

Clifford algebras and Maxwell's equations in Lipschitz domains *

Alan McIntosh and Marius Mitrea

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Abstract

We present a simple, Clifford algebra based approach to several key results in the theory of Maxwell's equations in non-smooth subdomains of \mathbf{R}^m . Among other things, we give new proofs to the boundary energy estimates of Rellich type for Maxwell's equations in Lipschitz domains from [20] and [10], discuss radiation conditions and the case of variable wave number.

1 Introduction.

It has been long recognized that there are fundamental connections between electromagnetism and Clifford algebras. Indeed, understanding Maxwell's equations was part of Clifford's original motivation.

A more recent trend concerns the treatment of such PDE's in domains with irregular boundaries. See [11] for an excellent survey of the state of the art in this field up to early 1990's and [13], [18] for the role of Clifford algebras in this context.

Following work in the context of smooth domains ([24], [2], [4], [23]), the three-dimensional Maxwell system

$$(\mathcal{M}_3) \quad \begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \mathbf{R}^3 \setminus \bar{\Omega}, \\ \operatorname{curl} H + ikE = 0 & \text{in } \mathbf{R}^3 \setminus \bar{\Omega}, \\ n \times E = f \in L^2(\partial\Omega), \end{cases}$$

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(plus appropriate radiation conditions at infinity) in the complement of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$ has been solved in [20], while the higher dimensional version (involving differential forms)

$$(\mathcal{M}_m) \quad \begin{cases} dE - ikH = 0 \text{ in } \mathbb{R}^m \setminus \bar{\Omega}, \\ \delta H + ikE = 0 \text{ in } \mathbb{R} \setminus \bar{\Omega}, \\ n \wedge E = f \in L^2(\partial\Omega), \end{cases}$$

(plus suitable radiation conditions) for arbitrary Lipschitz domains $\Omega \subset \mathbb{R}^m$, has been solved in [10] (precise definitions will be given shortly). See also [19], [16] for related developments.

One of the key ingredients of the approach in [20], [10] was establishing estimates to the effect that the so-called voltage-to-current map, taking $n \times E$ into $n \times H$ and $n \wedge E$ into $n \vee H$, respectively, is an isomorphism between appropriate boundary spaces. Ultimately, this comes down to proving norm equivalences of the following form

$$(1) \quad \|n \times E\|_{L^2(\partial\Omega)} + \|\langle n, H \rangle\|_{L^2(\partial\Omega)} \approx \|n \times H\|_{L^2(\partial\Omega)} + \|\langle n, E \rangle\|_{L^2(\partial\Omega)}$$

in \mathbb{R}^3 and, more generally,

$$(2) \quad \|n \wedge E\|_{L^2(\partial\Omega)} + \|n \wedge H\|_{L^2(\partial\Omega)} \approx \|n \vee E\|_{L^2(\partial\Omega)} + \|n \vee H\|_{L^2(\partial\Omega)}$$

in \mathbb{R}^m . As shown in [20, Theorem 4.1] (for $\text{Im } k > 0$) and [10, Theorem 8.5] (for arbitrary $k \neq 0$), the above estimates lead to the invertibility of certain magnetostatic (and electrostatic) boundary integral operators which, in turn, are used to solve (\mathcal{M}_m) , $m \geq 3$.

The aim of this paper is to shed new light on the basic estimates (1), (2) and to present an alternative, natural approach to proving such boundary energy estimates, developed in the framework of Clifford algebras. This is an extension of work in [18] where such an approach has first been used for certain PDE's in Lipschitz domains. Besides its intrinsic merit, the relevance of this work should be most apparent for the numerical treatment of electro-magnetic scattering problems in non-smooth domains. For the smooth context and different techniques see [8].

The departure point for us (which in fact goes back to Maxwell himself; it has also been reinvented by M.Riesz) is to write the Maxwell system as a single equation which, in fact, expresses the Clifford analyticity of a certain Clifford algebra-valued function. Specifically, if E and H are the

electric and magnetic components of an electro-magnetic wave in \mathbb{R}^3 , then the entire Maxwell system is equivalent to the Maxwell-Dirac equation

$$(3) \quad (D + ke_4)(H + iEe_4) = 0.$$

Here E and H are regarded as Clifford algebra-valued functions in \mathbb{R}^3 , and $D + ke_4$ is a perturbed Dirac operator.

This is both mathematically convenient and physically relevant. In fact, the same idea has been extensively used by physicists who employ the Lorentz metric in the four dimensional (flat Minkowskian) space-time domains in order to write the 3D Maxwell system for $E = (E_i)_{i=1,2,3}$ and $H = (H_i)_{i=1,2,3}$ solely in terms of the electromagnetic field-strength tensor. Due to its skew symmetry, the latter defines the exterior differential form (sometimes referred to as a Faraday bi-vector field)

$$\begin{aligned} \omega \quad := \quad & (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) dt \\ & + H_1 dx_2 dx_3 + H_2 dx_3 dx_1 + H_3 dx_1 dx_2 \end{aligned}$$

(typically, electrical forces are vectorial while magnetic ones are 2-tensors). See, e.g., the monographs [5], [31], [9], and [28].

The radiation conditions for E and H can also be naturally rephrased in terms of ω . Parenthetically, let us note that it is also possible to work with the non-homogeneous form $E + H$ and write Maxwell's equations in the form $(D + ik)(E + H) = 0$ (cf. [23], [10]) but this setup appears to be less physically relevant.

However, the most attractive aspect of this algebraic context is that it preserves many of the key features of the classical complex function theory. For us, Hardy spaces of monogenic functions in Lipschitz domains and Cauchy vanishing formulas will play a basic role in the sequel. Based on these, if $\Omega \subseteq \mathbb{R}^3$ is the complement of a bounded Lipschitz domain and if $n \equiv n_1 e_1 + n_2 e_2 + n_3 e_3$ denotes the outward unit normal to $\partial\Omega$, we shall show that for any solution $\omega = H + iEe_4$ of (3) there holds

$$\|n\omega \pm \omega n\|_{L^2(\partial\Omega)} \approx \|\omega\|_{L^2(\partial\Omega)},$$

modulo residual terms. The Rellich type estimate (1) follows from this. In fact, a similar argument applies in the higher dimensional case.

Before commencing the major developments we shall introduce here some notation. Call a bounded domain Ω Lipschitz if its boundary is locally given by graphs of Lipschitz functions in appropriate rectangular coordinates. We

let $d\sigma$ denote the surface measure on $\partial\Omega$ and set n for the outward unit normal to $\partial\Omega$. For a (possibly algebra-valued) function u defined in Ω , *the nontangential maximal function* u^* is given by $u^*(X) := \sup\{|u(Y)|; Y \in \Omega, |X - Y| \leq 2 \operatorname{dist}(Y, \partial\Omega)\}$. Also, *the nontangential boundary trace* on $\partial\Omega$ of a function u defined in Ω is taken as the pointwise nontangential limit almost everywhere with respect to the surface measure on the boundary.

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2 Clifford algebra rudiments.

Recall that *the (complex) Clifford algebra* associated with \mathbb{R}^m endowed with the usual Euclidean metric is the minimal enlargement of \mathbb{R}^m to a unitary complex algebra \mathcal{A}_m , which is not generated (as an algebra) by any proper subspace of \mathbb{R}^m and such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^m$. This identity readily implies that, if $\{e_j\}_{j=1}^m$ is the standard orthonormal basis in \mathbb{R}^m , then

$$e_j^2 = -1 \text{ and } e_j e_k = -e_k e_j \text{ for any } j \neq k.$$

In particular, we identify the canonical basis $\{e_j\}$ from \mathbb{R}^m with the algebraic basis of \mathcal{A}_m . Thus, any element $u \in \mathcal{A}_m$ can be uniquely represented in the form

$$(4) \quad u = \sum_{l=0}^m \sum'_{|I|=l} u_I e_I, \quad u_I \in \mathbb{C},$$

where e_I stands for the product $e_{i_1} e_{i_2} \dots e_{i_l}$ if $I = (i_1, i_2, \dots, i_l)$ (we make the convention that $e_\emptyset := 1$). For this multi-index I we call l the *length* of I and denote it by $|I|$. We shall adopt the convention that \sum' indicates that the sum is performed only over strictly increasing multi-indices I .

The *Clifford conjugation* on \mathcal{A}_m is defined as the unique complex-linear involution on \mathcal{A}_m for which $\bar{e}_I e_I = e_I \bar{e}_I = 1$ for any multi-index I . In particular, if $u = \sum_I u_I e_I \in \mathcal{A}_m$, then $\bar{u} = \sum_I u_I \bar{e}_I$. Note that

$$(5) \quad \bar{\bar{u}} = (-1)^{\frac{l(l+1)}{2}} u,$$

for any $u \in \Lambda^l$. For the *complex conjugation* on \mathcal{A}_m we set

$$(6) \quad u^c = \left(\sum_I u_I e_I \right)^c := \sum_I \bar{u}_I e_I.$$

We define *the scalar part* of $u = \sum_I u_I e_I \in \mathcal{A}_m$ as $u_0 := u_\emptyset$, and endow \mathcal{A}_m with the natural Euclidean metric $\langle u, \bar{u}^c \rangle = |u|^2 := (u \bar{u}^c)_0 = (\bar{u}^c u)_0$. Note that $(uv)_0 = (vu)_0 = \langle u, \bar{v} \rangle$, for any $u, v \in \mathcal{A}_m$. Another useful observation is that

$$(7) \quad |a u| = |u a| = |u| |a|$$

for any $u \in \mathcal{A}_m$ and $a \in \mathbb{R}^m$. We also define

$$(8) \quad u_\pm := \frac{1}{2} \{u \pm \bar{u}\}$$

for each $u \in \mathcal{A}_m$.

With u as in (4), define $\Pi_l u := \sum'_{|I|=l} u_I e_I$ and denote by Λ^l the range of $\Pi_l : \mathcal{A}_m \rightarrow \mathcal{A}_m$. Elements in Λ^l will be referred to as l -vectors or differential forms of degree l . The exterior and interior product of forms are defined for $\alpha \in \Lambda^1$, $u \in \Lambda^l$, respectively, by

$$(9) \quad \alpha \wedge u := \Pi_{l+1}(\alpha u) \quad \text{and} \quad \alpha \vee u := -\Pi_{l-1}(\alpha u).$$

By linearity, these operations extend to arbitrary $u \in \mathcal{A}_m$.

Two identities which are going to be of importance in the sequel are

$$(10) \quad \alpha u = \alpha \wedge u - \alpha \vee u,$$

and

$$(11) \quad u \alpha = (-1)^l (\alpha \wedge u + \alpha \vee u),$$

valid for arbitrary $\alpha \in \Lambda^1$ and $u \in \Lambda^l$.

We shall work with the Dirac operator $D := \sum_{j=1}^m e_j \frac{\partial}{\partial x_j}$. Then $\bar{D} = \sum_{j=1}^m \bar{e}_j \frac{\partial}{\partial x_j} = -D$, and $D^2 = -\Delta$, the negative of the Laplacian in \mathbb{R}^m . For reasons which will become more apparent shortly, we shall find it useful to embed everything into a larger Clifford algebra, say $\mathbb{R}^m \subseteq \mathcal{A}_m \subseteq \mathcal{A}_{m+1}$. If for $k \in \mathbb{C}$ we now set

$$(12) \quad \mathbb{D}_k := D + k e_{m+1}$$

then $-\mathbb{D}_k^2 = \Delta + k^2$. Furthermore,

$$(13) \quad \mathbb{D}_k^c = D + \bar{k} e_{m+1} = \mathbb{D}_{\bar{k}} \quad \text{and} \quad \bar{\mathbb{D}}_k = -D - k e_{m+1} = -\mathbb{D}_k.$$

If u is a \mathcal{A}_{m+1} -valued function defined in a region Ω of \mathbb{R}^m , set

$$\mathbb{D}_k u := \sum_{j=1}^m e_j \frac{\partial u}{\partial x_j} + k e_{m+1} u$$

and

$$u \mathbb{D}_k := \sum_{j=1}^m \frac{\partial u}{\partial x_j} e_j + k u e_{m+1}.$$

Call u *left (right, two-sided) k -monogenic* if $\mathbb{D}_k u$ ($u \mathbb{D}_k$, or both $\mathbb{D}_k u$ and $u \mathbb{D}_k$) = 0 in Ω . Note that each component of a k -monogenic function is annihilated by the Helmholtz operator $\Delta + k^2$. A function theory for the perturbed Dirac operator \mathbb{D}_k which is relevant for us here has been worked out in [21] and [22].

The exterior derivative operator d acts on a Λ^l -valued function u by $du := \Pi_{l+1}(Du)$. It extends by linearity to arbitrary \mathcal{A}_m -valued functions. Its formal transpose, δ , is given by $\delta u := \Pi_{l-1}(Du)$ if u is Λ^l -valued. Once again, δ extends by linearity to arbitrary \mathcal{A}_m -valued functions. Since $D = d + \delta$, it follows that $d^2 = 0$, $\delta^2 = 0$ and $-(d\delta + \delta d) = \Delta$, as is well known. More detailed accounts on these matters can be found in [1], [7], [12], [18]. Further references can be found in [14].

3 A distinguished fundamental solution.

The aim is to construct a convenient, explicit fundamental solution for the Dirac operator \mathbb{D}_k . The departure point is deriving an explicit expression for a fundamental solution of the Helmholtz operator $\Delta + k^2$ in \mathbb{R}^m . If $k = i$, this can be taken to be minus the kernel of the Bessel potential $\mathcal{J}_\alpha = (-\Delta + I)^{-\alpha/2}$ corresponding to $\alpha = 2$. This kernel is known (cf. [27], p. 131) to have the form

$$(14) \quad \Phi(X) = -\frac{1}{(4\pi)^{m/2}} \int_0^\infty \exp\left(-t - \frac{|X|^2}{4t}\right) \frac{dt}{t^{m/2}}.$$

For general $k \in i\mathbb{R}_+$ we simply re-scale the expression of Φ to obtain that

$$(15) \quad \Phi_k(X) := -\frac{1}{(4\pi)^{m/2}} \int_0^\infty \exp\left(k^2 t - \frac{|X|^2}{4t}\right) \frac{dt}{t^{m/2}}$$

is a fundamental solution for the operator $\Delta + k^2$ in \mathbb{R}^m . Clearly this also continues to hold true for all $k \in \mathbb{C}$ with $\text{Im } k > |\text{Re } k|$ and, in fact, the

above expression can be continued analytically to the open upper-half plane $\text{Im } k > 0$ with the property that $\Phi_k(X) = \mathcal{O}(\exp\{-|X| \text{Im } k\})$ as $|X| \rightarrow \infty$. For instance, when $\text{Im } k > 0$ and $m \geq 2$, one may take

$$\Phi_k(X) = -\frac{1}{\omega_m} \frac{(-ik)^{m-2}}{(m-2)!} \int_1^\infty e^{ik|X|t} (t^2 - 1)^{\frac{m-3}{2}} dt, \quad X \in \mathbb{R}^m \setminus 0,$$

where ω_m is the area of the unit sphere in \mathbb{R}^m . It is worth pointing out that the above definition can be further extended to $\{k \in \mathbb{C} \setminus 0; \text{Im } k \geq 0\}$ by setting, for $X \in \mathbb{R}^m \setminus 0$,

$$(16) \quad \Phi_k(X) = \begin{cases} \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r}\right)^{\frac{m-1}{2}} \left(\frac{1}{2ik} e^{ikr}\right) \Big|_{r=|X|}, & \text{if } m \text{ is odd,} \\ \left(-\frac{1}{2\pi r} \frac{\partial}{\partial r}\right)^{\frac{m-2}{2}} \left(-\frac{i}{4} H_0^{(1)}(kr)\right) \Big|_{r=|X|}, & \text{if } m \text{ is even,} \end{cases}$$

where $H_0^{(1)}$ is the usual zero-order Hankel function of the first kind. We shall continue to denote this extension by Φ_k . For this and a more detailed discussion see the excellent treatment in [6], pp.59-74.

It is now clear that

$$(17) \quad E_k := -\mathcal{D}_k \Phi_k = -\Phi_k \mathcal{D}_k = -\sum_{j=1}^m \frac{\partial \Phi_k}{\partial x_j} e_j - k e_{m+1} \Phi_k$$

is a (two-sided) fundamental solution for \mathcal{D}_k . When $k = 0$ then E_k reduces to the usual radial fundamental solution for the Laplace operator. If $\text{Im } k > 0$ then E_k decays exponentially at infinity. Furthermore, if $F_k(X)$ is so that $E_k(X) = F_k(X) e^{ik|X|}$ then

$$(18) \quad F_k(X) = \mathcal{O}\left(|X|^{-\frac{m-1}{2}}\right), \text{ as } |X| \rightarrow \infty.$$

This follows from the definition of Φ_k plus a straightforward computation. In the case when m is even, the classical asymptotic expansion (in the sense of Poincaré; cf., e.g., [26, (9.13.1), p.166])

$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{iz - i\pi/4} \left\{1 - i \frac{1}{1!2^3 z} + i^2 \frac{1 \cdot 9}{2!2^6 z^2} - i^3 \frac{1 \cdot 9 \cdot 25}{3!2^9 z^3} + \dots\right\}$$

as $|z| \rightarrow \infty$, is also used.

A more delicate analysis is required when $k \in \mathbb{R} \setminus 0$ because in this case there are two canonical decaying fundamental solutions. Specifically, for $k \in \mathbb{R} \setminus 0$, we define

$$(19) \quad E_k^\pm := -\mathbb{D}_k \Phi_{\pm k} = -\Phi_{\pm k} \mathbb{D}_k.$$

In passing let us note that $E_k^+ = E_k$. Now, if the functions F_k^\pm are so that $E_k^\pm(X) = F_k^\pm(X)e^{\pm ik|X|}$ then, along similar lines as before,

$$(20) \quad F_k^\pm(X) = c_{m,k} \frac{1}{|X|^{\frac{m-1}{2}}} \left(\pm \frac{X}{|X|} - ie_{m+1} \right) + o\left(|X|^{-\frac{m-1}{2}}\right), \text{ as } |X| \rightarrow \infty,$$

where $c_{m,k} \in \mathbb{C} \setminus 0$, and

$$(21) \quad \nabla F_k^\pm(X) = o\left(|X|^{-\frac{m-1}{2}}\right), \text{ as } |X| \rightarrow \infty.$$

In the physically relevant case when $m = 3$ and $\text{Im } k \geq 0$,

$$(22) \quad \Phi_k(X) = -\frac{1}{4\pi|X|} e^{ik|X|}$$

so that, for $k \in \mathbb{R} \setminus 0$,

$$(23) \quad E_k^\pm(X) = -\frac{1}{4\pi} \left(\frac{X}{|X|^3} \mp \frac{ikX}{|X|^2} - \frac{ke_4}{|X|} \right) e^{\pm ik|X|}.$$

In particular, (20)-(21) are trivially checked.

4 Function theory for perturbed Dirac operators.

Let Ω be a bounded Lipschitz domain in \mathbb{R}^m and let n stand for the outward unit normal to Ω . As in [18], [22], [7], we introduce the Hardy type space $\mathcal{H}_{left}^2(\Omega, k)$ of left k -monogenic functions in Ω by

$$\mathcal{H}_{left}^2(\Omega, k) := \{u : \Omega \rightarrow \mathcal{A}_{m+1}; u \text{ is left } k\text{-monogenic in } \Omega, u^* \in L^2(\partial\Omega)\}.$$

Similarly, we define $\mathcal{H}_{right}^2(\Omega, k)$, the Hardy space of right k -monogenic functions in Ω by

$$\mathcal{H}_{right}^2(\Omega, k) := \{v : \Omega \rightarrow \mathcal{A}_{m+1}; v \text{ is right } k\text{-monogenic in } \Omega, v^* \in L^2(\partial\Omega)\}.$$

We endow these spaces with the natural norms $\|u\|_{\mathcal{H}_{left}^2(\Omega, k)} := \|u^*\|_{L^2(\partial\Omega)}$, and $\|v\|_{\mathcal{H}_{right}^2(\Omega, k)} := \|v^*\|_{L^2(\partial\Omega)}$, respectively.

Equivalent norms are obtained by taking the L^2 norms of the boundary trace; see [12] and the theory of monogenic Hardy spaces developed in [18]. Also, note that $\mathcal{H}_{left}^2(\Omega, k) \cap \mathcal{H}_{right}^2(\Omega, k)$ consists precisely of *two-sided* k -monogenic functions in Ω which have a square-integrable non-tangential maximal function.

Proposition 4.1 *Let Ω be a bounded Lipschitz domain in R^m , $k \in C$, and assume that $u \in \mathcal{H}_{right}^2(\Omega, k)$, $v \in \mathcal{H}_{left}^2(\Omega, -k)$. Then u and v have (non-tangential) boundary limits at almost every point on $\partial\Omega$ and*

$$(24) \quad \int_{\partial\Omega} u n v d\sigma = 0.$$

Also, if $u \in \mathcal{H}_{right}^2(\Omega, k) \cap \mathcal{H}_{left}^2(\Omega, k)$ and $\theta \in C^1(R^m, \mathcal{A}_{m+1})$, then

$$(25) \quad \int_{\partial\Omega} u n u^c \theta d\sigma = \iint_{\Omega} u([D, \theta]u^c - 2(\operatorname{Re} k) e_{m+1} u^c \theta)$$

where $[D, \theta]$ is the commutator between D , acting from the left, and multiplication by θ to the right.

Formula (24) can be considered as the Clifford analogue of the classical Cauchy's vanishing theorem in complex analysis. The second identity, (25), is a perturbation of this result. We state it here since we shall need it shortly. In both cases, the integrands must be interpreted in the sense of multiplication in the Clifford algebra.

Proof.

The existence of the boundary trace for u and v is proved in the usual way. As for the vanishing formula, suppose first that u and v are smooth up to and including the boundary of Ω . Consequently, Gauss's formula gives

$$(26) \quad \int_{\partial\Omega} u n v d\sigma = \iint_{\Omega} (uD)v + u(Dv) = \iint_{\Omega} (u\mathbb{D}_k)v + u(\mathbb{D}_{-k}v).$$

In this case, (24) follows based on the monogenicity assumptions. The general case is then seen from this and a standard limiting argument (see, e.g., [30] for details in similar circumstances).

The proof of (25) is once again based on (26). This time, however,

$$(27) \quad \begin{aligned} \mathbb{D}_{-k}(u^c\theta) &= [\mathbb{D}_{-k}, \theta]u^c + (\mathbb{D}_{-k}u^c)\theta \\ &= [D, \theta]u^c - 2(\operatorname{Re} k) e_{m+1} u^c \theta. \end{aligned}$$

The proof is complete. \square

We shall also need the following Clifford algebra version of Cauchy's reproducing formula.

Proposition 4.2 *Let Ω be a bounded Lipschitz domain in R^m and $k \in C$. Then every function $u \in \mathcal{H}_{left}^2(\Omega, k)$ has (nontangential) boundary limits at almost every point on $\partial\Omega$ and*

$$(28) \quad u(X) = \int_{\partial\Omega} E_k(X - Y) n(Y) u(Y) d\sigma(Y), \quad X \in \Omega.$$

A similar statement is valid for functions in $\mathcal{H}_{right}^2(\Omega, k)$.

Proof.

In a smoother context, the proof goes along well known lines; cf., e.g., [1] for the case when $k = 0$. Note, however, that we utilize here the fact that $E_k(X - Y)\mathbb{D}_{-k,Y} = -E_k(X - Y)\mathbb{D}_{k,X} = 0$ for $X \neq Y$. Then, as before, passing to domains with Lipschitz boundaries is done via a routine limiting argument. \square

In the second part of this section we shall discuss the *exterior* domain version of Proposition 4.2. The novelty is that u should satisfy a (necessary) decay condition at infinity. In the case when $\text{Im } k > 0$, based on (18), it is not too difficult to see that

$$(29) \quad u(X) e^{-\text{Im } k |X|} = o(|X|^{-\frac{m-1}{2}}), \quad \text{as } |X| \rightarrow \infty,$$

is a (necessary and) sufficient condition to prove the exterior domain version of (28). In turn, the latter implies that actually u decays *exponentially* at infinity. Parenthetically, let us also note that, e.g., $u(X) = \mathcal{O}(1)$ at infinity always entails (29).

The behavior at infinity requires a more elaborate analysis when $k \in \mathbb{R} \setminus 0$ in which situation (29) is no longer the appropriate decay condition. In fact, the whole case when $k \in C \setminus 0$ has $\text{Im } k \geq 0$ can be covered by insisting that u satisfies a decay condition of radiation type. Specifically, we shall ask that

$$(30) \quad \left(1 - ie_{m+1}\hat{X}\right) u(X) = o(|X|^{-\frac{m-1}{2}}), \quad \text{as } |X| \rightarrow \infty,$$

where we set $\hat{X} := \frac{X}{|X|}$, $X \in \mathbb{R}^m \setminus 0$.

When $k \in \mathbb{R} \setminus 0$, the opposite choice of sign in (30) will also do. Thus, in this case, (30) may be replaced by

$$(31) \quad \left(1 \mp ie_{m+1} \hat{X}\right) u(X) = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty.$$

The source of (31) is the expression (20). As expected, for real non-zero k , the functions E_k^\pm satisfy (31) because

$$(32) \quad \left(1 \mp ie_{m+1} \hat{X}\right) \left(\pm \hat{X} - ie_{m+1}\right) = 0 = \left(1 \pm ie_{m+1} \hat{X}\right) \left(1 \mp ie_{m+1} \hat{X}\right).$$

Since $1 \mp ie_{m+1} \hat{X}$ are, as (32) shows, *zero-divisors*, one cannot deduce a decay condition of the same order for u itself based solely on (31) and algebraic manipulations. In fact, owing to the classical Rellich's lemma (cf. Appendix), any solution u of the Helmholtz equation $(\Delta + k^2)u = 0$ for non-zero real k which satisfies $u(X) = o(|X|^{-\frac{m-1}{2}})$ in a (connected) neighborhood of infinity in \mathbb{R}^m must vanish identically. Hence, generally speaking, for (31) to hold when $k \in \mathbb{R} \setminus 0$, certain internal *cancellations* must occur when multiplying u with $(1 \mp ie_{m+1} \hat{X})$. Nonetheless, we do have the following.

Proposition 4.3 *Let Ω be a bounded Lipschitz domain in \mathbb{R}^m and let $k \in \mathbb{C} \setminus 0$ have $\text{Im } k \geq 0$. Assume that $u \in C^1(\mathbb{R}^m \setminus \bar{\Omega}, \mathcal{A}_{m+1})$ has $u^* \in L^2(\partial\Omega)$ and satisfies (30) or, if k is real, either condition in (31). Also, suppose that $\mathcal{D}_k u = 0$ in $\mathbb{R}^m \setminus \bar{\Omega}$. Then we have the integral representation formula*

$$(33) \quad u(X) = \int_{\partial\Omega} E_k^\pm(X - Y) n(Y) u(Y) d\sigma(Y), \quad X \in \mathbb{R}^m \setminus \bar{\Omega},$$

where the choice of the sign depends on the form of the radiation condition.

In particular, the function u decays exponentially if $\text{Im } k > 0$ while, for non-zero real k ,

$$(34) \quad u(X) = \mathcal{O}\left(\frac{1}{|X|^{\frac{m-1}{2}}}\right), \text{ as } |X| \rightarrow \infty.$$

Similar results are valid for functions annihilated by the right-handed action of \mathcal{D}_k (with the radiation conditions appropriately adjusted; see below).

Proof.

First we shall prove a seemingly weaker decay condition for u , namely

$$(35) \quad \int_{|X|=R} |u|^2 d\sigma = \mathcal{O}(1), \text{ as } R \rightarrow \infty.$$

To this end, we note that

$$\begin{aligned}
|(1 \mp i e_{m+1} \hat{X}) u|^2 &= \operatorname{Re} \left[(1 \mp i e_{m+1} \hat{X}) u \bar{u}^c (1 \mp i e_{m+1} \hat{X}) \right]_0 \\
&= \operatorname{Re} \left[(1 \mp i e_{m+1} \hat{X})^2 u \bar{u}^c \right]_0 \\
&= \operatorname{Re} \left[(2 \mp 2i e_{m+1} \hat{X}) u \bar{u}^c \right]_0 \\
&= 2|u|^2 \mp 2 \operatorname{Im} \left(\bar{u}^c \hat{X} e_{m+1} u \right)_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{|X|=R} |(1 \mp i e_{m+1} \hat{X}) u|^2 d\sigma &= 2 \int_{|X|=R} |u|^2 d\sigma \\
(36) \quad &\mp 2 \operatorname{Im} \left(\int_{|X|=R} \bar{u}^c \hat{X} e_{m+1} u d\sigma \right)_0.
\end{aligned}$$

We use Gauss's formula (26) to further transform the last integral above. To this effect, for $0 < R_0 < R < \infty$ with R_0 sufficiently large and fixed, we write

$$\begin{aligned}
\int_{|X|=R} \bar{u}^c \hat{X} e_{m+1} u d\sigma &= \iint_{R_0 < |X| < R} [(\bar{u}^c \mathbb{D}_k)(e_{m+1} u) + \bar{u}^c \mathbb{D}_{-k}(e_{m+1} u)] \\
(37) \quad &+ \int_{|X|=R_0} \bar{u}^c \hat{X} e_{m+1} u d\sigma.
\end{aligned}$$

Next, observe that $\mathbb{D}_{-k}(e_{m+1} u) = -e_{m+1} \mathbb{D}_k u = 0$ and that $\mathbb{D}_k = -\bar{\mathbb{D}}_k^c + 2i(\operatorname{Im} k)e_{m+1}$. The latter identity implies $(\bar{u}^c \mathbb{D}_k)(e_{m+1} u) = -2i(\operatorname{Im} k)\bar{u}^c u$ so that the solid integral in (37) becomes $-2i(\operatorname{Im} k) \iint_{R_0 < |X| < R} \bar{u}^c u$. Utilizing this back in (36) and invoking the radiation condition gives

$$\begin{aligned}
\limsup_{|R| \rightarrow \infty} \int_{|X|=R} |u|^2 d\sigma &= \pm \operatorname{Im} \left(\int_{|X|=R_0} \bar{u}^c \hat{X} e_{m+1} u d\sigma \right)_0 \\
(38) \quad &\mp 2(\operatorname{Im} k) \iint_{|X| > R_0} |u|^2.
\end{aligned}$$

The last term above is zero when $k \in \mathbb{R}$ and negative when $\operatorname{Im} k > 0$. Thus, in any event, (35) is proved.

With (35) at hand, it is an easy matter to tackle (33). Specifically, working on the bounded Lipschitz domain $\{X \in \mathbb{R}^m \setminus \bar{\Omega}; |X| < R\}$ and invoking Proposition 4.2, matters are readily reduced to showing that

$$\begin{aligned} \int_{|Y|=R} |E_k^\pm(X-Y) \hat{Y} u(Y)| d\sigma &= \int_{|Y|=R} |F_k^\pm(X-Y) e^{\pm ik|X-Y|} \hat{Y} u(Y)| d\sigma \\ (39) \qquad \qquad \qquad &= o(1), \text{ as } R \rightarrow \infty, \end{aligned}$$

uniformly for X in compact subsets of \mathbb{R}^m . To this end, when $\text{Im } k > 0$, this follows trivially (for the “plus” choice of the sign) from (29), (35) and Schwarz’s inequality. On the other hand, when $k \in \mathbb{R} \setminus 0$, (39) becomes, thanks to (35) and (20)-(21), a simple exercise which we omit.

Finally, the decay of u at infinity is easily established from (33) and the decay of E_k^\pm . \square

Let us remark that, as inspection of the above proof shows, (31) can be weakened to

$$\int_{|X|=R} |(1 \mp ie_{m+1} \hat{X})u|^2 d\sigma = o(1), \quad \text{as } R \rightarrow \infty.$$

In passing, let us also point out that any Clifford-Cauchy type integral of the form $\int_{\partial\Omega} E_k^\pm(X-Y) f(Y) d\sigma(Y)$, corresponding to an absolutely integrable \mathcal{A}_{m+1} -valued function f on $\partial\Omega$, radiates at infinity. This follows from (20)-(21) and elementary estimates.

In the light of the above results, it seems natural to consider the *exterior* Hardy space

$$\begin{aligned} \mathcal{H}_{\text{ext}}^2(\mathbb{R}^m \setminus \bar{\Omega}, k) &:= \{u : \mathbb{R}^m \setminus \bar{\Omega} \rightarrow \mathcal{A}_{m+1}; \mathbb{D}_k u = 0 \text{ in } \mathbb{R}^m \setminus \bar{\Omega}, \\ &\quad u^* \in L^2(\partial\Omega) \text{ and } u \text{ satisfies (30)}\} \end{aligned}$$

for any $k \in \mathbb{C} \setminus 0$ with $\text{Im } k \geq 0$. As alluded to before, when $\text{Im } k > 0$, replacing (30) by (29) or, say, $u = \mathcal{O}(1)$ at infinity, yields the same space. However, if $k \in \mathbb{R} \setminus 0$, then there are *two* disjoint types of exterior Hardy spaces. Concretely, we set

$$\begin{aligned} \mathcal{H}_{\text{ext}}^{2,\pm}(\mathbb{R}^m \setminus \bar{\Omega}, k) &:= \{u : \mathbb{R}^m \setminus \bar{\Omega} \rightarrow \mathcal{A}_{m+1}; \mathbb{D}_k u = 0 \text{ in } \mathbb{R}^m \setminus \bar{\Omega}, \\ &\quad u^* \in L^2(\partial\Omega) \text{ and } u \text{ satisfies (31)}\}. \end{aligned}$$

In this setting, Proposition 4.3 becomes the natural exterior domain analogue of Proposition 4.2. Of course, similar considerations apply to the space $\mathcal{H}_{right}^{2,\pm}(R^m \setminus \bar{\Omega}, k)$, $k \in R \setminus 0$, provided (31) is replaced by

$$(40) \quad u(X) \left(1 \pm ie_{m+1} \hat{X}\right) = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty.$$

We conclude this section by discussing the exterior domain version of Cauchy's vanishing formula (24). Concretely, we have the following.

Proposition 4.4 *Let Ω be a bounded Lipschitz domain in R^m and $k \in R \setminus 0$. Then, for any $u \in \mathcal{H}_{right}^{2,+}(R^m \setminus \bar{\Omega}, -k)$, $v \in \mathcal{H}_{left}^{2,-}(R^m \setminus \bar{\Omega}, k)$ there holds*

$$(41) \quad \int_{\partial\Omega} u n v d\sigma = 0.$$

A similar conclusion is valid for any function $u \in \mathcal{H}_{right}^{2,-}(R^m \setminus \bar{\Omega}, -k)$, $v \in \mathcal{H}_{left}^{2,+}(R^m \setminus \bar{\Omega}, k)$.

Proof.

Making use of the Proposition 4.1 in the context of the bounded Lipschitz domain $\{X \in R^m; |X| \leq R, X \notin \bar{\Omega}\}$ gives

$$(42) \quad \begin{aligned} & \int_{\partial\Omega} u(X) n(X) v(X) d\sigma = \int_{|X|=R} u(X) \hat{X} v(X) d\sigma \\ & = \frac{1}{2} \int_{|X|=R} \left[u(X) \left(\hat{X} \pm ie_{m+1} \right) v(X) + u(X) \left(\hat{X} \mp ie_{m+1} \right) v(X) \right] d\sigma. \end{aligned}$$

Next, from the radiation conditions (31), (40) and the decay conditions $u(X) = \mathcal{O}(|X|^{-(m-1)/2})$, $v(X) = \mathcal{O}(|X|^{-(m-1)/2})$ at infinity we obtain

$$u(X) \left[\left(\hat{X} \mp ie_{m+1} \right) v(X) \right] = o(|X|^{-(m-1)}), \text{ as } |X| \rightarrow \infty$$

and

$$\left[u(X) \left(\hat{X} \pm ie_{m+1} \right) \right] v(X) = o(|X|^{-(m-1)}), \text{ as } |X| \rightarrow \infty.$$

These, in turn, prove that the last integral in (42) is $o(1)$ as $R \rightarrow \infty$. The desired conclusion now follows easily. \square

Remark. The definition of the spaces $\mathcal{H}_{left}^{2,\pm}(R^m \setminus \bar{\Omega}, k)$ can be naturally extended for $\pm \text{Im } k \geq 0$, whereas that of $\mathcal{H}_{right}^{2,\pm}(R^m \setminus \bar{\Omega}, k)$ extends for $\pm \text{Im } k \leq 0$. In particular, Proposition 4.4 continues to hold true for all $\text{Im } k \geq 0$.

5 Rellich type identities for Clifford k -monogenic functions.

In this section we present several Rellich-type identities which, in turn, will be used later to prove boundary L^2 -energy estimates. The starting point is the following theorem.

Theorem 5.1 *Let Ω be a bounded Lipschitz domain in R^m , $k \in C$ and $u \in \mathcal{H}_{left}^2(\Omega, k) \cap \mathcal{H}_{right}^2(\Omega, k)$. Then, for any C^1 -vector field θ with real components, also identified with a \mathcal{A}_m -valued function, we have the identities:*

$$(43) \quad \begin{aligned} \frac{1}{2} \int_{\partial\Omega} |u|^2 \langle n, \theta \rangle d\sigma &= \pm \operatorname{Re} \left(\int_{\partial\Omega} \theta u (nu)_{\pm}^c d\sigma \right)_0 \\ &\mp \operatorname{Re} \left(\iint_{\Omega} \frac{1}{2} u ([D, \theta]u^c) - (\operatorname{Re} k) u e_{m+1} u^c \theta \right)_0. \end{aligned}$$

Furthermore, the surface integral in the right side of (43) can be replaced by

$$\operatorname{Re} \left(\int_{\partial\Omega} (un)_{\pm}^c u \theta d\sigma \right)_0 = \operatorname{Re} \left(\int_{\partial\Omega} (\theta u)_{\pm}^c \bar{n} \bar{u} d\sigma \right)_0 = \operatorname{Re} \left(\int_{\partial\Omega} \bar{u} \bar{n} (u \theta)_{\pm}^c d\sigma \right)_0.$$

Proof.

We write $\langle n, \theta \rangle = \frac{1}{2}(\bar{n} \theta + \theta \bar{n})$ and $|u|^2 = (u \bar{u})_0$ so that

$$\begin{aligned} |u|^2 \langle n, \theta \rangle &= \frac{1}{2} (u \bar{u}^c \bar{n} \theta)_0 + \frac{1}{2} (u \bar{u}^c \theta \bar{n})_0 \\ &= \frac{1}{2} (u \bar{u}^c \bar{n} \theta)_0 + \frac{1}{2} (\bar{n} \theta u^c \bar{u})_0 \\ &= \frac{1}{2} (u \bar{u}^c \bar{n} \theta)_0 + \frac{1}{2} (u^c \bar{u} \bar{n} \theta)_0 \\ &= \operatorname{Re} (u \bar{u}^c \bar{n} \theta)_0. \end{aligned}$$

At this point, using the perturbed Cauchy vanishing formula (25) we may continue with

$$\begin{aligned} \int_{\partial\Omega} |u|^2 \langle n, \theta \rangle d\sigma &= \operatorname{Re} \left(\int_{\partial\Omega} u \bar{u}^c \bar{n} \theta d\sigma \right)_0 = \operatorname{Re} \left(\int_{\partial\Omega} u (\pm n u^c + \bar{u}^c \bar{n}) \theta d\sigma \right)_0 \\ &\mp \operatorname{Re} \left(\iint_{\Omega} u ([D, \theta]u^c) - 2 (\operatorname{Re} k) u e_{m+1} u^c \theta \right)_0. \end{aligned}$$

Now, since

$$\begin{aligned} \operatorname{Re} \left(\int_{\partial\Omega} u (\pm n u^c + \bar{u}^c \bar{n}) \theta \, d\sigma \right)_0 &= \pm 2 \operatorname{Re} \left(\int_{\partial\Omega} u (nu)_\pm^c \theta \, d\sigma \right)_0 \\ &= \pm 2 \operatorname{Re} \left(\int_{\partial\Omega} \theta u (nu)_\pm^c \, d\sigma \right)_0, \end{aligned}$$

the identity (43) follows.

The proof of the claim in the second part of the theorem is based on purely *algebraic* manipulations (as discussed in §2) and is left as an exercise to the reader. \square

The previous theorem has several important consequences. To be able to state them, we shall write $A \approx B \bmod C$ if there exists a positive constant M , independent of the relevant parameters, so that $A \leq M(B + C)$ and $B \leq M(A + C)$.

Corollary 5.2 *Assume that Ω is a bounded Lipschitz domain in R^m , $k \in C$, and $u \in \mathcal{H}_{left}^2(\Omega, k) \cap \mathcal{H}_{right}^2(\Omega, k)$. Then, for any real C^1 -vector field θ which is transversal to $\partial\Omega$, we have*

$$\begin{aligned} \|u\|_{L^2(\partial\Omega)} &\approx \|(nu)_\pm\|_{L^2(\partial\Omega)} \approx \|(un)_\pm\|_{L^2(\partial\Omega)} \\ &\approx \|(u\theta)_\pm\|_{L^2(\partial\Omega)} \approx \|(\theta u)_\pm\|_{L^2(\partial\Omega)} \quad \text{modulo } \|u\|_{L^2(\Omega)}. \end{aligned}$$

In particular, if u is Λ^{l+1} -valued for some $0 \leq l \leq m$, then

$$\begin{aligned} \|n \wedge u\|_{L^2(\partial\Omega)} &\approx \|n \vee u\|_{L^2(\partial\Omega)} \approx \|\theta \wedge u\|_{L^2(\partial\Omega)} \\ &\approx \|\theta \vee u\|_{L^2(\partial\Omega)} \approx \|u\|_{L^2(\partial\Omega)} \quad \text{modulo } \|u\|_{L^2(\Omega)}. \end{aligned}$$

An analogous statement is valid in the case when Ω is replaced by the complement of a bounded Lipschitz domain. More specifically, in this case, we take θ to be compactly supported and the corresponding equivalences are valid modulo $\|u\|_{L^2(\Omega \cap \operatorname{supp} \theta)}$.

Proof.

The first set of equivalences is a simple consequence of the previous theorem and the Cauchy-Schwarz inequality since, by assumption, $\langle n, \theta \rangle \geq c > 0$ a.e. on $\partial\Omega$. The fact that $[D, \theta]$ is a zero-order operator is also used here.

The second set of equivalences follows from the first one and the identities

$$(44) \quad (nu)_+ = \begin{cases} n \wedge u, & \text{if } \frac{l(l+1)}{2} \text{ is odd,} \\ -n \vee u, & \text{if } \frac{l(l+1)}{2} \text{ is even,} \end{cases}$$

$$(45) \quad (nu)_- = \begin{cases} n \wedge u, & \text{if } \frac{l(l+1)}{2} \text{ is even,} \\ -n \vee u, & \text{if } \frac{l(l+1)}{2} \text{ is odd.} \end{cases}$$

To see this, let us show, for instance, that if $\frac{l(l+1)}{2}$ is odd then $(nu)_+ = n \wedge u$. To this effect, recall that

$$(nu)_+ = \frac{1}{2}(nu + \overline{nu}) = \frac{1}{2}(nu + \bar{u}\bar{n}).$$

Now, since $u \in \Lambda^{l+1}$, $n \in \Lambda^1$, we have that $\bar{u} = (-1)^{\frac{(l+1)(l+2)}{2}}u$ and $\bar{n} = -n$. Also, $nu = n \wedge u - n \vee u$ and $un = (-1)^{(l+1)}[n \wedge u + n \vee u]$, by (10) and (11). From this and the fact that $\frac{1}{2}l(l+1)(l+2) + (l+1) + 1$ has the same parity as $\frac{1}{2}l(l+1) + 1 \pmod{2}$ the conclusion easily follows.

The case when Ω is replaced by the complement of a bounded Lipschitz domain is virtually identical. Note that, in this case, only the restriction of u to $\text{supp } \theta$ plays a role and, hence, the behavior of u at infinity is not an issue. \square

The interested reader may also consult the Rellich type estimates from [15].

6 A priori estimates and integral representation formulas for solutions of Maxwell's equations.

Here we indicate how our results from § 3 can be utilized in connection with Maxwell's equations in Lipschitz domains. Our main interest is to derive a priori boundary estimates as well as integral representation formulae.

First, we derive an integral representation formula.

Lemma 6.1 *Let Ω be a bounded Lipschitz domain in R^m and consider $k \in C$. Then for any $u \in \mathcal{H}_{\text{eff}}^2(\Omega, k)$ we have*

$$\begin{aligned} u(X) &= \int_{\partial\Omega} E_k(X - Y)(n \wedge u)(Y) d\sigma(Y) \\ &\quad - \int_{\partial\Omega} E_k(X - Y)(n \vee u)(Y) d\sigma(Y), \quad X \in \Omega. \end{aligned}$$

A similar statement is valid for functions in $\mathcal{H}_{\text{eff}}^2(R^m \setminus \bar{\Omega}, k)$ provided $k \in C \setminus 0$ has $\text{Im } k \geq 0$ and functions in $\mathcal{H}_{\text{eff}}^{2,\pm}(R^m \setminus \bar{\Omega}, k)$ if $k \in R \setminus 0$.

Moreover, an analogous result holds for functions annihilated by the right handed version of \mathcal{D}_k .

Proof.

This follows from the Cauchy-Clifford reproducing formulas (28), (33) and the fact that $nu = n \wedge u - n \vee u$. \square

In order to proceed, we make the following key observation. Let E, H be two smooth differential forms of degrees l and $l+1$, respectively, $0 \leq l \leq m$, defined in some open subset of \mathbb{R}^m . To these, we associate an \mathcal{A}_{m+1} -valued function u by setting

$$(46) \quad u := H - ie_{m+1}E = H + i(-1)^{l+1}Ee_{m+1}.$$

For instance, if $m = 3$ and $l = 1$, the above reduces to $u = H + iEe_4$.

Proposition 6.1 *With the above notation, u is (left or right) k -monogenic if and only if E, H satisfy the Maxwell system*

$$(\mathcal{M}_m) \quad \begin{cases} dE - ikH = 0, \\ \delta E = 0, \\ \delta H + ikE = 0, \\ dH = 0. \end{cases}$$

Moreover, u satisfies the radiation condition (30) if and only if E, H satisfy the Silver-Müller type radiation conditions

$$(47) \quad \begin{cases} \hat{X} \wedge E - H = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty, \\ \hat{X} \vee H - E = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty. \end{cases}$$

Proof.

A straightforward calculation shows that

$$\mathbb{D}_k u = i(-1)^{l+1}(dE + \delta E - ikH)e_{m+1} + (dH + \delta H + ikE).$$

The claim in the first part of the proposition follows from this based on simple degree considerations.

Similarly, $(1 - ie_{m+1}\hat{X})u = (H - \hat{X}E) - ie_{m+1}(\hat{X}H - E)$ so that the claim in the second part of the proposition is seen from (30) and (10). \square

Remark. The radiation conditions (31) for u translate into the Silver-Müller type radiation conditions for E, H

$$(48) \quad \begin{cases} \hat{X} \wedge E \mp H = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty, \\ \hat{X} \vee H \mp E = o(|X|^{-\frac{m-1}{2}}), \text{ as } |X| \rightarrow \infty. \end{cases}$$

Remark. If k is nonzero then, because $d^2 = 0$ and $\delta^2 = 0$, the equations $\delta E = 0$, $dH = 0$ become superfluous. Nonetheless, as for $k = 0$ Maxwell's equations decouple (i.e. E and H become unrelated), it is precisely this case for which these two equations are relevant. Also, when $m = 3$, $l = 1$ (and $k \neq 0$), the formulae (\mathcal{M}_m) reduce to the more familiar system of equations

$$(\mathcal{M}_3) \quad \begin{cases} \nabla \times E - ikH = 0, \\ \nabla \times H + ikE = 0, \end{cases}$$

while (47) becomes the classical Silver-Müller radiation condition for vector fields; see, e.g., [20] and the references therein.

We are now ready to present the estimates and the integral representation formulas alluded to earlier. They have been first proved in [10] with a different approach (a real variable argument).

Theorem 6.2 *Let Ω be a bounded Lipschitz domain in R^m and assume that $k \in C \setminus 0$. Also, suppose that the forms $E \in C^\infty(\Omega, \Lambda^l)$, $H \in C^\infty(\Omega, \Lambda^{l+1})$ solve the Maxwell system (\mathcal{M}_m) and have $E^*, H^* \in L^2(\partial\Omega)$. Then, for any real C^1 -vector field θ which is transversal to $\partial\Omega$ we have*

$$\begin{aligned} \|E\|_{L^2(\partial\Omega)} + \|H\|_{L^2(\partial\Omega)} &\approx \|n \wedge E\|_{L^2(\partial\Omega)} + \|n \wedge H\|_{L^2(\partial\Omega)} \\ &\approx \|n \vee E\|_{L^2(\partial\Omega)} + \|n \vee H\|_{L^2(\partial\Omega)} \\ &\approx \|\theta \wedge E\|_{L^2(\partial\Omega)} + \|\theta \wedge H\|_{L^2(\partial\Omega)} \\ &\approx \|\theta \vee E\|_{L^2(\partial\Omega)} + \|\theta \vee H\|_{L^2(\partial\Omega)} \\ &\quad \text{modulo } \|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)}. \end{aligned}$$

Furthermore, for $X \in \Omega$,

$$\begin{aligned} E(X) &= -d \left(\int_{\partial\Omega} \Phi_k(X - Y) (n \vee E)(Y) d\sigma(Y) \right) \\ &\quad + \delta \left(\int_{\partial\Omega} \Phi_k(X - Y) (n \wedge E)(Y) d\sigma(Y) \right) \\ (49) \quad &\quad - ik \int_{\partial\Omega} \Phi_k(X - Y) (n \vee H)(Y) d\sigma(Y), \end{aligned}$$

and

$$H(X) = -ik \int_{\partial\Omega} \Phi_k(X - Y) (n \wedge E)(Y) d\sigma(Y)$$

$$(50) \quad \begin{aligned} & +\delta \left(\int_{\partial\Omega} \Phi_k(X - Y)(n \wedge H)(Y) d\sigma(Y) \right) \\ & -d \left(\int_{\partial\Omega} \Phi_k(X - Y)(n \vee H)(Y) d\sigma(Y) \right). \end{aligned}$$

Finally, a similar set of conclusions is valid with Ω replaced by the complement of a bounded Lipschitz domain. In this latter case, as far as (49)-(50) are concerned, we assume that $\text{Im } k \geq 0$ and, in addition, that E, H satisfy the (generalized) Silver-Müller radiation condition (47).

Proof.

If we set $u := H + i(-1)^{l+1}Ee_{m+1}$, then everything follows by straightforward calculations from Proposition 6.1, Corollary 5.2, and Lemma 6.1. \square

Let us now pause and explain how these estimates relate to, for instance, the study of boundary value problems for the Maxwell system in \mathbb{R}^3 , the Laplace operator and the Helmholtz operator. First, if Ω is a Lipschitz domain in \mathbb{R}^3 and (E, H) solve the Maxwell system (\mathcal{M}_3) in Ω , then the above theorem yields

$$\|n \times E\|_{L^2(\partial\Omega)} + \|\langle n, H \rangle\|_{L^2(\partial\Omega)} \approx \|n \times H\|_{L^2(\partial\Omega)} + \|\langle n, E \rangle\|_{L^2(\partial\Omega)} \\ \text{modulo } \|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)}.$$

This has been first proved by real variable methods in [20] and the importance of this estimate is discussed there in detail.

Furthermore, taking this time $k := 0$, $E := 0$ and $H := du$, where u is a function harmonic in the Lipschitz domain $\Omega \subseteq \mathbb{R}^m$, we obtain

$$\|\nabla_{\tan} u\|_{L^2(\partial\Omega)} \approx \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\partial\Omega)} \quad \text{modulo } \|\nabla u\|_{L^2(\Omega)}.$$

See, e.g., [30], [11] for the relevance of this estimate in connection with boundary value problems for the Laplacian.

Finally, if the complex-valued function U satisfies $(\Delta + k^2)U = 0$ in $\Omega \subseteq \mathbb{R}^m$, then for any transversal vector $\theta \in \mathbb{R}^m$ to $\partial\Omega$ one has

$$\|U\|_{L^2(\partial\Omega)} + \|\nabla U\|_{L^2(\partial\Omega)} \approx \left\| \frac{\partial U}{\partial n} \right\|_{L^2(\partial\Omega)} \approx \left\| \frac{\partial U}{\partial \theta} \right\|_{L^2(\partial\Omega)} \\ \approx \|\nabla_{\tan} U\|_{L^2(\partial\Omega)} + \|U\|_{L^2(\partial\Omega)} \\ \text{modulo } \|\nabla U\|_{L^2(\Omega)},$$

where ∇_{tan} stands for the tangential gradient on $\partial\Omega$. This is immediately seen by applying the second part of Corollary 5.2 to the two-sided k -monogenic function $u := \mathbb{D}_k U$ in Ω . In fact, one can easily check that u satisfies (30) if and only if U satisfies the higher dimensional analogue of the classical Sommerfeld radiation condition, i.e.

$$\langle \nabla U(X), \hat{X} \rangle - ikU(X) = o(|X|^{-\frac{m-1}{2}}), \quad \text{as } |X| \rightarrow \infty.$$

We omit the straightforward details. See also [22], [29] for a fuller treatment of the Helmholtz equation.

7 More general wave numbers.

Consider first the propagation of an electromagnetic wave in an isotropic, inhomogeneous medium. In this case, the Maxwell system reads

$$(51) \quad \begin{cases} dE - ik_1 H = 0, \\ \delta H + ik_2 E = 0, \end{cases}$$

where $k_1, k_2 \in C^2(\mathbb{R}^m, \mathbb{C})$ are *variable* parameters. As before, E, H are smooth differential forms of degree l and $l+1$, respectively.

The general idea is that much of our previous analysis carries over to this context even though the various identities and estimates are going to be valid only modulo certain residual terms. Our aim is to prove the following.

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^m$ be an arbitrary, bounded Lipschitz domain and let $k_1, k_2 \in C^1(\mathbb{R}^m)$ be complex-valued, non-vanishing functions. Then the map assigning $(n \wedge E, n \wedge H) \in L^2(\partial\Omega) \oplus L^2(\partial\Omega)$ to each pair (E, H) which solves (51) in Ω and also has $E^*, H^* \in L^2(\partial\Omega)$, has a closed range and a finite dimensional kernel. In particular, it is semi-Fredholm between appropriate spaces.*

Proof.

The idea is to show that the mapping in the statement of the theorem is bounded from below modulo compact operators. To this end, assume for a moment that the estimate

$$(52) \quad \begin{aligned} \|E^*\|_{L^2(\partial\Omega)} + \|H^*\|_{L^2(\partial\Omega)} &\leq C\|n \wedge E\|_{L^2(\partial\Omega)} + C\|n \wedge H\|_{L^2(\partial\Omega)} \\ &\quad + C\|E\|_{L^2(\Omega)} + C\|H\|_{L^2(\Omega)} \end{aligned}$$

holds uniformly in (E, H) as in the statement of the theorem. Then, since E and H (separately) satisfy a strongly elliptic, second order, variable coefficient PDE with a formally self-adjoint principal part, it follows (cf. the results in § 3 of [17]) that

$$(53) \quad \|E\|_{W^{1/2,2}(\Omega)} + \|H\|_{W^{1/2,2}(\Omega)} \leq C\|E^*\|_{L^2(\partial\Omega)} + C\|H^*\|_{L^2(\partial\Omega)}.$$

Now, the desired conclusion follows from the compactness of the embedding $W^{1/2,2}(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, it remains to prove the a priori estimate (52).

To this effect, fix (E, H) as before and once again consider the homogeneous form $u := H - ie_{m+1}E$. Since, this time,

$$(54) \quad dH = -\frac{dk_1}{k_1} \wedge H \quad \text{and} \quad \delta E = -\frac{dk_2}{k_2} \vee E \quad \text{in } \Omega,$$

it follows that

$$(55) \quad \left| \mathbb{D}_{k_j} u \right| + \left| u \mathbb{D}_{k_j} \right| \leq C|u| \quad \text{pointwise in } \Omega, \quad j = 1, 2.$$

In particular, a natural adaptation of the Cauchy reproducing formula from § 4 yields in this case

$$(56) \quad \|u^*\|_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\partial\Omega)} + C\|u\|_{L^2(\Omega)}.$$

Next, we turn attention to the Rellich type identities of § 5. The first observation is that, as is well known, it is possible to choose a compactly supported, C^1 -variable vector field θ with real components in \mathbf{R}^m and so that $\langle n, \theta \rangle \geq c > 0$ almost everywhere on $\partial\Omega$. Using this, invoking (55) and paralleling the proofs of Theorem 5.1 and Corollary 5.2, we arrive at

$$(57) \quad \|u\|_{L^2(\partial\Omega)}^2 \leq C\|u\|_{L^2(\partial\Omega)}\|n \wedge u\|_{L^2(\partial\Omega)} + C\|u\|_{L^2(\Omega)}^2.$$

Using the usual trick to the effect that $ab \leq \varepsilon a^2 + (4\varepsilon)^{-1}b^2$ for any $\varepsilon > 0$, we finally get

$$(58) \quad \|u\|_{L^2(\partial\Omega)} \leq C\|n \wedge u\|_{L^2(\partial\Omega)} + C\|u\|_{L^2(\Omega)}.$$

Now, the estimate (52) follows by combining (56) and (58). This completes the proof of the theorem. \square

Our next result deals with the case when (51) is equipped with a pair of boundary conditions involving an arbitrary (Lipschitz) transversal field in place of the exterior unit normal.

Theorem 7.2 *Let Ω and k_1, k_2 be as before and assume that θ is a Lipschitz vector field in R^m which is transversal to $\partial\Omega$. Then the operator assigning $(\theta \wedge E, \theta \wedge H) \in L^2(\partial\Omega) \oplus L^2(\partial\Omega)$ to each pair (E, H) which solves (51) in Ω and also satisfies $E^*, H^* \in L^2(\partial\Omega)$, has a closed range and a finite dimensional kernel. In particular, this operator is also semi-Fredholm.*

Proof.

Essentially the same pattern of reasoning as in the previous theorem applies. One notable exception is that the role of (58) is now played by

$$(59) \quad \|u\|_{L^2(\partial\Omega)} \leq C\|\theta \wedge u\|_{L^2(\partial\Omega)} + C\|u\|_{L^2(\Omega)}$$

which, in turn, follows by paralleling the corresponding argument in Corollary 5.2. \square

Finally, let us point out that analogous results are valid in the case of *anisotropic, inhomogeneous* media, in which case k_1, k_2 in (51) are matrix-valued functions. For example, when $m = 3$, the problem becomes

$$\begin{cases} \operatorname{curl} E - i\omega\mu H = 0 & \text{in } \Omega, \\ \operatorname{curl} H + i\omega\epsilon E = 0 & \text{in } \Omega, \end{cases}$$

where $\omega \in \mathbb{C}$ plays the role of the frequency, while $\mu = (\mu_{ij}), \epsilon = (\epsilon_{ij})$ are complex invertible matrices corresponding to the permeability and the dielectricity of the medium, respectively. We leave the details of this matter to the motivated reader.

8 Appendix.

Here, for the convenience of the reader, we recall a short proof of the following.

Lemma 8.1 (Rellich's lemma [25]) *Assume that a complex-valued function u , defined in the complement of some ball in R^m , satisfies $(\Delta + k^2)u = 0$ for some non-zero real k as well as*

$$(60) \quad \int_{|X|=R} |u|^2 d\sigma = o(1), \quad \text{as } R \rightarrow \infty.$$

Then u must vanish identically.

Proof.

Let $u(X) = \sum \sum a_{n,\nu}(|X|)Y_{n,\nu}(\hat{X})$ be the expansion of u in spherical harmonics, where

$$(61) \quad a_{n,\nu}(r) := \int_{S^{m-1}} u(r\omega) \bar{Y}_{n,\nu}(\omega) d\omega.$$

Since the Laplacian in \mathbb{R}^m is related to the Laplacian on the unit sphere $S^{m-1} \subseteq \mathbb{R}^m$ by

$$\Delta_{\mathbb{R}^m} = \frac{\partial^2}{\partial r^2} + \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{m-1}},$$

it follows from $(\Delta + k^2)u = 0$, $\Delta_{S^{m-1}}Y_{n,\nu} = -n(n+m-2)Y_{n,\nu}$, (61) and integration by parts that each $a_{n,\nu}$ satisfies the second order, linear ODE

$$(62) \quad \frac{\partial^2 a_{n,\nu}}{\partial r^2} + \frac{m-1}{r} \frac{\partial a_{n,\nu}}{\partial r} + \left(k^2 - \frac{n(n+m-2)}{r^2}\right) a_{n,\nu} = 0.$$

As is well known (cf. [3, (20)-(23), p.96]), for each fixed n , the solutions of (62) are linear combinations of $r^{1-m/2}J_{\pm\mu}(kr)$, where $\mu := [(m/2 - 1)^2 + n(n+m-2)]^{1/2}$ and J_s is the Bessel function. In particular, the coefficients $a_{n,\nu}$ have the asymptotic behavior

$$(63) \quad a_{n,\nu}(r) = c_{k,m,n,\nu} e^{\pm ikr} r^{\frac{1-m}{2}} \left\{ 1 + \mathcal{O}\left(\frac{1}{r}\right) \right\}, \quad \text{as } r \rightarrow \infty.$$

On the other hand, Parseval's identity gives

$$(64) \quad r^{m-1} \sum \sum |a_{n,\nu}(r)|^2 = \int_{|X|=r} |u(X)|^2 d\sigma_r$$

so that, by (60), $r^{m-1}|a_{n,\nu}(r)|^2 = o(1)$ as $r \rightarrow \infty$. Utilizing this back in (63) finally yields $a_{n,\nu} = 0$ for each n, ν . Thus, $u = 0$ as desired. \square

References

- [1] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman Advanced Publ. Program, 1982.
- [2] A. P. Calderón, *The multipole expansion of radiation fields*, J. Rat. Mech. Anal., 3 (1954), 523-537.
- [3] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, San Francisco, London, 1997.

- [4] D. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley, New York, 1983.
- [5] H. Flanders, *Differential forms with applications to the physical sciences*, A.P. Press, 1963.
- [6] H.G. Garnir, *Les problèmes aux limites de la physique mathématique*, Basel, Birkhäuser Verlag, 1958.
- [7] J. Gilbert, and M. A. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Studies in Advanced Mathematics, 1991.
- [8] K. Gürlebeck and W. Sprößig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhäuser Verlag, Basel, 1990.
- [9] B. Jancewicz, *Multivectors and Clifford Algebras in Electrodynamics*, World Scientific, 1988, Singapore/ New Jersey/ London.
- [10] B. Jawerth and M. Mitrea, *Higher dimensional scattering theory on C^1 and Lipschitz domains*, American Journal of Mathematics, Vol. 117, No. 4 (1995), 929–963.
- [11] C. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, CBMS Series in Mathematics, No. 83, Amer. Math. Soc., 1994.
- [12] C. Li, A. McIntosh and S. Semmes, *Convolution singular integrals on Lipschitz surfaces*, J. Amer. Math. Soc., 5 (1992), 455–481.
- [13] A. McIntosh, *Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains*, in the Proceedings of the Conference on Clifford Algebras in Analysis, J. Ryan ed., Studies in Advanced Mathematics, C.R.C. Press Inc., 1995, pp. 33–88.
- [14] A. McIntosh, *Review of “Clifford algebra and spinor-valued functions, a function theory for the Dirac operator”, by R. Delanghe, F. Sommen and V. Souček*, Bull. of A.M.S., Vol. 32, No. 3, (1995), 344–348.
- [15] A. McIntosh, D. Mitrea and M. Mitrea, *Rellich type identities for one-sided monogenic functions in Lipschitz domains and applications*, in “Analytical and Numerical Methods in Quaternionic and Clifford Algebras”, W. Sprössig and K. Gürlebeck eds., the Proceedings of the Seiffen Conference, Germany, 1996, pp. 135–143.
- [16] D. Mitrea, M. Mitrea and J. Pipher, *Vector potential theory on non-smooth domains in R^3 and applications to electromagnetic scattering*, Journal of Fourier Analysis and Applications, Vol. 3, No. 2 (1997), 131–192.
- [17] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge laplacian and global boundary problems in non-smooth Riemannian manifolds*, preprint, (1997).

- [18] M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy Spaces*, Lecture Notes in Mathematics, No. 1575, Springer-Verlag, Berlin Heidelberg, 1994.
 - [19] M. Mitrea, *Electromagnetic scattering on nonsmooth domains*, Mathematical Research Letters, Vol.1, No.6 (1994), 639–646.
 - [20] M. Mitrea, *The method of layer potentials in electro-magnetic scattering theory on non-smooth domains*, Duke Math. Journal, Vol.77, No. 1 (1995), 111–133.
 - [21] M. Mitrea, *Hypercomplex variable techniques in Harmonic Analysis*, in Proceedings of the Conference on Clifford Algebras in Analysis, J. Ryan ed., Studies in Advanced Mathematics, C.R.C. Press Inc., 1995, pp. 103–128.
 - [22] M. Mitrea, *Boundary value problems and Hardy spaces for the Helmholtz equation in Lipschitz domains*, Jour. Math. Anal. Appl., Vol.202 (1996), 819–842.
 - [23] M. Mitrea, R. Torres, and G. Welland, *Regularity and approximation results for the Maxwell problem on C^1 and Lipschitz domains*, in Proceedings of the Conference on Clifford Algebras in Analysis, J. Ryan ed., Studies in Advanced Mathematics, C.R.C. Press Inc., 1995, pp. 297–308.
 - [24] C. Müller, *Foundations of the mathematical theory of electromagnetic waves*, Springer-Verlag, New York, 1969.
 - [25] F. Rellich, *Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten*, Jber. Deutsch. Math. Verein., 53 (1943), 57–65.
 - [26] B. Spain and M. G. Smith, *Functions of Mathematical Physics*, Van Nostrand Reinhold Co., London, New York, 1970.
 - [27] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
 - [28] M. Taylor, *Partial Differential Equations*, Springer-Verlag, 1996.
 - [29] R. Torres and G. Welland, *The Helmholtz equation and transmission problems with Lipschitz interfaces*, Indiana Univ. Math. J. 42 (1993), 1457–1485.
 - [30] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), 572–611.
 - [31] C. Von Westenholz, *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam, 1978.
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Alan McIntosh:

Macquarie University, Sydney
NSW 2109, Australia
e-mail: alan@mpce.mq.edu.au

Marius Mitrea:
Institute of Mathematics
of the Romanian Academy,
P.O. Box 1-764
RO-70700 Bucharest, Romania

and

Department of Mathematics
University of Missouri-Columbia
202 Mathematical Sciences Building
Columbia, MO 65211
e-mail: marius@math.missouri.edu