HOLOMORPHIC FUNCTIONAL CALCULI OF OPERATORS, QUADRATIC ESTIMATES AND INTERPOLATION

Pascal Auscher, Alan M^{c}Intosh and Andrea Nahmod

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Abstract.

We develop some connections between interpolation theory and the theory of bounded holomorphic functional calculi of operators in Hilbert spaces, via quadratic estimates. In particular we show that an operator T of type ω has a bounded holomorphic functional calculus if and only if the Hilbert space is the complex interpolation space midway between the completion of its domain and of its range. We also characterise the complex interpolation spaces of the domains of all the fractional powers of T, whether or not T has a bounded functional calculus. This treatment extends earlier ones for self–adjoint and maximal accretive operators. This work is motivated by the study of first order elliptic systems which are related to the square root problem for non–degenerate second order operators under boundary conditions on an interval. See our subsequent paper [AM^{c}N].

§1. Introduction.

It is often of interest to know whether an operator T of type ω in a Hilbert space $\mathcal{H}$ has a bounded H_∞ functional calculus. This is related to the property of whether T satisfies quadratic estimates in $\mathcal{H}$. Indeed T does have a bounded H_∞ functional calculus if and only if the quadratic norm

$$||u||_T = \left\{ \int_0^\infty ||\psi(tT)u||^2 \ \tfrac{dt}{t} \right\}^{\frac{1}{2}}$$

is equivalent to the given norm of $\mathcal{H}$ (where ψ is a holomorphic function which decays at 0 and ∞.) This result of M^{c}Intosh [M^{c}2] was based on earlier work by Yagi [Y1].

It was shown in these papers that these properties hold if and only if the domains of the fractional powers of T interpolate by the complex method. In this case $\mathcal{H}$ is the complex interpolation space midway between the completion $\mathcal{D}_T$ of its domain $\mathcal{D}(T)$ under the norm $\|Tu\|$, and the completion $\mathcal{R}_T$ of its range $\mathcal{R}(T)$ under $\left\|T^{-1}u\right\|$.

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A principal result of the current paper, Theorem 4.2, is that $\| u \|_T$ is the norm of this complex interpolation space midway between $\mathcal{D}_T$ and $\mathcal{R}_T$, whether or not T has a bounded H_∞ functional calculus.

A consequence is the fact that if two operators S and T satisfy $\mathcal{D}(S) = \mathcal{D}(T)$ with $\| Su \| \approx \| Tu \|$ and $\mathcal{R}(S) = \mathcal{R}(T)$ with $\left\| S^{-1}u \right\| \approx \left\| T^{-1}u \right\|$, then $\| u \|_S \approx \| u \|_T$. Therefore, if T satisfies the quadratic estimate $\| u \|_T \le c \| u \|$, then $\| u \|_S \le c \| u \|$. We also present a more direct proof of this result as Theorem 3.1.

These results also hold for operators of type S_ω which have spectrum in the union of two sectors. In this case they can be used to show how known quadratic estimates for the operators $a\frac{d}{dx}$ and $b\frac{d}{dx}$ imply quadratic estimates for a particular system T, which in turn give the bound $||L^{\frac{1}{2}}u||_2 \approx ||\frac{du}{dx}||_2$ for all $u \in H^1(\mathbb{R})$, originally proved by Kenig and Meyer [KM]. Here $a, b \in L_\infty(\mathbb{R})$ with $\mathrm{Re}\, a \ge \kappa$ and $\mathrm{Re}\, b \ge \kappa$ for some $\kappa > 0$, and $Lu = -a\frac{d}{dx}(b\frac{du}{dx})$.

In order to prove such a bound when L includes lower order terms and is defined under boundary conditions on an interval, we need a further development of the connections between quadratic estimates, interpolation spaces, and fractional powers. Such connections are developed in this paper, while the application is presented in a second paper [AMcN].

The results of this paper have a long history, so that it is not possible to give credit for all the underlying ideas. First, there is the operator theory developed in the study of semi–groups as well as fractional powers. Then there is the harmonic analysis related to Littlewood–Paley estimates as developed by Stein and many others. This is related to the interpolation theory initiated by Calderón, Lions and Peetre, and to the theory of Sobolev and Besov spaces.

These concepts all lie behind the main interpolation results, Theorems 4.2 and 5.3. Although these theorems include many known results as specific cases, they are, so far as we are aware, new as stated, as are the consequences presented in Theorems 7.3 and 8.5. Our interest stems from the fact that they provides a useful tool for the study of functional calculi of operators in Hilbert spaces.

Let us describe the contents. We review the basic background on operator theory in Section 2. In Section 3 we present a direct approach to obtaining quadratic estimates for an operator from known estimates for a related operator. This material is re-obtained and strengthened from the point of view of interpolation theory in Section 4. In Section 5 we characterise the real interpolation spaces of domains of fractional powers, while we show in Section 6 that these are the same as the complex interpolation spaces.

We turn our attention in Section 7 to applying this material to a class of operators obtained from multiplicative perturbations of positive self–adjoint operators. In Section 8 we show how to extend these results to operators of type S_ω, whose spectrum is in the union of two sectors, since that is what is needed for the motivating example in Section 9 and its generalisation in [AMcN]. We describe in Section 10 some results connecting the square root problem of Kato to quadratic estimates for related operators. We conclude in Section 11 with some peturbation results.

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Whenever it is stated that two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are equal, $\mathcal{H} = \mathcal{K}$, it is implied that their norms are equivalent. Often we write a statement such as "$\mathcal{X} \subset \mathcal{Y}$ with $\| u \|_{\mathcal{Y}} \le c \| u \|_{\mathcal{X}}$" concerning two normed spaces $\mathcal{X}$ and $\mathcal{Y}$ to mean that "$\mathcal{X} \subset \mathcal{Y}$ and there exists a constant c such that $\| u \|_{\mathcal{Y}} \le c \| u \|_{\mathcal{X}}$ for all $u \in \mathcal{X}$".

§2. Operators of type ω, functional calculi and quadratic estimates.

Here is a brief survey of some known results about functional calculi and quadratic estimates of operators of type ω in a Hilbert space, as developed in the papers [Y1], [M^{c}2] and [M^{c}Y]. See the lecture notes [ADM^c] for a more detailed presentation of the material in this section, and for further references.

Throughout this paper $\mathcal{H}$ denotes a complex Hilbert space. By an *operator* in $\mathcal{H}$ we mean a linear mapping $T : \mathcal{D}(T) \to \mathcal{H}$ whose *domain* $\mathcal{D}(T)$ is a linear subspace of $\mathcal{H}$. The *norm* of T is the (possibly infinite) number

$$\| T \| = \sup \{ \| Tu \| : u \in \mathcal{D}(T) , \ \| u \| = 1 \} .$$

We say that T is *bounded on* $\mathcal{H}$ if $\mathcal{D}(T) = \mathcal{H}$ and $\| T \| < \infty$. The Banach algebra of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. We call T *closed* if its graph, $\{(u, Tu) : u \in \mathcal{D}(T)\}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

The *spectrum* $\sigma(T)$ of T is the set of all complex λ for which $(T - \lambda I)^{-1} \notin \mathcal{L}(\mathcal{H})$, together with ∞ if $T \notin \mathcal{L}(\mathcal{H})$.

For $0 \leq \omega < \mu < \pi$, define the closed and open sectors in the (extended) complex plane $\mathbb{C}$:

$$\begin{aligned} S_{\omega+} &= \{\zeta \in \mathbb{C} : |\arg\zeta| \leq \omega\} \cup \{0, \infty\} \\ S^0_{\mu+} &= \{\zeta \in \mathbb{C} : \zeta \neq 0, |\arg\zeta| < \mu\}. \end{aligned}$$

A closed operator T is said to be of *type* ω if $\sigma(T) \subset S_{\omega+}$ and for each $\mu > \omega$ there exists C_μ such that

$$\| (T - \zeta I)^{-1} \| \leq C_\mu |\zeta|^{-1}, \quad \zeta \notin S_{\mu+}.$$

Traditionally, such operators are called *type* (ω, M) where $M = \sup\{ |\xi| \, \| (T + \xi I)^{-1} \| : \xi > 0 \}$ [K1]. See [T] for basic results. For a treatment closer to our needs, see the lecture notes [ADMc], in which the notation *type* S_ω is used in place of *type* ω.

We remark that every one–one operator T of type ω in $\mathcal{H}$ has dense domain $\mathcal{D}(T)$ and dense range $\mathcal{R}(T) = \{Tu : u \in \mathcal{D}(T)\}$ [CDMcY]. If T is not one–one, then $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{H}_0$ where $\mathcal{N}(T)$ is the nullspace of T and $\mathcal{H}_0$ is the closure of $\mathcal{R}(T)$, and moreover, the restriction of T to $\mathcal{H}_0$ is a one–one operator of type ω in $\mathcal{H}_0$. So the general theory can readily be reduced to that of one–one operators. The direct sum $\oplus$ is in general not orthogonal.

In investigating the holomorphic functional calculus of T, we employ the following subspaces of the space $H(S^0_{\mu+})$ of all holomorphic functions on $S^0_{\mu+}$.

$$H_\infty(S^0_{\mu+}) = \{ f \in H(S^0_{\mu+}) : \| f \|_\infty < \infty \}$$

where $\| f \|_\infty = \sup\{ |f(\zeta)| : \zeta \in S^0_{\mu+} \}$,

$$\Psi(S^0_{\mu+}) = \{ \psi \in H(S^0_{\mu+}) : |\psi(\zeta)| \leq C|\zeta|^s (1 + |\zeta|^{2s})^{-1} \text{ for some } C, \, s > 0 \}$$

and

$$F(S^0_{\mu+}) = \{ f \in H(S^0_{\mu+}) : |f(\zeta)| \leq C(|\zeta|^{-s} + |\zeta|^s) \text{ for some } C, \, s > 0 \},$$

so that

$$\Psi(S^0_{\mu+}) \subset H_\infty(S^0_{\mu+}) \subset F(S^0_{\mu+}) \subset H(S^0_{\mu+}).$$

Every one–one operator T of type ω in $\mathcal{H}$ has a unique holomorphic functional calculus which is consistent with the usual definition of polynomials of an operator. By this we mean that for each $\mu > \omega$ and for each $f \in F(S^0_{\mu+})$, there corresponds a closed operator $f(T)$ in $\mathcal{H}$. Further, if $f, g \in F(S^0_{\mu+})$ and $\alpha \in \mathbb{C}$, then

$$\begin{aligned} \alpha f(T) + g(T) &= (\alpha f + g)(T)|_{\mathcal{D}(f(T)) \cap \mathcal{D}((\alpha f + g)(T))} \\ g(T) f(T) &= (gf)(T) \,|_{\mathcal{D}(f(T)) \cap \mathcal{D}((gf)(T))}. \end{aligned}$$

In particular, if $\psi \in \Psi(S^0_{\mu+})$, then $\psi(T) = \frac{1}{2\pi i}\int_\delta (T-\zeta I)^{-1}\psi(\zeta)\,d\zeta \in \mathcal{L}(\mathcal{H})$ where the integral is on the unbounded contour $\delta = \{\zeta = re^{\pm i\theta} : r \geq 0\}$, parametrised clockwise around $S_{\omega+}$, and $\omega < \theta < \mu$.

Note that, if f is a function which can be expressed as $f(\zeta) = \alpha(\zeta+1)^{-1} + \beta + \psi(\zeta)$ with $\alpha, \beta \in \mathbb{C}$ and $\psi \in \Psi(S^0_{\mu+})$, then $f(T) \in \mathcal{L}(\mathcal{H})$. For example, when $\omega < \frac{\pi}{2}$, then, taking $f_z(\zeta) = e^{-z\zeta}$, we see that $e^{-zT} \in \mathcal{L}(\mathcal{H})$ if $|\arg z| < \frac{\pi}{2} - \omega$. By the properties of the functional calculus, $-T$ generates the holomorphic semigroup $\{e^{-zT} : |\arg(z)| < \frac{\pi}{2} - \omega\}$.

Of concern to us is the question whether, for every function $f \in H_\infty(S^0_{\mu+})$, the operator $f(T) \in \mathcal{L}(\mathcal{H})$ with $||f(T)|| \leq c_\mu ||f||_\infty$. When this holds, we say that T has a *bounded* H_∞ *functional calculus in* $\mathcal{H}$. This holds if and only if the given norm $||u||$ on $\mathcal{H}$ is equivalent to the norm $||u||_T$ which we now define [M^{c}2].

Given a non–zero function $\psi \in \Psi(S^0_{\mu+})$, let $\psi_t(\zeta) = \psi(t\zeta)$, and define the norm

$$||u||_T = \left\{ \int_0^\infty ||\psi_t(T)u||^2 \ \frac{dt}{t} \right\}^{\frac{1}{2}}$$

on the space of all those $u \in \mathcal{H}$ for which the right hand side is finite. Complete this space under the norm $||u||_T$ to form the Hilbert space $\mathcal{H}_T$. It is an important fact that different choices of μ and ψ give rise to equivalent norms, and to the same space $\mathcal{H}_T$. Further, $||f(T)u||_T \leq c_\mu ||f||_\infty ||u||_T$ for all $f \in H_\infty(S^0_{\mu+})$ [M^{c}Y, ADM^c].

Let T' be the dual of T (also a one–one operator of type ω) in a Hilbert space $\mathcal{K}$ which is related to $\mathcal{H}$ by a bilinear or sesquilinear duality $\langle u, v\rangle$. (If $\mathcal{K} = \mathcal{H}$ and the duality is given by the inner product (u, v), then T' is the adjoint T^*.) We remark that $|\langle u, v\rangle| \leq c \,\| u \|_T \,\| v \|_{T'}$ for all $u \in \mathcal{H}$, $v \in \mathcal{K}$. To see this, choose a function $\psi \in \Psi(S^0_{\mu+})$ such that $\int_0^\infty \psi^2(\tau)\frac{d\tau}{\tau} = 1$ and write

$$|\langle u, v\rangle| \ = \ \left|\langle \int_0^\infty {\psi_t}^2(T)u \ \frac{dt}{t} \,,\, v\rangle\right| \ = \ \left|\int_0^\infty \langle \psi_t(T)u \,,\, \overline{\psi}_t(T')v\rangle \ \frac{dt}{t}\right| \ \leq \ c \,\| u \|_T \,\| v \|_{T'}$$

where $\overline{\psi}(\zeta) = \overline{\psi(\overline{\zeta})}$ when the pairing is sesquilinear, and $\overline{\psi} = \psi$ when the pairing is bilinear. Indeed more is true, as was essentially proved in [M^{c}Q], and explicitly in [ADM^c]:

Theorem 2.1. *Let T, T' be one–one operators of type ω in a dual pair of Hilbert spaces $\mathcal{H}, \mathcal{K}$ such that $\langle Tu, v\rangle = \langle u, T'v\rangle$ for all $u \in \mathcal{D}(T)$, $v \in \mathcal{D}(T')$. Then $\langle \mathcal{H}_T, \mathcal{K}_{T'}\rangle$ is a dual pair of Hilbert spaces under an extension of the same pairing.*

Consequently $\mathcal{H}_T \subset \mathcal{H}$ with $||u|| \leq c||u||_T$ if and only if $\mathcal{K} \subset \mathcal{K}_{T'}$ with $||v||_{T'} \leq c'||v||$.

Inequalities of this kind are referred to as *quadratic estimates* and have long played a role in harmonic analysis. The main result of the paper [M^{c}2] is that T has a bounded

H_∞ functional calculus if and only if it satisfies quadratic estimates. Here is a precise statement.

Theorem 2.2. *Let T be a one–one operator of type ω in $\mathcal{H}$, and let $0 \le \omega < \mu < \pi$. Then the following statements are equivalent.*

(a) $\mathcal{H}_T = \mathcal{H}$;

(b) $\mathcal{H} \subset \mathcal{H}_T$ *with* $||u||_T \le c||u||$ *and* $\mathcal{K} \subset \mathcal{K}_{T'}$ *with* $||v||_{T'} \le c||v||$;

(c) $f(T) \in \mathcal{L}(\mathcal{H})$ *and* $\| f(T) \| \le c_\mu \| f \|_\infty$ *for all* $f \in H_\infty(S^0_{\mu+})$.

We remark that (c) holds for one value of μ if and only if it holds for any other value. Thus it is unambiguous to say that an operator T which satisfies (c) has a bounded H_∞ functional calculus in $\mathcal{H}$, without specifying the angle μ. The next result is an easy one which is stated for later use.

Proposition 2.3. *Let T be a one–one operator of type ω in $\mathcal{H}$, let $0 \le \omega < \mu < \pi$, and let V be an isomorphism from $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. Then $S = VTV^{-1}$ is type ω in $\mathcal{K}$ and $f(S) = Vf(T)V^{-1}$ for all $f \in F(S^0_{\mu+})$. Moreover, $\mathcal{K}_S = V\mathcal{H}_T$ with $\| u \|_T \approx \| Vu \|_S$. Hence $\mathcal{K}_S = \mathcal{K}$ if and only if $\mathcal{H}_T = \mathcal{H}$.*

We conclude this section with some comments about self–adjoint and accretive operators.

An operator T in $\mathcal{H}$ is called *self–adjoint* if $T = T^*$, in which case $(Tu, u) \in \mathbb{R}$ for all $u \in \mathcal{D}(T)$. If in addition, $(Tu, u) > 0$ for all non–zero $u \in \mathcal{D}(T)$, then T is *positive self–adjoint.* Such an operator is a one–one operator of type 0 and satisfies $\mathcal{H}_T = \mathcal{H}$. Indeed T has a Borel functional calculus with bound 1. In particular, $\| f(T) \| \le \| f \|_\infty$ for all $f \in H_\infty(S^0_{\mu+})$ and all $\mu > 0$.

An operator T in $\mathcal{H}$ is called *maximal accretive* if $\mathrm{Re}\,(Tu, u) \ge 0$ and $\mathrm{Re}\,\sigma(T) \ge 0$. Such an operator is type $\frac{\pi}{2}$. If, in addition, T is one–one, then $\mathcal{H}_T = \mathcal{H}$, and indeed T has a bounded H_∞ functional calculus with bound 1. That is, $\| f(T) \| \le \| f \|_\infty$ for all $f \in H_\infty(S^0_{\mu+})$ and all $\mu > \frac{\pi}{2}$. (Actually, a one–one operator T of type $\frac{\pi}{2}$ satisfies $\| f(T) \| \le \| f \|_\infty$ if and only if it is maximal accretive.)

Given $\omega < \frac{\pi}{2}$, let us call A a *bounded ω-accretive operator on $\mathcal{H}$* if $A \in \mathcal{L}(\mathcal{H})$ and $|\arg(Au, u)| \le \omega$ for all $u \in \mathcal{H}$. For such an operator, $\sigma(A) \subset S_{\omega+}$. We shall often consider bounded invertible ω-accretive operators A, meaning in addition, that A has a bounded inverse, or, equivalently, that $\mathrm{Re}\,(Au, u) \ge \kappa \| u \|^2$ for some $\kappa > 0$.

The *numerical range* of an operator A is $\{(Au, u) : \| u \| = 1\}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is a bounded invertible ω-accretive operator on $\mathcal{H}$ for some $\omega < \frac{\pi}{2}$, if and only if the closure of its numerical range is a compact subset of the open right half plane.

§3. A direct approach to quadratic estimates.

Let us present various ways in which quadratic estimates for one operator can be deduced from quadratic estimates for a related operator. The first result, modified for operators of type S_ω, is precisely what is needed for the application in Section 9. We follow it with a strengthened version which is more in the nature of an interpolation result.

Theorem 3.1. *Let S and T be two one–one operators of type ω in $\mathcal{H}$.*
(i) Assume that $\mathcal{D}(T) \subset \mathcal{D}(S)$ with $\| Su \| \le c \| Tu \|$, and $\mathcal{D}(S^*) \subset \mathcal{D}(T^*)$ with $\| T^* v \| \le c \| S^* v \|$. If $\mathcal{H} \subset \mathcal{H}_T$ with $\| u \|_T \le c \| u \|$, then $\mathcal{H} \subset \mathcal{H}_S$ with $\| u \|_S \le c' \| u \|$.
(ii) Assume that $\mathcal{D}(T) = \mathcal{D}(S)$ with $\| Su \| \approx \| Tu \|$, and $\mathcal{D}(S^*) = \mathcal{D}(T^*)$ with $\| T^* v \| \approx \| S^* v \|$. If $\mathcal{H} = \mathcal{H}_T$ with $\| u \|_T \approx \| u \|$, then $\mathcal{H} = \mathcal{H}_S$ with $\| u \|_S \approx \| u \|$.

Proof. (i) For $\omega < \mu < \pi$, choose $\psi \in \Psi(S^0_{\mu+})$ such that $\int_0^\infty \psi(\tau) \frac{d\tau}{\tau} = 1$ and $\psi^{(1)}, \psi^{(2)} \in \Psi(S^0_{\mu+})$ also, where $\psi(z) = z\psi^{(1)}(z)$ and $\psi^{(2)}(z) = z\phi(z) = z \int_1^\infty \psi_t(z) \frac{dt}{t}$. Note that $\phi_r(z) = \phi(rz) = \int_r^\infty \psi_t(z) \frac{dt}{t}$ and $r \frac{d}{dr} \phi_r(z) = -\psi_r(z)$. Since $\phi_t(S^*)$ and $\phi_t(T)$ converge strongly to I as $t \to 0$ and to 0 as $t \to \infty$, the fundamental theorem of calculus gives

$$\| u \|^2 = - \int_0^\infty \tfrac{d}{dt} (\phi_t(S^*)u \, , \, \phi_t(T)u) \, dt \ .$$

Hence

$$\begin{aligned}
\| u \|^2 &= \int_0^\infty \{ (\psi_t(S^*)u, \, \phi_t(T)u) + (\phi_t(S^*)u, \, \psi_t(T)u) \} \, \tfrac{dt}{t} \\
&= \int_0^\infty (tS^* \psi_t^{(1)}(S^*)u \, , \, \phi_t(T)u) \tfrac{dt}{t} \ + \ \int_0^\infty (\phi_t(S^*)u \, , \, tT\psi_t^{(1)}(T)u) \, \tfrac{dt}{t} \\
&\le \left\{ \int_0^\infty \left\| \psi_t^{(1)}(S^*)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \left\{ \int_0^\infty \| tS\phi_t(T)u \|^2 \, \tfrac{dt}{t} \right\}^{\frac12} \\
&\quad + \left\{ \int_0^\infty \| tT^* \phi_t(S^*)u \|^2 \, \tfrac{dt}{t} \right\}^{\frac12} \left\{ \int_0^\infty \left\| \psi_t^{(1)}(T)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \\
&\le c \left\{ \int_0^\infty \left\| \psi_t^{(1)}(S^*)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \left\{ \int_0^\infty \left\| \psi_t^{(2)}(T)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \\
&\quad + c \left\{ \int_0^\infty \left\| \psi_t^{(2)}(S^*)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \left\{ \int_0^\infty \left\| \psi_t^{(1)}(T)u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} \\
&\le 2c \, \| u \|_{S^*} \| u \|_T \\
&\le c'' \| u \|_{S^*} \| u \| \qquad (\text{since } \| u \|_T \le c \| u \|) \ .
\end{aligned}$$

Therefore $\| u \| \le c'' \| u \|_{S^*}$ and so, by Theorem 2.1, $\| u \|_S \le c' \| u \|$.

(ii) This follows on applying the above estimate twice, first as stated, and then with S and T replaced by S^* and T^*. $\square$

Remark. The assumptions on $\mathcal{D}(T^*)$ and $\mathcal{D}(S^*)$ can be replaced by assumptions on $\mathcal{R}(T)$ and $\mathcal{R}(S)$. In fact, $\mathcal{R}(T) \subset \mathcal{R}(S)$ with $\| S^{-1}u \| \leq c \| T^{-1}u \|$ if and only if $\mathcal{D}(S') \subset \mathcal{D}(T')$ with $\| T'v \| \leq c \| S'v \|$ where S' and T' denote duals of S and T with respect to any duality.

Let us present a second version of the above result which includes these replacements, and is strengthened as well. In this form it becomes an interpolation theorem. In the next section we shall see that $\mathcal{H}_T$ is a real interpolation space midway between $\mathcal{D}_T$ and $\mathcal{R}_T$.

Theorem 3.2. *Let S and T be one–one operators of type ω in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and let V be a bounded linear map from $\mathcal{K}$ to $\mathcal{H}$. Suppose that $V\mathcal{D}(T) \subset \mathcal{D}(S)$ with $\| SVu \| \leq c \| Tu \|$, and $V\mathcal{R}(T) \subset \mathcal{R}(S)$ with $\| S^{-1}Vu \| \leq c \| T^{-1}u \|$. Then $V\mathcal{K}_T \subset \mathcal{H}_S$ with $\| Vu \|_S \leq c' \| u \|_T$.*

Proof. Let S' denote the dual of S in some dual Hilbert space $\mathcal{K}'$. Proceeding as in the above proof, we obtain

$$|\langle v, Vu \rangle| \;=\; \left| \int_0^\infty \tfrac{d}{dt} \langle \phi_t(S')v \,,\, V\phi_t(T)u \rangle \, dt \right| \leq \;\ldots\; \leq \; c'' \| v \|_{S'} \| u \|_T$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{K}'$. Therefore, by Theorem 2.1,

$$\| Vu \|_S \;\approx\; \sup_{v \neq 0} \left\{ \frac{|\langle v, Vu \rangle|}{\| v \|_{S'}} \right\} \;\leq\; c' \| u \|_T \,. \qquad \square$$

Here is a consequence which is useful in practice. See [AMcN].

Corollary 3.3. *Let S and T be one–one operators of type ω in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and suppose there exist bounded linear transformations $V : \mathcal{K} \to \mathcal{H}$ and $W : \mathcal{H} \to \mathcal{K}$ such that $WV = I_{\mathcal{K}}$,*

$$\begin{aligned} &V\mathcal{D}(T) \subset \mathcal{D}(S) \text{ with } \| SVu \| \leq c \| Tu \| \text{, and} \\ &\qquad\qquad V\mathcal{R}(T) \subset \mathcal{R}(S) \text{ with } \| S^{-1}Vu \| \leq c \| T^{-1}u \| \text{, and} \\ &W\mathcal{D}(S) \subset \mathcal{D}(T) \text{ with } \| TWu \| \leq c \| Su \| \text{, and} \\ &\qquad\qquad W\mathcal{R}(S) \subset \mathcal{R}(T) \text{ with } \| T^{-1}Wu \| \leq c \| S^{-1}u \| \,. \end{aligned}$$

If S has a bounded H_∞ functional calculus in $\mathcal{H}$, then T has a bounded H_∞ functional calculus in $\mathcal{K}$.

Proof. By Theorem 2.2, $\mathcal{H}_S = \mathcal{H}$ with equivalence of norms. Combining this with Theorem 3.2, we find that $V\mathcal{K}_T \subset \mathcal{H}$ with $\| Vu \| \le c' \| u \|_T$ and $W\mathcal{H} \subset \mathcal{K}_T$ with $\| Wu \|_T \le c' \| u \|$. Therefore

$$\mathcal{K}_T \subset \mathcal{K} \text{ with } \| u \| = \| WVu \| \le \| W \| \| Vu \| \le c' \| W \| \| u \|_T \quad \text{and}$$
$$\mathcal{K} \subset \mathcal{K}_T \text{ with } \| u \|_T = \| WVu \|_T \le c' \| Vu \| \le c' \| V \| \| u \| .$$

Thus $\mathcal{K}_T = \mathcal{K}$. The result follows on again applying Theorem 2.2. □

§4. Quadratic estimates and interpolation theory.

Throughout this section and the next, T denotes a one–one operator of type ω in a Hilbert space $\mathcal{H}$, where $0 \le \omega < \pi$. In them we develop the connection between interpolation theory and quadratic estimates, thus re-deriving and strengthening the results in Section 3.

For each $s \in \mathbb{R}$, define the quadratic norm

$$\| u \|_{T,s} = \left\{ \int_0^\infty \left\| t^{-s} \psi_t(T) u \right\|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

where $\psi \in \Psi(S^0_{\mu+})$ has sufficient decay at 0 and ∞ to ensure that the function $\zeta^{-s}\psi(\zeta)$ of ζ is also in $\Psi(S^0_{\mu+})$. Let $\mathcal{H}_{T,s}$ be the Hilbert space with this norm, formed by completing the set of all $u \in \mathcal{H}$ for which $\| u \|_{T,s}$ is finite. In particular, $\mathcal{H}_{T,0} = \mathcal{H}_T$ with $\| u \|_{T,0} = \| u \|_T$. It is straightforward to modify the theory for the case $s = 0$ to obtain the following facts.

Theorem 4.1. *Let T be a one–one operator of type ω in $\mathcal{H}$ and let $s \in \mathbb{R}$. Then*
(i) different choices of μ and ψ give rise to equivalent norms $\| u \|_{T,s}$ and to the same space $\mathcal{H}_{T,s}$;
(ii) $f(T) \in \mathcal{L}(\mathcal{H}_{T,s})$ for all $f \in H_\infty(S^0_{\mu+})$ with $\| f(T)u \|_{T,s} \le c_{\mu,s} \| f \|_\infty \| u \|_{T,s}$;
(iii) if T' is dual to T with respect to a dual pair $\langle \mathcal{H}, \mathcal{K} \rangle$ of Hilbert spaces, then $\langle \mathcal{H}_{T,s}, \mathcal{K}_{T',-s} \rangle$ is a dual pair of Hilbert spaces under the same pairing;
(iv) If $S = VTV^{-1}$, where V is an isomorphism from $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, then $\mathcal{K}_{S,s} = V\mathcal{H}_{T,s}$ and $\| u \|_{T,s} \approx \| Vu \|_{S,s}$.

We come now to the main result which relates quadratic estimates to interpolation theory. We follow the real interpolation method of Lions and Peetre as presented in Chapter 3 of Bergh and Löfström's book [BL] though we abbreviate the symbol $(\mathcal{H}\,,\ \mathcal{K})_{\alpha,2}$ for the usual real interpolation space to $(\mathcal{H}\,,\ \mathcal{K})_\alpha$. Later we shall remark that these spaces are identical to the complex interpolation spaces $[\mathcal{H}\,,\ \mathcal{K}]_\alpha$.

Recall that $\mathcal{D}_T$ is the completion of the domain $\mathcal{D}(T)$ under the norm $\| Tu \|$ and that $\mathcal{R}_T$ is the completion of the range $\mathcal{R}(T)$ under $\left\| T^{-1}u \right\|$. These spaces are compatible in the sense that each is continuously embedded in a larger Banach space $\mathcal{X}$. For example, we could choose $\mathcal{X} = \mathcal{R}_{\psi(T)}$ where $\psi(\zeta) = \zeta^2(1+\zeta^2)^{-2}$.

Theorem 4.2. *Let T be a one–one operator of type ω in $\mathcal{H}$. Then*

$$\mathcal{H}_T = (\mathcal{R}_T,\, \mathcal{D}_T)_{\frac{1}{2}}$$

with equivalence of norms. More generally, if $0<\alpha<1$, then $\mathcal{H}_{T,2\alpha-1} = (\mathcal{R}_T, \mathcal{D}_T)_\alpha$.

Proof. We prove the first statement and leave the generalisation to the reader.

Let $u \in \mathcal{H}_T \subset \mathcal{D}_T + \mathcal{R}_T$. Recall $K(t,u) = \inf\left\{\left\| T^{-1}u_0 \right\| + \| tTu_1 \| \;:\; u = u_0 + u_1\right\}$. For ψ with sufficient decay at 0 and ∞, normalised so that $\int_0^\infty \psi(t)\,\frac{dt}{t} = 1$, choose $u_0 = \int_0^{\sqrt{t}} \psi_\tau(T)u\,\frac{d\tau}{\tau}$ and $u_1 = \int_{\sqrt{t}}^\infty \psi_\tau(T)u\,\frac{d\tau}{\tau}$, where $\psi_\tau(\zeta) = \psi(\tau\zeta)$. Let $\psi^{(0)}(\zeta) = \zeta^{-1}\int_0^1 \psi_\tau(\zeta)\,\frac{d\tau}{\tau}$ and $\psi^{(1)}(\zeta) = \zeta\int_1^\infty \psi_\tau(\zeta)\,\frac{d\tau}{\tau}$ so that $t^{-1}K(t^2,u) \le \left\| \psi_t^{(0)}(T)u \right\| + \left\| \psi_t^{(1)}(T)u \right\|$.

Then, using the K–method of interpolation, $u \in (\mathcal{R}_T, \mathcal{D}_T)_{\frac{1}{2}}$ with norm

$$\begin{aligned}
\| u \|_{\frac{1}{2},K} &= \left\{ \int_0^\infty t^{-1}K(t,u)^2\,\frac{dt}{t} \right\}^{\frac{1}{2}} = \left\{ 2\int_0^\infty t^{-2}K(t^2,u)^2\,\frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\le \left\{ 2\int_0^\infty \left\{ \left\| \psi_t^{(0)}(T)u \right\| + \left\| \psi_t^{(1)}(T)u \right\| \right\}^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\le c\, \| u \|_T \qquad (\text{since } \psi^{(0)},\, \psi^{(1)} \in \Psi(S^0_{\mu+}))\,.
\end{aligned}$$

Use of the J–method, together with the equivalence of the two methods, gives the reverse inequality, as we now show. With respect to the functional J defined by $J(t,v) = \max\left(\left\| T^{-1}v \right\|,\, t\,\| Tv \|\right)$, the interpolation space $(\mathcal{R}_T, \mathcal{D}_T)_{\frac{1}{2}}$ has norm

$$\| u \|_{\frac{1}{2},J} = \inf\left\{ \left\{ \int_0^\infty t^{-1}J(t,u(t))^2\,\frac{dt}{t} \right\}^{\frac{1}{2}} : u = \int_0^\infty u(t)\,\frac{dt}{t},\; u(t) \in \mathcal{D}_T \cap \mathcal{R}_T \right\}.$$

For a particular decomposition $u = \int_0^\infty u(t)\,\frac{dt}{t}$ with $u(t) \in \mathcal{D}_T \cap \mathcal{R}_T$, write $u(t) = \psi_{\sqrt{t}}(T)\,w(t)$ where $w(t) = (\sqrt{t}\,T)^{-1}u(t) + \sqrt{t}\,Tu(t)$ and $\psi(\zeta) = \zeta(1+\zeta^2)^{-1}$. Therefore, applying Schur's lemma,

$$\begin{aligned}
\| u \|_T^2 &= \tfrac{1}{2}\int_0^\infty \left\| \psi_{\sqrt{\tau}}(T)\int_0^\infty u(t)\,\frac{dt}{t} \right\|^2 \frac{d\tau}{\tau} \\
&= \tfrac{1}{2}\int_0^\infty \left\| \int_0^\infty \psi_{\sqrt{\tau}}(T)\psi_{\sqrt{t}}(T)w(t)\,\frac{dt}{t} \right\|^2 \frac{d\tau}{\tau} \\
&\le \tfrac{1}{2}\sup_t\left\{ \int_0^\infty \| (\psi_{\sqrt{\tau}}\psi_{\sqrt{t}})(T) \| \,\frac{d\tau}{\tau} \right\} \sup_\tau\left\{ \int_0^\infty \| (\psi_{\sqrt{\tau}}\psi_{\sqrt{t}})(T) \| \,\frac{dt}{t} \right\} \int_0^\infty \| w(t) \|^2\,\frac{dt}{t} \\
&\le c\int_0^\infty t^{-1}J(t,u(t))^2\,\frac{dt}{t}\,.
\end{aligned}$$

The integral bounds on $(\psi_{\sqrt{\tau}}\psi_{\sqrt{t}})(T)$ are obtained by contour integrals as in Lemma E of [ADM^c].

On taking the infimum over all decompositions of u, we conclude that $(\mathcal{R}_T,\, \mathcal{D}_T)_{\frac{1}{2}} \subset \mathcal{H}_T$ with

$$\| u \|_T \;\le\; c\, \| u \|_{\frac{1}{2},J}\,. \qquad \square$$

Remark. An adaptation of this proof leads to the identities

$$\mathcal{H}_{T,\alpha} \;=\; (\mathcal{H},\, \mathcal{D}_T)_\alpha \quad \text{and} \quad \mathcal{H}_{T,-\alpha} \;=\; (\mathcal{H},\, \mathcal{R}_T)_\alpha$$

when $0 < \alpha < 1$. These characterisations of $\mathcal{H}_{T,\alpha}$ and $\mathcal{H}_{T,-\alpha}$ as interpolation spaces are due to Komatsu [Kom1, Kom2, Kom3]. In the case when $\omega < \frac{\pi}{2}$, such characterisations were also obtained by Berens and Butzer [BB1] and by Grisvard [G1]. See also Sections 3.5 and 3.6.5 of [BB2]. In these papers, specific quadratic functionals are used. Our more general approach to quadratic estimates allow us to simplify the exposition and to interpolate between $\mathcal{R}_T$ and $\mathcal{D}_T$.

The significance of Theorem 4.2 is that it provides a method for determining whether T has a bounded H_∞ functional calculus. Indeed this is equivalent to $\mathcal{H}$ being interpolated midway between $\mathcal{R}_T$ and $\mathcal{D}_T$. For both statements are equivalent to $\mathcal{H}_T = \mathcal{H}$.

Remark. The authors mentioned above all consider real interpolation in Banach spaces as well as in Hilbert spaces. We do not do so here, because in Banach spaces the property of T having a bounded H_∞ functional calculus is related to complex rather than to real interpolation. Some results along these lines are contained in [CDM^{c}Y] and in [Y2].

As an initial application, we present an alternative proof of Theorem 3.2 and hence of Theorem 3.1.

Proof of Theorem 3.2. By assumption, $V \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_S)$ and $V \in \mathcal{L}(\mathcal{R}_T, \mathcal{R}_S)$, and so, by interpolation, $V \in \mathcal{L}(\mathcal{H}_T, \mathcal{H}_S)$. $\square$

§5. Fractional powers of operators.

Our aim now is to give a fuller version of Theorem 4.2 involving fractional powers of T. For each $\alpha \in \mathbb{C}$, the powers T^α are defined by $T^\alpha = f_\alpha(T)$, where $f_\alpha(\zeta) = \zeta^\alpha$. These are one–one closed operators on $\mathcal{H}$ with dense domain $\mathcal{D}(T^\alpha)$ in $\mathcal{H}$. Properties to be expected of powers follow from the identities for a functional calculus. This definition is consistent with those employed by other authors.

Proposition 5.1. *For all $\alpha \in \mathbb{C}$, T^α is an isomorphism from $\mathcal{H}_{T,\mathrm{Re}\alpha}$ to $\mathcal{H}_T$ with $\| T^\alpha u \|_T \;\approx\; \| u \|_{T,\mathrm{Re}\alpha}$. If $\mathcal{H}_T = \mathcal{H}$, then $\mathcal{D}_{T,\alpha} = \mathcal{H}_{T,\mathrm{Re}\alpha}$.*

Proof. For ψ with sufficient decay at 0 and ∞,

$$\begin{aligned}\| T^\alpha u \|_T &= \left\{ \int_0^\infty \| (tT)^{-\alpha} \psi_t(T) T^\alpha u \|^2 \tfrac{dt}{t} \right\}^{\frac12} \\ &= \left\{ \int_0^\infty \left\| t^{-\alpha} \psi_t(T) u \right\|^2 \tfrac{dt}{t} \right\}^{\frac12} = \| u \|_{T,\mathrm{Re}\alpha} .\end{aligned}$$

The final statement follows from the estimates $\| T^\alpha u \| \approx \| T^\alpha u \|_T \approx \| u \|_{T,\mathrm{Re}\alpha}$. □

The first part of the following result is well known [K1], [T].

Proposition 5.2. *If $0 < \alpha < \frac{\pi}{\omega}$, then T^α is a one–one operator of type $\alpha\omega$, and $\mathcal{H}_{T^\alpha} = \mathcal{H}_T$. More generally, if $s \in \mathbb{R}$, then $\mathcal{H}_{T^\alpha,s} = \mathcal{H}_{T,\alpha s}$.*

Proof. Suppose $\mu > \omega$ and $\lambda \notin S_{\alpha\mu+}$. Then

$$(T^\alpha - \lambda I)^{-1} = \lambda^{-1} \left\{ -|\lambda|^{\frac{1}{\alpha}} (T + |\lambda|^{\frac{1}{\alpha}})^{-1} + \psi_{(\lambda)}(T) \right\}$$

where

$$\psi_{(\lambda)}(\zeta) = \lambda(\zeta^\alpha - \lambda)^{-1} + |\lambda|^{\frac{1}{\alpha}} (\zeta + |\lambda|^{\frac{1}{\alpha}})^{-1} = (\lambda\zeta + |\lambda|^{\frac{1}{\alpha}} \zeta^\alpha)(\zeta^\alpha - \lambda)^{-1}(\zeta + |\lambda|^{\frac{1}{\alpha}})^{-1} .$$

On expressing $\psi_{(\lambda)}(T)$ as a contour integral as described in Section 2, we have

$$\| \psi_{(\lambda)}(T) \| = \frac{1}{2\pi} \left\| \int_\delta (T - \zeta I)^{-1} \psi_{(\lambda)}(\zeta)\, d\zeta \right\| \le c \int_\delta \left| \psi_{(\lambda)}(\zeta) \right| \tfrac{|d\zeta|}{|\zeta|} \le c$$

where the constant c is independent of λ when $|\arg(\lambda)| \ge \alpha\mu$. (Prove this first for $|\lambda| = 1$, and then use the scale invariance $\psi_{(s^\alpha\lambda)}(s\zeta) = \psi_{(\lambda)}(\zeta)$, $s > 0$, to handle other values of λ.) Therefore

$$\left\| (T^\alpha - \lambda I)^{-1} \right\| \le c_\mu |\lambda|^{-1} .$$

That T^α is one–one, we leave to the reader.

Given $\psi \in \Psi(S^0_{\mu+})$, then $\psi^{(\alpha)}(\zeta) = \psi(\zeta^\alpha) \in \Psi(S^0_{\alpha\mu+})$ provided $\alpha\mu < \pi$, so we can take

$$\| u \|_{T^\alpha} = \left\{ \int_0^\infty \| \psi_t(T^\alpha) u \|^2 \tfrac{dt}{t} \right\}^{\frac12} = \sqrt{\alpha} \left\{ \int_0^\infty \left\| \psi^{(\alpha)}_\tau(T) u \right\|^2 \tfrac{d\tau}{\tau} \right\}^{\frac12} = \sqrt{\alpha}\, \| u \|_T$$

where we set $t = \tau^\alpha$. Thus $\mathcal{H}_{T^\alpha} = \mathcal{H}_T$. □

In the following material we write $\mathcal{D}_{T,s}$ in place of $\mathcal{D}_{T^s}$ for notational convenience. That is, the Hilbert space $\mathcal{D}_{T,s}$ is the completion of the domain $\mathcal{D}(T^s)$ of the power T^s under the norm $\| T^s u \|$. In particular, $\mathcal{D}_{T,1} = \mathcal{D}_T$ under the norm $\| Tu \|$, $\mathcal{D}_{T,0} = \mathcal{H}$, while $\mathcal{D}_{T,-1} = \mathcal{R}_T$ under $\left\| T^{-1} u \right\|$.

We are now ready for the main result of this section. As before, each pair of spaces considered is compatible.

Theorem 5.3. *Let T be a one–one operator of type ω in $\mathcal{H}$. Let $s,t \in \mathbb{R}$, $s \neq t$, $0 < \alpha < 1$. Then*

$$\mathcal{H}_{T,s+\alpha(t-s)} = (\mathcal{D}_{T,s}\,,\,\mathcal{D}_{T,t})_\alpha$$

with equivalence of norms. In particular, $\mathcal{H}_{T,\alpha} = (\mathcal{H},\mathcal{D}_T)_\alpha$ and $\mathcal{H}_{T,-\alpha} = (\mathcal{H},\mathcal{R}_T)_\alpha$ and $\mathcal{H}_{T,2\alpha-1} = (\mathcal{R}_T,\mathcal{D}_T)_\alpha$ and $\mathcal{H}_{T,0} = (\mathcal{R}_T,\mathcal{D}_T)_{\frac{1}{2}}$. Moreover,

$$\mathcal{H}_{T,s+\alpha(t-s)} = (\mathcal{D}_{T,s}\,,\,\mathcal{H}_{T,t})_\alpha = (\mathcal{H}_{T,s}\,,\,\mathcal{D}_{T,t})_\alpha = (\mathcal{H}_{T,s}\,,\,\mathcal{H}_{T,t})_\alpha\ .$$

Proof. Proceed as before. The final statement is a consequence of the reiteration theorem. For example, on choosing $\tau > t$ and $\gamma = (t-s)(\tau-s)^{-1}$, we have

$$(\mathcal{D}_{T,s}\,,\,\mathcal{H}_{T,t})_\alpha = \Big(\mathcal{D}_{T,s}\,,\,(\mathcal{D}_{T,s}\,,\,\mathcal{D}_{T,\tau})_\gamma\Big)_\alpha = (\mathcal{D}_{T,s}\,,\,\mathcal{D}_{T,\tau})_{\alpha\gamma} = \mathcal{H}_{T,s+\alpha(t-s)}\ . \quad \square$$

On combining this interpolation result with Proposition 5.1, we obtain the following fact.

Theorem 5.4. *Let T be a one–one operator of type ω for which $\mathcal{H}_T = \mathcal{H}$. Then the spaces $\mathcal{D}_{T,s}$ interpolate by the real method.*

We shall see in a moment that these spaces also interpolate by the complex method. This result dates back to the early days of interpolation theory in the particular case when T is positive self–adjoint. In the case when T is maximal accretive, it was first proved by Lions [L]. For general operators of type ω satisfying $\mathcal{H}_T = \mathcal{H}$, see [Y1, M^{c}2]. The result follows from the boundedness of the imaginary powers T^{is}, $s \in \mathbb{R}$ (which is a particular case of T having a bounded H_∞ functional calculus) and the three lines theorem.

Another consequence of Theorem 5.3 is the following result of Yagi [Y1]. Recall that T^* denotes the adjoint of T using the given inner product on $\mathcal{H}$.

Corollary 5.5. *(i) Suppose there exists $s > 0$ such that $\|\,{T^*}^s u\,\| \leq c\,\|\,T^s u\,\|$ for all $u \in \mathcal{H}$. Then $\mathcal{H}_T \subset \mathcal{H} \subset \mathcal{H}_{T^*}$ and $\|\,u\,\|_{T^*} \leq c_1\,\|\,u\,\| \leq c_2\,\|\,u\,\|_T$ for all u. Suppose further that there exists $t > 0$ such that $\|\,T^t u\,\| \leq c\,\big\|\,{T^*}^t u\,\big\|$. Then $\mathcal{H}_T = \mathcal{H}$.*

(ii) Suppose $(\mathcal{H}\,,\,\mathcal{D}_T)_s = (\mathcal{H}\,,\,\mathcal{D}_{T^*})_s$ for some $s > 0$. Then $\mathcal{H}_T = \mathcal{H}$.

Proof. (i) By completion of the appropriate spaces, we have $\mathcal{D}_{T,s} \subset \mathcal{D}_{T^*,s}$, and, by duality, $\mathcal{D}_{T,-s} \subset \mathcal{D}_{T^*,-s}$. Therefore, by interpolation, $\mathcal{H}_T \subset \mathcal{H}_{T^*}$ and $\|\,u\,\|_{T^*} \leq c_3\,\|\,u\,\|_T$, from which it follows that $\|\,u\,\|^2 = \langle u,u\rangle \leq c_4\,\|\,u\,\|_T\,\|\,u\,\|_{T^*} \leq c_3 c_4 (\|\,u\,\|_T)^2$ and hence, by Theorem 2.1, that $\|\,u\,\|_{T^*} \leq c_1\,\|\,u\,\|$.

(ii) By Theorem 4.2, $\mathcal{H}_{T,s} = \mathcal{H}_{T^*,s}$, and, by duality, $\mathcal{H}_{T,-s} = \mathcal{H}_{T^*,-s}$. Therefore, applying Theorem 5.3, $\mathcal{H}_T = \mathcal{H}_{T^*}$. Now proceed as before to conclude that $\mathcal{H}_T = \mathcal{H}$. □

Yagi used this result to show that, if L is an m'th order elliptic operator with smooth coefficients on a smooth domain Ω, such that $T - I = L + \lambda I$ is type ω in $\mathcal{H} = L_2(\Omega)$ for some positive λ, when defined using appropriate smooth boundary conditions, then $\mathcal{H}_T = \mathcal{H}$. For then $(\mathcal{H}\ ,\ \mathcal{D}_T)_s$ and $(\mathcal{H}\ ,\ \mathcal{D}_{T^*})_s$ are both precisely the Sobolev space $H^{sm}(\Omega)$ when $0 < s < \frac{1}{2m}$, as was shown previously by Grisvard [G2].

§6. Real and complex interpolation of Hilbert spaces.

It is time to observe that the real interpolation spaces of Theorem 5.3 are equal to the corresponding complex interpolation spaces. Indeed there is always an equivalence between real and complex interpolation for pairs of Hilbert spaces whose intersection is dense in each of them. This is a special case of a result of Peetre [P].

As Peetre's aims are much more general, it may be worth presenting a direct proof of this fact. The result is well known when one space is densely embedded in the other. See e.g. pp141–143 of [Tr].

Here and elsewhere $[\mathcal{H}, \mathcal{K}]_s$ denotes the usual complex interpolation space.

Theorem 6.1. *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are two compatible Hilbert spaces with intersection $\mathcal{H} \cap \mathcal{K}$ dense in $\mathcal{H}$ and dense in $\mathcal{K}$. Then*

$$[\mathcal{H}, \mathcal{K}]_s = (\mathcal{H}, \mathcal{K})_s , \quad 0 < s < 1 .$$

Proof. Let $\mathcal{V} = \mathcal{H} \cap \mathcal{K}$. There exists a positive self–adjoint operator S with domain $\mathcal{V}$ in the Hilbert space $\mathcal{H}$ such that $||u||_{\mathcal{K}} = ||Su||_{\mathcal{H}}$ for all $u \in \mathcal{K}$, and thus $\mathcal{K} = \mathcal{D}_S$. Therefore, by the known results for self–adjoint operators,

$$[\mathcal{H}, \mathcal{K}]_s = [\mathcal{H}, \mathcal{D}_S]_s = D_{S,s} = (\mathcal{H}, \mathcal{D}_S)_s = (\mathcal{H}, \mathcal{K})_s .$$

The existence of such an operator S can be seen as follows. The positive sesquilinear form $J[u, v] = (u, v)_{\mathcal{K}}$ with domain $\mathcal{V} \times \mathcal{V}$ in the Hilbert space $\mathcal{H}$ is closed, meaning that $\mathcal{V}$ is complete under the norm $J[u, u] + (u, u)_{\mathcal{H}}$. Therefore the operator L associated with J is a positive self–adjoint operator [K]. Now let S be the positive square root of L. Then $\mathcal{D}(S) = \mathcal{V}$ and $J[u, v] = (Su, Sv)$ for all $u, v \in \mathcal{V}$ as claimed. □

As a consequence of this result, the interpolation spaces in all the preceding results can be taken as complex interpolation spaces.

§7. Special classes of operators.

Aside from maximal accretive operators, the operators of type ω which are most commonly encountered are multiplicative perturbations AS or SA of positive self–adjoint operators S by bounded invertible ω-accretive operators A. The important thing to note is that they do not necessarily have bounded H_∞ functional calculi [McY]. In other words, there exist such operators S and A for which $\mathcal{H}_{AS} \neq \mathcal{H}$.

In practice, a lot of work can be involved in proving the quadratic estimates required to obtain $\mathcal{H}_T = \mathcal{H}$ for specific operators.

Here are some results which are automatically satisfied by operators AS and SA, starting with the known fact [Mc2] that they are of type ω.

Proposition 7.1. *Let $T = AS$ and $\underline{T} = SA$, where S is a positive self–adjoint operator in $\mathcal{H}$, and A is a bounded invertible ω-accretive operator on $\mathcal{H}$. Then T and $\underline{T}$ are one–one operators of type ω. Here $\mathcal{D}(T) = \mathcal{D}(S)$ and $\mathcal{D}(\underline{T}) = A^{-1}\mathcal{D}(S)$.*

Proof. For $\zeta \notin S_{\omega+}$ and $u \in \mathcal{D}(T)$ we have

$$\begin{aligned}\| A^{-1} \| \, \| (T-\zeta I)u \| \, \| u \| \;&\geq\; |(A^{-1}(T-\zeta I)u, u)| \;=\; |(Su,u) - \zeta(A^{-1}u,u)| \\ &=\; |(A^{-1}u,u)| \left| \frac{(Su,u)}{(A^{-1}u,u)} - \zeta \right| \;\geq\; |(A^{-1}u,u)| \operatorname{dist}(\zeta, S_{\omega+})\end{aligned}$$

because $(Su,u) > 0$ and $(A^{-1}u,u) \in S_{\omega+}$. Hence, for some constant c,

$$c \, \| (T-\zeta I)u \| \;\geq\; \operatorname{dist}(\zeta, S_{\omega+}) \; \| u \|$$

so that $(T-\zeta I)$ is one–one with closed range.

The dual of $(T-\zeta I)$ with respect to the pairing $(u,v)_A = (A^{-1}u, v)$ of $\mathcal{H}$ with itself, is $(T' - \overline{\zeta} I)$ where $T' = A^*S$. Since T' has the same form as T, we also have

$$c \left\| (T' - \overline{\zeta} I)u \right\| \;\geq\; \operatorname{dist}(\zeta, S_{\omega+}) \; \| u \| \;.$$

Thus $\mathcal{R}(T-\zeta I) = \mathcal{H}$. It follows that $\zeta \in \rho(T)$ and

$$\left\| (T-\zeta I)^{-1} \right\| \;\leq\; \frac{c}{\operatorname{dist}(\zeta, S_{\omega+})} \;.$$

Therefore T is type ω.

The result for $\underline{T}$ is obtained by taking adjoints (using the given inner product), with A^* in place of A. $\square$

Remark. If the condition on S is relaxed to the statement that S is a one–one maximal accretive operator with numerical range in $S_{\nu+}$ where $\nu < \pi - \omega$, then T

and $\underline{T}$ are one–one operators of type $\omega+\nu$. This is proved in a similar way by first obtaining the estimate

$$\|(T-\zeta I)^{-1}\| \leq \frac{c}{\operatorname{dist}(\zeta, S_{(\omega+\mu)+})}$$

for all $\zeta \notin S_{(\omega+\mu)+}$. There is a similar estimate for the resolvent of $\underline{T}$.

We know that $\mathcal{H}_S = \mathcal{H}$, so that $\mathcal{H}_{S,s} = \mathcal{D}_{S,s}$ for all $s \in \mathbb{R}$. Let us determine the interpolation spaces for T in terms of these spaces when we can.

Theorem 7.2. *Let $T = AS$ where S is a positive self–adjoint operator in $\mathcal{H}$, and A is a bounded invertible ω-accretive operator on $\mathcal{H}$. If $0 < s < 1$, then $\mathcal{H}_{T,s} = \mathcal{D}_{S,s}$ with $\|u\|_{T,s} \approx \|S^s u\|$, while, if $-1 < s < 0$, then $\mathcal{H}_{T,s} = A\mathcal{D}_{S,s}$ with $\|u\|_{T,s} \approx \|S^s A^{-1} u\|$.*

Moreover, if $\underline{T} = SA$ and if $0 < s < 1$, then $\mathcal{H}_{\underline{T},s} = A^{-1}\mathcal{D}_{S,s}$ with $\|u\|_{\underline{T},s} \approx \|S^s A u\|$, while, if $-1 < s < 0$, then $\mathcal{H}_{\underline{T},s} = \mathcal{D}_{S,s}$ with $\|u\|_{\underline{T},s} \approx \|S^s u\|$.

Proof. Since $\mathcal{D}_T = \mathcal{D}_S$, interpolation gives $\mathcal{H}_{T,s} = \mathcal{H}_{S,s} = \mathcal{D}_{S,s}$ when $0 < s < 1$. Moreover, $\mathcal{R}_T = A\mathcal{R}_S$ and clearly $\mathcal{H} = A\mathcal{H}$ so, interpolating again, $\mathcal{H}_{T,s} = A\mathcal{H}_{S,s} = A\mathcal{D}_{S,s}$ when $-1 < s < 0$. The proof for $\underline{T}$ is similar. $\square$

This theorem informs us what $\mathcal{H}_{T,s}$ is, for every value of $s \in (-1,1)$ except for the most important value $s = 0$. Note that $\mathcal{D}_{S,0} = \mathcal{H} = A\mathcal{D}_{S,0}$. Nevertheless, as noted above, there exist operators S and A for which $\mathcal{H}_{T,0} = \mathcal{H}_T \neq \mathcal{H}$.

Remark. In the case when A is a bounded positive self–adjoint operator (i.e. $\omega = 0$) then $\mathcal{H}_T = \mathcal{H}$ because T is itself self–adjoint with respect to the inner product $(u,v)_A = (A^{-1}u, v)$ on $\mathcal{H}$.

We conclude this section by stating an important result which depends on the understanding of interpolation provided by Theorem 5.3. See [AM^cN] and [M^cN] for applications.

Theorem 7.3. *Let $T = AS$ where S is a positive self–adjoint operator in $\mathcal{H}$, and A is a bounded invertible ω-accretive operator on $\mathcal{H}$, and let $r > 0$. Define $T_r = AS^r$. Then T_r is a one–one operator of type ω and $\mathcal{H}_{T_r} = \mathcal{H}_T$. Similarly, if $\underline{T} = SA$ and $\underline{T}_r = S^r A$, then $\mathcal{H}_{\underline{T}_r} = \mathcal{H}_{\underline{T}}$.*

Proof. By Proposition 7.1, T_r is a one–one operator of type ω. Choose a positive number s so that $s < 1$ and $rs < 1$. Then $\mathcal{H}_{T_r,s} = \mathcal{D}_{S^r,s} = \mathcal{D}_{S,rs} = \mathcal{H}_{T,rs}$. Also $\mathcal{H}_{T_r,-s} = A\mathcal{D}_{S^r,-s} = A\mathcal{D}_{S,-rs} = \mathcal{H}_{T,-rs}$. Therefore

$$\mathcal{H}_{T_r} = [\mathcal{H}_{T_r,s}, \mathcal{H}_{T_r,-s}]_{\frac{1}{2}} = [\mathcal{H}_{T,rs}, \mathcal{H}_{T,-rs}]_{\frac{1}{2}} = \mathcal{H}_T . \qquad \square$$

§8. Operators of type S_ω .

There is a straightforward generalisation of the theory of operators of type ω to operators with spectrum in the union of two sectors. Let us review this, and record some analogues of the preceding theory in this case.

For $0 \le \omega < \mu < \frac{\pi}{2}$, define $S_{\omega-} = -S_{\omega+}$ and $S^0_{\mu-} = -S^0_{\mu+}$. Then define the closed and open double sectors $S_\omega = S_{\omega-} \cup S_{\omega+}$ and $S^0_\mu = S^0_{\mu-} \cup S^0_{\mu+}$, and the function spaces

$$\Psi(S^0_\mu) \subset H_\infty(S^0_\mu) \subset F(S^0_\mu) \subset H(S^0_\mu)$$

on them.

Let $0 \le \omega < \frac{\pi}{2}$. An operator T in $\mathcal{H}$ is said to be of *type* S_ω if $\sigma(T) \subset S_\omega$ and, for each $\mu > \omega$,

$$\left\| (T - \zeta I)^{-1} \right\| \le C_\mu |\zeta|^{-1} , \quad \zeta \notin S_\mu .$$

The results of Sections 2 and 3 generalise directly to the case when T is a one–one operator of type S_ω , with the following modifications.

(i) The functions ψ used to define the norms $\| u \|_T$ are not identically zero on either sector.

(ii) The unbounded contour δ used in the formula $\psi(T) = \frac{1}{2\pi i} \int_\delta (T - \zeta I)^{-1} \psi(\zeta)\, d\zeta$ when $\psi \in \Psi(S^0_{\mu+})$, consists of four rays from the origin in S^0_μ which enclose S_ω .

(iii) The only powers of T which we consider are integer powers.

We shall not write out the results in detail, though we draw attention to the fact that, if $\mu > \omega$, then $\| f(T)u \|_T \le c_\mu \| f \|_\infty \| u \|_T$ for all $f \in H_\infty(S^0_\mu)$. As before, T has a bounded H_∞ functional calculus if and only if $\mathcal{H}_T = \mathcal{H}$. In such a case it follows that $\| \mathrm{sgn}(T)u \| \le c \| u \|$ where sgn is the holomorphic function on S^0_μ defined by $\mathrm{sgn}(z) = 1$ when $z \in S^0_{\mu+}$ and $\mathrm{sgn}(z) = -1$ when $z \in S^0_{\mu-}$.

Also define $|T|_* = (T^2)^{\frac{1}{2}}$. (Note that $|T|_*$ is not equal to the operator $|T|$ used in the polar decomposition of operators except when T is self–adjoint.)

Proposition 8.1. *Let T be a one–one operator of type S_ω in $\mathcal{H}$. Then T^2 is type 2ω and $|T|_*$ is type ω . Moreover, $\mathcal{H}_{|T|_*} = \mathcal{H}_{T^2} = \mathcal{H}_T$.*

Proof. For $\mu > \omega$ and $\zeta \notin S_{2\mu+}$, then $\pm\sqrt{\zeta} \notin S_\mu$, so $(T^2 - \zeta I)^{-1} = (T - \sqrt{\zeta} I)^{-1}$ $(T + \sqrt{\zeta} I)^{-1} \in \mathcal{L}(H)$ and $\left\| (T^2 - \zeta I)^{-1} \right\| \le {c_\mu}^2 \, |\zeta|^{-1}$. Therefore T^2 is type 2ω , and by Proposition 5.2, $|T|_*$ is type ω .

The identity $\mathcal{H}_{|T|_*} = \mathcal{H}_{T^2}$ follows from Proposition 5.2, while $\mathcal{H}_{T^2} = \mathcal{H}_T$ is proved in a similar way. □

Let us draw attention to the fact that such proofs repeatedly use the equivalence of the norms $\| u \|_T$ defined with different choices of ψ .

An important consequence of our theory is provided by the following result. It is applied in Theorem 9.1, and also in the paper [AMcN].

Theorem 8.2. *Suppose that T is a one–one operator of type S_ω in $\mathcal{H}$ with the property that $\mathcal{H}_T = \mathcal{H}$. Then $\mathcal{D}(|T|_*) = \mathcal{D}(T)$ with $\| \, |T|_* u \, \| \approx \| Tu \|$.*

Proof. A consequence of the assumption that $\mathcal{H}_T = \mathcal{H}$ is the fact that $\operatorname{sgn}(T) \in \mathcal{L}(\mathcal{H})$. The result then follows from the identities $|T|_* = \operatorname{sgn}(T)T$ and $T = \operatorname{sgn}(T)\,|T|_*$. □

The interpolation results of Sections 4 and 5 can also be adapted to the case when T is a one–one operator of type S_ω. In this case we define the spaces $\mathcal{H}_{T,s}$ for all values of $s \in \mathbb{R}$, but only consider the powers T^s when s is an integer. Note that if $\mu > \omega$ and $s \in \mathbb{R}$, then $\| f(T)u \|_{T,s} \leq c_{\mu,s} \| f \|_\infty \| u \|_{T,s}$ for all $f \in H_\infty(S^0_\mu)$.

Theorem 5.3 remains true for those values of s and t for which it makes sense. Moreover $\mathcal{H}_{T,s+\alpha(t-s)} = \left[\mathcal{D}_{|T|_*,s} \, , \, \mathcal{D}_{|T|_*,t}\right]_\alpha$.

Here are the analogues to Proposition 7.1 and Theorem 7.2, obtained by dropping the positivity assumption on S.

Theorem 8.3. *Let $T = AS$ and $\underline{T} = SA$, where S is a one–one self–adjoint operator in $\mathcal{H}$ and A is a bounded invertible ω-accretive operator on $\mathcal{H}$. Then T and $\underline{T}$ are one–one operators of type S_ω. Here $\mathcal{D}(T) = \mathcal{D}(S)$ and $\mathcal{D}(\underline{T}) = A^{-1}\mathcal{D}(S)$.*

If $0 < s < 1$, then $\mathcal{H}_{T,s} = \mathcal{D}_{|S|,s}$ with $\| u \|_{T,s} \approx \| \, |S|^s u \, \|$, while, if $-1 < s < 0$, then $\mathcal{H}_{T,s} = A\mathcal{D}_{|S|,s}$ with $\| u \|_{T,s} \approx \left\| \, |S|^s A^{-1} u \right\|$.

Moreover, if $0 < s < 1$, then $\mathcal{H}_{\underline{T},s} = A^{-1}\mathcal{D}_{S,s}$ with $\| u \|_{\underline{T},s} \approx \| \, |S|^s Au \, \|$, while, if $-1 < s < 0$, then $\mathcal{H}_{\underline{T},s} = \mathcal{D}_{|S|,s}$ with $\| u \|_{\underline{T},s} \approx \| \, |S|^s u \, \|$.

Let us also state a variant of Proposition 8.3 for use in [AMcN].

Proposition 8.4. *Let $T = AS$ where A and S have the properties specified in Proposition 8.3, let $B \in \mathcal{L}(\mathcal{H})$, and let $\mu > \omega$. Denote $\inf \left\{ \left|(A^{-1}u, u)\right| : \| u \| = 1 \right\} = \kappa > 0$. If $\zeta \notin S_\mu$ and $|\zeta| \geq \frac{2}{\kappa} \left\| A^{-1}B \right\| \operatorname{cosec}(\mu - \omega)$, then $(T + B - \zeta I)$ has an inverse in $\mathcal{L}(\mathcal{H})$ and*

$$\left\| (T + B - \zeta I)^{-1} \right\| \; \leq \; 2\kappa^{-1} \left\| A^{-1} \right\| \operatorname{cosec}(\mu - \omega) \, |\zeta|^{-1} \; .$$

The same result holds with $\underline{T} = SA$ replacing T provided the condition on $\zeta \notin S_\mu$ is replaced by $|\zeta| \geq \frac{2}{\kappa} \left\| BA^{-1} \right\| \operatorname{cosec}(\mu - \omega)$.

Proof. For $\zeta \notin S_\mu$ and $u \in \mathcal{D}(T)$, then

$$\begin{aligned}\|A^{-1}\|\,\|(T+B-\zeta I)u\|\,\|u\| &\ge |(A^{-1}(T+B-\zeta I)u,u)|\\ &\ge |\zeta|\sin(\mu-\omega)\left|(A^{-1}u,u)\right| - \left\|A^{-1}B\right\|\|u\|^2\\ &\ge \tfrac{1}{2}\kappa\,|\zeta|\sin(\mu-\omega)\,\|u\|^2\end{aligned}$$

provided $\left\|A^{-1}B\right\| \le \frac{1}{2}\kappa\,|\zeta|\sin(\mu-\omega)$. Now proceed as in the proof of Proposition 7.1. □

We conclude with a result similar to Theorem 7.3.

Theorem 8.5. *Let S be a one–one self–adjoint operator in $\mathcal{H}$, and A a bounded invertible ω-accretive operator on $\mathcal{H}$. Then $\mathcal{H}_{AS^2} = \mathcal{H}_{AS}$ and $\mathcal{H}_{S^2A} = \mathcal{H}_{SA}$.*

Proof. Let $T = AS$ and $T_2 = AS^2$ Choose a number s such that $0 < s < \frac{1}{2}$. Then $\mathcal{H}_{T_2,s} = \mathcal{D}_{|S|^2,s} = \mathcal{D}_{|S|,2s} = \mathcal{H}_{T,2s}$. Also $\mathcal{H}_{T_2,-s} = A\mathcal{D}_{|S|^2,-s} = A\mathcal{D}_{|S|,-2s} = \mathcal{H}_{T,-2s}$. Therefore

$$H_{T_2} = [\mathcal{H}_{T_2,s}\,,\,\mathcal{H}_{T_2,-s}]_{\frac{1}{2}} = [\mathcal{H}_{T,2s}\,,\,\mathcal{H}_{T,-2s}]_{\frac{1}{2}} = \mathcal{H}_T$$

as required. □

§9. First order systems and square roots of second order differential operators in $L_2(\mathbb{R})$.

Here is the result which stimulated the current investigation.

In this section $0 \le \omega < \pi/2$, and a and b denote bounded ω-accretive functions on $\mathbb{R}$ with bounded reciprocals, meaning that $a, b, \frac{1}{a}, \frac{1}{b} \in L_\infty(\mathbb{R},\mathbb{C})$ and $|\arg a|, |\arg b| \le \omega$. The operator of multiplication by b is a bounded invertible ω-accretive operator on $\mathcal{H} = L_2(\mathbb{R})$, as is multiplication by a.

First consider the operator $-ib\frac{d}{dx}$ in $\mathcal{H} = L_2(\mathbb{R})$ with domain $\mathcal{D}(-ib\frac{d}{dx}) = H^1(\mathbb{R}) = \{\, u \in L_2(\mathbb{R}) : \frac{du}{dx} \in L_2(\mathbb{R})\,\}$ where the derivative is in the weak or distributional sense. The operator $-i\frac{d}{dx}$ with domain $H^1(\mathbb{R})$ is a one–one self–adjoint operator. Thus, by Proposition 8.3, $-ib\frac{d}{dx}$ is a one–one operator of type S_ω in $\mathcal{H}$.

For this operator it is well known that $\mathcal{H}_{-ib\frac{d}{dx}} = L_2(\mathbb{R})$. This is the same as having quadratic estimates for $D_\gamma = -i\frac{d}{dz}$ in $L_2(\gamma)$, where the Lipschitz curve γ in $\mathbb{C}$ is parametrised by $z = g(x)$ and $\frac{dg}{dx} = \frac{1}{b}$. These results are intimately connected with the L_2 boundedness of the Cauchy integral C_γ on $L_2(\gamma)$ which was first proved by Calderón [C] when $\operatorname{Re} g(x) = x$ and $\|\operatorname{Im} b\|_\infty$ is small, and by Coifman, McIntosh and Meyer [CMcM] in the general case. There are now many proofs of this fact. See

[M^cQ] for a treatment of the connections between such estimates and the holomorphic functional calculus of D_γ. There is also some discussion of this in [ADM^c].

Second, consider $S = -ia\frac{d}{dx} \oplus -ib\frac{d}{dx}$ in $\mathcal{K} = L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$. There is no essential change, and so we find that $\mathcal{K}_S = \mathcal{K}$.

Third, consider

$$T = \begin{bmatrix} 0 & -ia\frac{d}{dx} \\ -ib\frac{d}{dx} & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -i\frac{d}{dx} \\ -i\frac{d}{dx} & 0 \end{bmatrix} = B\mathbf{D} .$$

Now $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a bounded invertible ω-accretive operator on $\mathcal{K}$, while $\mathbf{D} = \begin{bmatrix} 0 & -i\frac{d}{dx} \\ -i\frac{d}{dx} & 0 \end{bmatrix}$ is a one–one self–adjoint operator. So, by Proposition 8.3, T is a one–one operator of type S_ω. Clearly $\mathcal{D}(T) = \mathcal{D}(S)$ with $\| Tu \| \approx \| Su \|$ and $\mathcal{R}(T) = \mathcal{R}(S)$ with $\| T^{-1}u \| \approx \| S^{-1}u \|$, where S is defined in the previous paragraph. Therefore, by the analogue of Theorem 3.1 for operators of type S_ω, $\mathcal{K}_T = \mathcal{K}$.

The application that we have in mind is the following result of Kenig and Meyer [KM].

Theorem 9.1. *Suppose that a and b are functions with the properties given above, and let L denote the operator in $\mathcal{H}$ defined by $Lu = -a\frac{d}{dx}(b\frac{du}{dx})$ with domain $\mathcal{D}(L) = \{u \in H^1(\mathbb{R}) : b\frac{du}{dx} \in H^1(\mathbb{R})\}$. Then L is one–one of type 2ω in $L_2(\mathbb{R})$, its square root $L^{\frac{1}{2}}$ has domain $\mathcal{D}(L^{\frac{1}{2}}) = H^1(\mathbb{R})$, and $\left\| L^{\frac{1}{2}}u \right\|_2 \approx \left\| \frac{du}{dx} \right\|_2$ for all $u \in H^1(\mathbb{R})$.*

Proof. By Proposition 8.1, $T^2 = \begin{bmatrix} -a\frac{d}{dx}b\frac{d}{dx} & 0 \\ 0 & -b\frac{d}{dx}a\frac{d}{dx} \end{bmatrix}$ is type 2ω in $\mathcal{K}$ and

$$|T|_* \;=\; (T^2)^{\frac{1}{2}} \;=\; \begin{bmatrix} (-a\frac{d}{dx}b\frac{d}{dx})^{\frac{1}{2}} & 0 \\ 0 & (-b\frac{d}{dx}a\frac{d}{dx})^{\frac{1}{2}} \end{bmatrix} .$$

Applying Theorem 8.2 and the fact that $\mathcal{K}_T = \mathcal{K}$, it follows that $\mathcal{D}(|T|_*) = \mathcal{D}(T)$ and $\| |T|_* u \| \approx \| Tu \|$ for all $u \in \mathcal{D}(T)$.

Therefore $-a\frac{d}{dx}b\frac{d}{dx}$ is type 2ω in $L_2(\mathbb{R})$, and $\left\| (-a\frac{d}{dx}b\frac{d}{dx})^{\frac{1}{2}}u \right\| \approx \left\| \frac{du}{dx} \right\|$ (and of course $\left\| (-b\frac{d}{dx}a\frac{d}{dx})^{\frac{1}{2}}u \right\| \approx \left\| \frac{du}{dx} \right\|$ as well). □

The fact that this result can be deduced from the same quadratic estimates as those already known for $-ib\frac{d}{dx}$, was first proved by Pipher using direct arguments involving integration by parts. Her work led us to state and prove Theorem 3.1, the key idea of which is the first expression for $\|u\|^2$, which is a reformulation of her more explicit formula.

We remark that the interpolation spaces for the operator T are $\mathcal{K}_{T,s} = \dot{H}^s(\mathbb{R}) \oplus \dot{H}^s(\mathbb{R})$ when $0 < s < 1$, and $\mathcal{K}_{T,s} = a\dot{H}^s(\mathbb{R}) \oplus b\dot{H}^s(\mathbb{R})$ when $-1 < s < 0$. Here $\dot{H}^s(\mathbb{R})$ denotes the homogeneous Sobolev space. This is a consequence of Theorem 8.3 and does not depend on quadratic estimates for $-ib\frac{d}{dx}$. These are only needed for the key case $s = 0$.

Let us record a related result of independent interest. It is a consequence of the fact that $\mathcal{K}_{T^2} = \mathcal{K}_T = \mathcal{K}$ and the diagonal nature of T^2.

Theorem 9.2. *The operator $L = -a\frac{d}{dx}b\frac{d}{dx}$ defined in Theorem 9.1 satisfies $\mathcal{H}_L = \mathcal{H}$. Thus L has a bounded H_∞ functional calculus in $L_2(\mathbb{R})$.*

For a more comprehensive treatment of these topics, see [AM^{c}N].

§10. Remarks about sesquilinear forms and square roots.

Let J be the sesquilinear form in $\mathcal{H}$ given by

$$J[u,v] = (ASu, Sv)\ , \quad u, v \in \mathcal{V} = \mathcal{D}(S)$$

where S is a positive self–adjoint operator in $\mathcal{H}$, and A is a bounded invertible ω-accretive operator on $\mathcal{H}$ for some $\omega < \pi/2$. Then J is a regularly accretive form in $\mathcal{H}$, and the operator associated with it, namely $L = SAS$, is a maximal accretive operator of type ω [K]. (The operator *associated with* a sesquilinear form J on $\mathcal{V} \times \mathcal{V}$ is the operator L in $\mathcal{H}$ with largest domain $\mathcal{D}(L) \subset \mathcal{V}$ which satisfies $(Lu, v) = J[u, v]$ for all $u \in \mathcal{D}(L)$ and all $v \in \mathcal{V}$.)

Actually every regularly accretive form can be represented in this way, provided $J[u,u] \neq 0$ when $u \neq 0$.

It is a consequence of the fact of L being maximal accretive, that $\mathcal{H}_L = \mathcal{H}$, and that $\mathcal{D}_{L,\frac{1}{2}} = \mathcal{H}_{L,\frac{1}{2}} = [\mathcal{H}, \mathcal{D}_L]_{\frac{1}{2}}$ with equivalence of norms.

The operator $T = AS$ is type ω, as was shown in Proposition 7.1. We indicate now how quadratic estimates for T are related to the square root problem for J.

The *square root problem* for a particular regularly accretive form J is the problem of determining whether $\mathcal{D}(L^{\frac{1}{2}}) = \mathcal{V}$ with either the equivalence $\|L^{\frac{1}{2}}u\| + \|u\| \approx \|Su\| + \|u\|$ or the stronger equivalence $\|L^{\frac{1}{2}}u\| \approx \|Su\|$. This question was originally posed by Kato for all forms [K2], though an example of a form for which it does not hold was subsequently presented in [M^{c}1].

Theorem 10.1. *Suppose that* J, S, A, L *and* T *have the properties specified above. Then* $\mathcal{D}_{L,\frac{1}{2}} = \mathcal{H}_{T,1}$ *with* $\| L^{\frac{1}{2}} u \| \approx \| u \|_{T,1} \approx \| Tu \|_T$. *Consequently the following statements are equivalent.*
(i) $\mathcal{V} \subset \mathcal{D}(L^{\frac{1}{2}})$ *with* $\| L^{\frac{1}{2}} u \| \le c_1 \| Su \|$
(ii) $\mathcal{H}_{S,1} = \mathcal{D}_S \subset \mathcal{H}_{T,1}$ *with* $\| u \|_{T,1} \le c_2 \| Su \|$
(iii) $\mathcal{H} \subset \mathcal{H}_T$ *with* $\| u \|_T \le c_3 \| u \|$
(iv) $\mathcal{H}_{T^*} \subset \mathcal{H}$ *with* $\| u \| \le c_4 \| u \|_{T^*}$
(v) $\mathcal{D}(L^{*\frac{1}{2}}) \subset \mathcal{V}$ *with* $\| Su \| \le c_5 \| L^{*\frac{1}{2}} u \|$ *where* $L^* = SA^*S$.
The relationship of the constants to each other depends only on ω *and* $\| A \| \, \| A^{-1} \|$.

It follows that $\mathcal{D}(L^{\frac{1}{2}}) = \mathcal{V}$ *with* $\| L^{\frac{1}{2}} u \| \approx \| Su \|$ *if and only if* T *has a bounded* H_∞ *functional calculus in* $\mathcal{H}$.

Proof. The operator T^2 is type 2ω and $\mathcal{D}_{T^2} = \mathcal{D}_L$, so $\mathcal{D}_{L^{\frac{1}{2}}} = [\mathcal{H}, \mathcal{D}_L]_{\frac{1}{2}} = [\mathcal{H}, \mathcal{D}_{T^2}]_{\frac{1}{2}} = \mathcal{H}_{T,1}$. The other statements follow. □

The fact that $\mathcal{V} \subset \mathcal{D}(L^{\frac{1}{2}})$ if and only if $\mathcal{D}(L^{*\frac{1}{2}}) \subset \mathcal{V}$ is due to Lions [L] and Kato [K3]. These statements are also equivalent to $\mathcal{D}(L^{*\frac{1}{2}}) \subset \mathcal{D}(L^{\frac{1}{2}})$.

For example, if S is the operator $S = -i\frac{d}{dx}$ acting in $\mathcal{H} = L_2(\mathbb{R})$ with $\mathcal{D}(S) = \mathcal{V} = H^1(\mathbb{R})$ and b is a bounded ω-accretive function on $\mathbb{R}$ with $\mathrm{Re}\, b \ge \kappa > 0$ as in Section 9, then we re-derive the known equivalence of $-ib\frac{d}{dx}$ having a bounded H_∞ functional calculus with the square root problem for the second order operator $L = -\frac{d}{dx} b \frac{d}{dx}$.

An interesting corollary of this theorem and Theorem 7.3 is the following result. It provides a way to derive estimates for the square root of one operator from those of a related operator. See [AT] for an application.

Theorem 10.2. *Suppose that* J, S, A, L *and* T *have the properties specified above and let* $r > 0$. *Define the related regularly accretive form* J_r *by* $J_r[u,v] = (AS^r u, S^r v)$, $u, v \in \mathcal{V}_r = \mathcal{D}(S^r)$. *Its associated operator is* $L_r = S^r A S^r$. *Then* $\mathcal{V} \subset \mathcal{D}(L^{\frac{1}{2}})$ *with* $||L^{\frac{1}{2}} u|| \le c||Su||$ *if and only if* $\mathcal{V}_r \subset \mathcal{D}({L_r}^{\frac{1}{2}})$ *with* $||{L_r}^{\frac{1}{2}} u|| \le c'||S^r u||$.

Proof. Both statements are equivalent to the statement that $\mathcal{H}_{AS^r} = \mathcal{H}$ with $\| u \|_{AS^r} \approx \| u \|$. □

§11. Some perturbation results.

In this section we show that if an operator satisfies quadratic estimates, then so do certain related operators. These results are all useful in applications. See [AM^{c}N].

Proposition 11.1. *Let T be a one–one operator of type ω in $\mathcal{H}$, and let $\kappa > 0$. Then $T+\kappa I$ is type ω with $\| u \|_{T+\kappa I} \leq c\{ \| u \|_T + \| u \| \}$. Moreover, if $\mathcal{H}_T = \mathcal{H}$, then $\mathcal{H}_{T+\kappa I} = \mathcal{H}$.*

Proof. It is easy to see that $T+\kappa I$ is type ω. Let us use $\psi(\zeta) = \zeta(\zeta+1)^{-2}$ in the definition of $\| u \|_T$ and $\| u \|_{T+\kappa I}$, so that

$$\begin{aligned}
\| u \|_{T+\kappa I} &= \left\{ \int_0^\infty \| \psi_t(T+\kappa I)u \|^2 \, \tfrac{dt}{t} \right\}^{\frac{1}{2}} \\
&= \left\{ \int_0^\infty \left\| t(T+\kappa I)(tT+t\kappa I+I)^{-2}u \right\|^2 \, \tfrac{dt}{t} \right\}^{\frac{1}{2}} \\
&\leq \left\{ \int_0^\infty \left\| f^2(T)tT(tT+I)^{-2}u \right\|^2 \, \tfrac{dt}{t} \right\}^{\frac{1}{2}} \\
&\qquad + \ c\left\{ \int_0^\infty \| u \|^2 \, ((t\kappa)(t\kappa+1)^{-2})^2 \, \tfrac{dt}{t} \right\}^{\frac{1}{2}} \\
&\leq \ c\{ \| u \|_T + \| u \| \}
\end{aligned}$$

where $f(\zeta) = \frac{t\zeta+1}{t\zeta+t\kappa+1}$, because

$$\| f(T) \| = \left\| (tT+I)(tT+t\kappa I+I)^{-1} \right\| = \left\| I - t\kappa(tT+t\kappa I+I)^{-1} \right\| \leq c \, .$$

If $\mathcal{H}_T = \mathcal{H}$, then $\mathcal{H} \subset \mathcal{H}_{T+\kappa I}$ with $\| u \|_{T+\kappa I} \leq c \| u \|$. Further, $\mathcal{H}_{T^*} = \mathcal{H}$, so $\mathcal{H} \subset \mathcal{H}_{T^*+\kappa I}$ with $\| u \|_{T^*+\kappa I} \leq c \| u \|$. Therefore, by Theorem 2.1, $\mathcal{H}_{T+\kappa I} = \mathcal{H}$. □

Here are other ways to obtain quadratic estimates for one operator from those for a related operator.

Proposition 11.2. *Let $T = SA$ and $T_1 = (S+I)A$, where S is a positive self–adjoint operator and A is a bounded invertible ω-accretive operator on $\mathcal{H}$. If $\mathcal{H}_T = \mathcal{H}$ with $\| u \|_T \approx \| u \|$ then $\mathcal{H}_{T_1} = \mathcal{H}$ with $\| u \|_{T_1} \approx \| u \|$.*

Proof. First note that, if $u \in \mathcal{D}(T)$, then

$$\| Tu \| + \| u \| \ = \ \left\| (I-(T+I)^{-1})(T+I)u \right\| + \| u \| \ \leq \ c \| (T+I)u \| \ .$$

So $\mathcal{D}(T+I) = \mathcal{D}(T_1)$ with

$$\| (T+I)u \| \ \approx \ \| Tu \| + \| u \| \ \approx \ \| SAu \| + \| Au \| \ \approx \ \| T_1 u \| \ .$$

Similarly, $\mathcal{D}((T+I)^*) = \mathcal{D}(T_1^*)$ with $\| (T+I)^* u \| \approx \| T_1^* u \|$.

Recall from Proposition 11.1 that $\mathcal{H}_{T+I} = \mathcal{H}$ with $\| u \|_{T+I} \approx \| u \|$. Therefore, by Theorem 3.1, $\mathcal{H}_{T_1} = \mathcal{H}$ with $\| u \|_{T_1} \approx \| u \|$. □

Lemma 11.3. *Let S be a one–one self–adjoint operator in $\mathcal{H}$, and let A be a bounded invertible ω-accretive operator on $\mathcal{H}$. Suppose that $\mathcal{H}_{SA} = \mathcal{H}$ with $\| u \|_{SA} \approx \| u \|$. Then $\mathcal{H}_{SA^*} = \mathcal{H}$ with $\| u \|_{SA^*} \approx \| u \|$.*

Proof. Apply the analogue of Proposition 2.3 for operators of type S_ω to see that $\mathcal{H}_{AS} = \mathcal{H}$, and then the analogue of Theorem 2.1 to obtain the result. □

Theorem 11.4. *Let S be a self–adjoint operator in $\mathcal{H}$ such that $\| Su \| \geq \| u \|$ for all $u \in \mathcal{D}(S)$, and let A be a bounded invertible ω-accretive operator on $\mathcal{H}$. Suppose that $\mathcal{H}_{SA} = \mathcal{H}$ with $\| u \|_{SA} \approx \| u \|$. If U is another bounded invertible ω-accretive operator on $\mathcal{H}$, such that $U = A + X + Y$ where X, Y, SX and SY^* are all bounded on $\mathcal{H}$, then $\mathcal{H}_{SU} = \mathcal{H}$ with $\| u \|_{SU} \approx \| u \|$.*

Proof. (i) First consider the special case when $Y = 0$. Then

$$\| SUu \| \ \approx\ \| SUu \| + \| u \| \ \approx\ \| SAu \| + \| u \| \ \approx\ \| SAu \|$$

and

$$\| (SU)^* v \| \ \approx\ \| Sv \| \ \approx\ \| (SA)^* v \|$$

so, by the analogue of Theorem 3.1, $\mathcal{H}_{SU} = \mathcal{H}$ with $\| u \|_{SU} \approx \| u \|$.

(ii) Next consider the special case when $X = 0$. By Lemma 11.3, $\mathcal{H}_{SA^*} = \mathcal{H}$. Now apply part (i) to $U^* = A^* + Y^*$ to obtain $\mathcal{H}_{SU^*} = \mathcal{H}$. Apply Lemma 11.3 again to conclude that $\mathcal{H}_{SU} = \mathcal{H}$.

(iii) Let us now consider the general case. Let C be the closed convex hull of the numerical ranges of A and U, and choose an integer n large enough that $C_n = \{z + \zeta : z \in C, |\zeta| \leq \frac{\| X \|}{n}\}$ is also a compact subset of the open right half plane. Let

$$\begin{aligned} U_k &= A + \tfrac{k}{n} X + \tfrac{k}{n} Y = \tfrac{n-k}{n} A + \tfrac{k}{n} U, && 0 \leq k \leq n, \text{ and} \\ V_k &= A + \tfrac{k+1}{n} X + \tfrac{k}{n} Y = U_k + \tfrac{1}{n} X, && 0 \leq k \leq n-1. \end{aligned}$$

The numerical range of each operator U_k is a subset of C, and so the numerical range of each operator V_k is a subset of C_n. Thus they are all bounded invertible μ-accretive operators for some $\mu < \frac{\pi}{2}$.

Therefore parts (i) and (ii) can be applied repeatedly with X and Y replaced by $\frac{1}{n}X$ and $\frac{1}{n}Y$ respectively. First, by (i), we find that $\mathcal{H}_{SV_0} = \mathcal{H}$, then by (ii) we get $\mathcal{H}_{SU_1} = \mathcal{H}$, and again using (i) we get $\mathcal{H}_{SV_1} = \mathcal{H}$. Proceeding in this way, we conclude that $\mathcal{H}_{SU} = \mathcal{H}_{SU_n} = \mathcal{H}$. (The reason for not taking just two steps, is that $A + X$ might not be accretive.) □

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Faculté de mathématiques et informatique, Université de Picardie–Jules Verne, 33 rue Saint-Leu, F-80039 Amiens, FRANCE; School of Mathematics, Physics, Computing and Electronics, Macquarie University, NSW 2109, AUSTRALIA; Department of Mathematics, University of Texas at Austin, Austin, TX 78712, USA

E-mail address: auscher@mathinfo.u-picardie.fr, alan@mpce.mq.edu.au, nahmod@math.utexas.edu