{\partial t}(t) + \sum B_{0,k} \partial_k F \right) = w \in L^2(\mathbb{R}^n)$$

where $B = \begin{bmatrix} B_{0,0} & B_{0,k} \\ B_{j,0} & B_{j,k} \end{bmatrix} \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C} \oplus \mathbb{C}^n))$, $\mathrm{Re}(B(\mathbf{x})) \geq \kappa I > 0$

Definition: we say **N is well-posed** if there exists a unique solution F (up to constants) with $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x_j} \in C^1_b(\mathbb{R}^+, L^2(\mathbb{R}^n))$ and

$$\left\| \frac{\partial F}{\partial t}(t) \right\|_2 + \|\nabla F(t)\|_2 \lesssim \|w\|_2$$

Neumann Problem with t-independent coefficients

Theorem (Auscher-Axelsson-McIntosh 2010)

N *is well posed if and only if the mapping*

$$\begin{aligned} \mathcal{H}_{D\hat{B},+} &\to L^2(\mathbb{R}^n) \\ u &\mapsto u_0 = -w \end{aligned}$$

is an isomorphism, where

$$\hat{B} = \begin{bmatrix} I & 0 \\ B_{j,0} & B_{j,k} \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,k} \\ 0 & I \end{bmatrix}^{-1} .$$

Neumann Problem with t-independent coefficients

Corollary: The set of $B \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^{1+n}))$ for which **N** is well posed, is an open set.

Case 1: When $B = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$, then $\hat{B} = B$ and $u = \begin{bmatrix} u_0 \\ -\nabla L^{-1/2} u_0 \end{bmatrix}$ so the fact that $\|u\| \approx \|u_0\|$ is a consequence of the Kato Square Root Problem for $L = -\mathbf{div}\, A\nabla$.

Case 2: When $B = B^*$, then a simple Rellich argument allows us to prove that $\|u\| \approx \|u_0\|$ and hence that **N** is well-posed.

Historical Comments: When B is real-symmetric, then the well-posedness of **N** was established in [Kenig-Pipher 1993]. Earlier results on related problems were established by Dahlberg, Jerison and Kenig. See the book [Kenig 1994].

Counterexample: There exist B for which **N** is NOT well-posed [Kenig-Rule 2009].

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