Hodge decompositions on weakly Lipschitz domains

Andreas Axelsson, Alan McIntosh

Abstract. We survey the L_2 theory of boundary value problems for exterior and interior derivative operators $d_{k_1} = d + k_1 e_0 \wedge$ and $\delta_{k_2} = \delta + k_2 e_0 \lrcorner$ on a bounded, weakly Lipschitz domain $\Omega \subset \mathbf{R}^n$, for $k_1, k_2 \in \mathbf{C}$. The boundary conditions are that the field be either normal or tangential at the boundary. The well-posedness of these problems is related to a Hodge decomposition of the space $L_2(\Omega)$ corresponding to the operators d and δ. In developing this relationship, we derive a theory of nilpotent operators in Hilbert space.

1. Introduction

The aim of this paper is to survey and further develop the Hilbert space theory of boundary value problems (BVP's) for the exterior (d) and interior derivative (δ) operators in a bounded domain $\Omega \subset \mathbf{R}^n$ with a boundary $\Sigma = \partial\Omega$ of minimal regularity. The BVP we have in mind is the following. Given a $(j-1)$-vector field $G \in L_2(\Omega; \wedge^{j-1})$ which is k_2-divergence free, i.e. $\delta_{k_2} G = 0$, find a j-vector field $F \in L_2(\Omega; \wedge^j)$ such that

$$(1) \qquad \begin{cases} d_{k_1} F = 0 & \text{in } \Omega, \\ \delta_{k_2} F = G & \text{in } \Omega, \\ \nu \wedge f = 0 & \text{on } \Sigma. \end{cases}$$

Here $f := F|_\Sigma$ and ν denotes the outward pointing unit vector field on Σ. Thus the boundary condition $\nu \wedge f = 0$ means that F is normal on the boundary. The differential operators are the zero order perturbations $d_{k_1} = d + k_1 e_0 \wedge$ and $\delta_{k_2} = \delta + k_2 e_0 \lrcorner$ of the operators d and δ defined in Section 2, where $k_i \in \mathbf{C}$ are the wave numbers and $e_0 \in \wedge^1$ is the time-like vector. An important property of these operators is that they are nilpotent, i.e. $d_{k_1}^2 = \delta_{k_2}^2 = 0$.

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Example 1.1. When $j = 1$ and $k_1 = k_2 = k$, the BVP (1) is essentially the Dirichlet BVP for the Helmholtz equation $(\Delta + k^2)U = G$, where $U : \Omega \to \wedge^0 = \mathbf{C}$. To see this, let $F = d_k U = \nabla U + k e_0 U$ so that

$$\delta_k F = \delta_k d_k U = (\Delta + k^2)U = G.$$

Since $u = U|_\Sigma = 0$ implies that f is normal, i.e. $\nu \wedge f = 0$ on Σ, we see that the two BVP's are equivalent. Note that since G in this case is a scalar function, the condition $\delta_{k_2} G = 0$ is automatically satisfied.

Similarly, the Neumann problem corresponds to (1) with tangential boundary conditions, i.e. $\nu \lrcorner f = 0$.

Example 1.2. When $n = 3$ and $j = 2$, the BVP (1) coincides with the electromagnetic BVP for time-harmonic ($\frac{\partial}{\partial t} = -i\omega$) Maxwell's equations with frequency $\omega \in \mathbf{C}$ and $\Omega^- = \mathbf{R}^3 \setminus \overline{\Omega}$ being a perfect conductor. Assuming that Ω is composed of a linear, homogeneous, isotropic, possibly conducting material with permittivity $\epsilon > 0$, permeability $\mu > 0$ and conductivity $\sigma \geq 0$, we consider the electromagnetic field

$$F = \sqrt{\epsilon_*}(-ie_0) \wedge (E_1 e_1 + E_2 e_2 + E_3 e_3) \\ + \tfrac{1}{\sqrt{\mu}}(B_1 e_2 \wedge e_3 + B_2 e_3 \wedge e_1 + B_3 e_1 \wedge e_2) : \Omega \longrightarrow \wedge^2 \mathbf{R}^4,$$

where E and B are the electric and magnetic fields and $\epsilon_* := \epsilon + i\sigma/\omega$. If we let $k_1 = k_2 = k = \omega\sqrt{\epsilon_* \mu}$, then Faraday's induction law and the magnetic Gauss' law combine to $d_k F = 0$ whereas Maxwell's–Ampère's law and Gauss' law combine to $\delta_k F = G$, where the four-current $G = \frac{i}{\sqrt{\epsilon_*}}\rho e_0 - \sqrt{\mu} J$ satisfies the continuity equation $\delta_k G = 0$.

In this paper we investigate BVP's from the point of view of *splittings of function spaces* following our earlier work Axelsson–Grognard–Hogan–McIntosh [4], Axelsson [2] and [1]. The splittings relevant to this paper are Hodge type decompositions of the Hilbert space $L_2(\Omega; \wedge)$. For simplicity, assume $k_2 = -{k_1}^{c}$. Then the operators $d_{k_1,\overline{\Omega}}$ and $\delta_{k_2,\Omega}$ are adjoint, where $d_{k_1,\overline{\Omega}}$ denotes d_{k_1} with normal boundary conditions and $\delta_{k_2,\Omega}$ denotes δ_{k_2} without boundary conditions in Ω, as in Definition 4.1. Consider the following diagram.

$$\begin{array}{ccccccc} L_2(\Omega;\wedge) & = & \mathsf{R}(\delta_{k_2,\Omega}) & \oplus & \mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}}) & \oplus & \mathsf{R}(d_{k_1,\overline{\Omega}}) \\ & & & \delta_{k_2,\Omega} \swarrow \quad \searrow d_{k_1,\overline{\Omega}} & & & \\ L_2(\Omega;\wedge) & = & \mathsf{R}(\delta_{k_2,\Omega}) & \oplus & \mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}}) & \oplus & \mathsf{R}(d_{k_1,\overline{\Omega}}) \end{array}$$

What is needed here is to prove a Hodge decomposition, i.e. that the space $\mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}})$ of "harmonic forms" is finite dimensional and that the ranges $\mathsf{R}(\delta_{k_2,\Omega})$ and $\mathsf{R}(d_{k_1,\overline{\Omega}})$ are closed, or equivalently that $\mathsf{R}(\Gamma)$ is a closed subset of finite codimension in the null space $\mathsf{N}(\Gamma)$ for both choices $\delta_{k_2,\Omega}$ and $d_{k_1,\overline{\Omega}}$ for Γ.

Note that Fredholm well-posedness of the BVP (1), put into operator theoretic language means that

$$\delta_{k_2,\Omega} : \mathsf{N}(d_{k_1,\overline{\Omega}}) \longrightarrow \mathsf{N}(\delta_{k_2,\Omega})$$

is a Fredholm map. Clearly this holds if we have a Hodge decomposition as above. The Hodge decomposition in the case $k_1 = k_2 = 0$ for a general weakly Lipschitz domain is due to Picard [21], and the extension to the case $k_2 = -{k_1}^{\mathrm{c}}$ is straightforward. Although the Hodge decomposition is not valid for general k_1, $k_2 \in \mathbf{C}$, nevertheless the BVP (1) is well-posed in the Fredholm sense. Indeed the following result will be proved in Section 4.

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded weakly Lipschitz domain, as in Definition 2.1, and let k_1, $k_2 \in \mathbf{C}$. Then $\mathsf{R}(\delta_{k_2,\Omega})$ and $\mathsf{R}(d_{k_1,\overline{\Omega}})$ are closed subspaces of finite codimension in $\mathsf{N}(\delta_{k_2,\Omega})$ and $\mathsf{N}(d_{k_1,\overline{\Omega}})$ respectively. The maps*

$$\delta_{k_2,\Omega} : \mathsf{N}(d_{k_1,\overline{\Omega}}) \longrightarrow \mathsf{N}(\delta_{k_2,\Omega}) \tag{2}$$

$$d_{k_1,\overline{\Omega}} : \mathsf{N}(\delta_{k_2,\Omega}) \longrightarrow \mathsf{N}(d_{k_1,\overline{\Omega}}) \tag{3}$$

are Fredholm maps with compact Fredholm inverses.

For the Hodge decomposition with tangential boundary conditions, i.e. with $\delta_{k_2,\Omega}$ and $d_{k_1,\overline{\Omega}}$ replaced by $\delta_{k_2,\overline{\Omega}}$ and $d_{k_1,\Omega}$ as in Definition 4.1, the corresponding result holds.

W.V.D. Hodge's pioneering work on harmonic integrals on Riemannian manifolds during the 1930's was published in his book [12]. The splitting of a differential form into its exact, coexact and harmonic parts, now referred to as the Hodge decomposition, was in this book proved using Fredholm's theory of linear integral equations. The connection between splittings of function spaces such as the Hodge decomposition and boundary value problems in potential theory was early recognised by Weyl [27]. Here it was shown how the classical Dirichlet minimum principle could be replaced by the construction of orthogonal projections in Hilbert space.

In the present paper, we treat Hodge decompositions from a purely first order, operator theoretic point of view. By first order we mean that the focus is on nilpotent operators (see Definition 3.1 below) such as the exterior derivative d and not on the Hodge–Laplace operator $\Delta = d\delta + \delta d$. An early investigation along these lines is Friedrichs [8], where the operators d and δ were introduced as closed unbounded operators. Other references we would like to mention are Kodaira [14], where the weak Hodge decomposition (14) appears, and Gaffney [9] and [10] which introduced the a priori estimate

$$\|F\|_{W_2^1} \lesssim \|dF\|_{L_2} + \|\delta F\|_{L_2} + \|F\|_{L_2}. \tag{4}$$

For a domain with boundary, we discuss this inequality in Theorem 4.10. For further early literature on the Hodge decomposition, we refer to Chapter 7 in Morrey [20].

On a domain with non-smooth boundary, the Gaffney–Friedrich inequality (4) is in general not valid. The first proof of the Hodge decomposition on domains with non-smooth boundaries without using (4) is Weck [26]. The extension to general weakly Lipschitz domains is due to Picard [21].

Returning the Example 1.2, we remark that the standard approach to the Maxwell BVP uses the Maxwell operator M acting on a pair of divergence-free vector fields. An investigation of M on domains with non-smooth boundary, from the point of view of the Weyl decomposition (essentially the Hodge decomposition of vector fields) can be found in Birman–Solomyak [5], [6]. They show (in the language of the present paper) how M constitute part of the elliptic Dirac operator $\mathbf{D}_{\Omega^\perp}$ from Example 4.6.

For further literature on the connection between Hodge decompositions and BVP's, we refer to Schwarz [23] in the case of smooth domains and to Mitrea–Mitrea [17] in the case of strongly Lipschitz domains.

The key idea in this paper is that not only do we treat Hodge decompositions from a pure first order point of view, but we show that by investigating the "half-elliptic" operators d and δ separately, one can easily prove the Hodge decomposition on a domain with weakly Lipschitz boundary. Indeed, it is not necessary to use the given adjoint δ operator in proving that $\mathsf{R}(d_\Omega)$ is closed and of finite codimension in $\mathsf{N}(d_\Omega)$. As in Remark 3.12, we may equally well choose to work with the adjoint given by a metric in which Ω has a smooth boundary.

The first step in the proof of Theorem 1.3 uses the duality theorem 3.3 from general operator theory. As we show in Proposition 3.11, this duality result proves that the maps (2) and (3) have the same properties concerning closed range and compact inverse. The second step in the proof of Theorem 1.3 is Lemma 3.13, where we use the basic differential geometric fact that the exterior derivative is independent of the Riemannian metric, here given in the form of Proposition 2.6. These two steps show that the general case of a weakly Lipschitz domain in Theorem 1.3 can be reduced to the case of a smooth domain Ω. This reduction technique has been used by Picard [21]. We also provide some basic density results for the d and δ operators in Proposition 4.3 and construct extension maps in Proposition 4.8. Finally, we survey three different ways to prove Theorem 1.3 under certain additional regularity and topological assumptions on Σ.

- Theorem 4.10: The classical Gaffney–Friedrichs a priori estimate technique, which gives optimal $W_2^1(\Omega;\wedge)$ regularity for fields in $\mathsf{D}(d_{\overline{\Omega}})\cap\mathsf{D}(\delta_\Omega)$ if the domain has a smooth boundary.
- Theorem 4.13: The boundary integral equation method, which gives optimal regularity $W_2^{1/2}(\Omega;\wedge)$ in the class of strongly Lipschitz domains by using Rellich estimates.
- Theorem 4.17: A path integral method for a star shaped domain. This method, which is based on the classical Poincaré lemma, seems new. Although it does not give optimal regularity, it has the advantage of being entirely explicit.

2. Preliminaries

Throughout this paper $\Omega = \Omega^+ \subset \mathbf{R}^n$ denotes a *bounded* open set, separated from the exterior domain $\Omega^- = \mathbf{R}^n \setminus \overline{\Omega}^+$ by a weakly Lipschitz interface $\Sigma = \partial\Omega^+ = \partial\Omega^-$, defined as follows.

Definition 2.1. The interface Σ is *weakly Lipschitz* if, for all $y \in \Sigma$, there exists a neighbourhood $V_y \ni y$ and a global bilipschitz map $\rho_y : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\begin{aligned} \Omega^+ \cap V_y &= \rho_y(\mathbf{R}^n_+) \cap V_y, \\ \Sigma \cap V_y &= \rho_y(\mathbf{R}^{n-1}) \cap V_y, \\ \Omega^- \cap V_y &= \rho_y(\mathbf{R}^n_-) \cap V_y. \end{aligned}$$

In this case Ω is called a bounded weakly Lipschitz domain.

If $\rho : \mathbf{R}^n \to \mathbf{R}^n$ is a global bilipschitz map, then $\rho(\mathbf{R}^{n-1})$ is called a *special weakly Lipschitz* surface/interface and $\rho(\mathbf{R}^n_\pm)$ are called special weakly Lipschitz domains.

By Rademacher's theorem, a weakly Lipschitz surface Σ has a tangent plane and an outward (into Ω^-) pointing unit normal $\nu(y)$ at almost every $y \in \Sigma$.

Example 2.2. We now give two examples of weakly Lipschitz surfaces which are not strongly Lipschitz, i.e. not locally the graph of a Lipschitz function.

(i) Let $\rho_0 : S^{n-1} \to S^{n-1}$ be a bilipschitz homeomorphism of the unit sphere. Consider the conical surface

$$\Sigma := \{x \in \mathbf{R}^n \setminus \{0\} \,;\, x/|x| \in \rho_0(S^{n-1} \cap \mathbf{R}^{n-1})\} \cup \{0\}.$$

The natural parametrisation here is $\rho(r\omega) := r\rho_0(\omega)$, $r \geq 0$, $\omega \in S^{n-1}$. Using the identity $|r\omega - r'\omega'|^2 = |r-r'|^2 + rr'|\omega-\omega'|^2$, it is straightforward to show that $\rho : \mathbf{R}^n \to \mathbf{R}^n$ is a bilipschitz map. Thus Σ is a weakly Lipschitz surface.

An important special case is the "two brick" domain, defined as the interior of

$$\{(x,y,z) \in \mathbf{R}^3 \,;\, y \leq 0,\ z \leq 0\} \cup \{(x,y,z) \in \mathbf{R}^3 \,;\, x \leq 0,\ z \geq 0\}.$$

Indeed, the intersection with the unit sphere S^2 is a two dimensional strongly Lipschitz domain. Nevertheless, the boundary of the two brick domain is not locally a graph of a Lipschitz function around 0.

(ii) Let $a_i > 0$ and $e^{-2\pi}a_2 < a_1 < a_2$ and consider the *logarithmic spiral*

$$\Omega := \{re^{i\theta} \,;\, r > 0,\ \theta \in \mathbf{R},\ a_1 e^{-\theta} < r < a_2 e^{-\theta}\} \subset \mathbf{R}^2.$$

To see that Ω is a special weakly Lipschitz domain, define the maps

$$\rho_s(x,y) := (x\cos(s\ln r) - y\sin(s\ln r), x\sin(s\ln r) + y\cos(s\ln r)),$$

where $r^2 = x^2 + y^2$, or in complex notation $\rho_s : z \mapsto ze^{is\ln|z|}$. We see that $\rho_s \circ \rho_t = \rho_{s+t}$, $s,t \in \mathbf{R}$ and that $|\nabla \otimes \rho_s| \leq C$. In particular $\rho_{-1} : \mathbf{R}^2 \to \mathbf{R}^2$

is a bilipschitz map. Since

$$\Omega = \rho_{-1}(\{z \in \mathbf{C} \,;\, \ln a_1 < \arg(z) < \ln a_2\}),$$

this shows that Ω is a special weakly Lipschitz domain. But clearly $\partial\Omega$ is not locally a graph of a Lipschitz function around 0.

In this paper we make use of the three differential operators d, δ and $\mathbf{D}$ as described below. These operators act on functions $F : \Omega \to \wedge$ which take values in an exterior algebra $\wedge$, sometimes referred to as (multivector-)fields. Boundary traces and fields on Σ will be written with small letters, for example f. We here use the complexified exterior algebra

$$\wedge = \wedge_{\mathbf{C}}\mathbf{R}^{n+1} = \wedge^0 \oplus \wedge^1 \oplus \ldots \oplus \wedge^{n+1}$$

for $\mathbf{R}^n$ spacetime. Let $\{e_s \,;\, s \subset \{0, 1, \ldots, n\}\}$ be the standard basis for $\wedge_{\mathbf{C}}\mathbf{R}^{n+1}$. Here $e_0 \in \wedge^1$ is interpreted as a (imaginary time-like) vector and the space of j-vectors $\wedge^j$ is the span of $\{e_s \,;\, |s| = j\}$. Furthermore, let $\langle\cdot,\cdot\rangle$ denote the standard complex bilinear pairing, u^{c} denote component-wise complex conjugation and $u^{\neg}$ denote *involution.* Concretely, if we expand the multivectors u, $v \in \wedge_{\mathbf{C}}\mathbf{R}^{n+1}$ as $u = \sum_s u_s e_s$ and $v = \sum v_s e_s$, then

$$\begin{aligned} \langle u, v\rangle &= \sum u_s v_s, \\ u^{\mathrm{c}} &= \sum u_s^{\mathrm{c}} e_s, \\ u^{\neg} &= \sum (-1)^{|s|} u_s e_s. \end{aligned}$$

Definition 2.3. Introduce the counting function $\sigma(s,t) := \#\{(s_i, t_j) \,;\, s_i > t_j\}$, where $s = \{s_i\}$, $t = \{t_j\} \subset \{0, 1, \ldots, n\}$. Basic complex bilinear products on the algebra $\wedge$ are the following.

(i) The *exterior product* of two basis multivectors e_s and e_t is

$$e_s \wedge e_t = (-1)^{\sigma(s,t)}\, e_{s\cup t} \quad \text{if } s \cap t = \emptyset \text{ and otherwise zero.}$$

(ii) The left (right) *interior product* $u \,\lrcorner\, v$ ($u \,\llcorner\, v$) is the unique bilinear (non-associative) product for which $\langle u \lrcorner x, y\rangle = \langle x, u \wedge y\rangle$ and $\langle x \llcorner u, y\rangle = \langle x, y \wedge u\rangle$ respectively for all u, x, $y \in \wedge$. The action on two basis vectors e_s and e_t is

$$e_s \lrcorner e_t = (-1)^{\sigma(s, t\setminus s)}\, e_{t\setminus s}, \qquad e_t \llcorner e_s = (-1)^{\sigma(t\setminus s, s)}\, e_{t\setminus s},$$

if $s \subset t$ and otherwise zero.

(iv) The *Clifford product* of two basis multivectors e_s and e_t is

$$e_s \vartriangle e_t = (-1)^{\sigma(s,t)}\, e_{s \vartriangle t},$$

where $\vartriangle$ denotes the symmetric difference when acting on index sets. When there is no risk of confusion we will use the standard short-hand notation $uv := u \vartriangle v$ for the Clifford product.

Proposition 2.4. *For a vector $a \in \wedge^1$ and for general multivectors u, v and $w \in \wedge$ the following hold.*

$$u \lrcorner (v \lrcorner w) = (v \wedge u) \lrcorner w \tag{5}$$

$$a \vartriangle u = a \lrcorner u + a \wedge u \tag{6}$$

$$a \lrcorner u = -u^{\neg} \llcorner a = \tfrac{1}{2}(a \vartriangle u - u^{\neg} \vartriangle a) \tag{7}$$

$$a \wedge u = u^{\neg} \wedge a = \tfrac{1}{2}(a \vartriangle u + u^{\neg} \vartriangle a) \tag{8}$$

$$a \lrcorner (u \wedge v) = (a \lrcorner u) \wedge v + u^{\neg} \wedge (a \lrcorner v) \tag{9}$$

These basic geometric algebra identities are essentially well known and we omit the proof. Here (5) is the associativity property of the interior product. The formulae (7) and (8), which are inverse to (6), are sometimes referred to as *Riesz' formulae.* The formula (9) is the derivation property for the interior product. A classical example of (9) is when a, $b = u$ and $c = v$ are vectors in a three-dimensional space. Using the Hodge complement $u^{\perp} := u \lrcorner e_{123}$ (usually called the Hodge star $*u$), and the vector product $b \times c = (b \wedge c)^{\perp}$, we get the well known identity

$$-a \times (b \times c) = a \lrcorner (b \wedge c) = \langle a, b\rangle c - \langle a, c\rangle b.$$

Throughout this paper we make use of the nabla symbol $\nabla = \sum_{j=1}^{n} e_j \partial_j$. We recall that the products $\wedge$, $\lrcorner$ and $\vartriangle$ induce differential operators

$$dF(x) := \nabla \wedge F(x) = \sum_{j=1}^{n} e_j \wedge (\partial_j F)(x),$$

$$\delta F(x) := \nabla \lrcorner F(x) = \sum_{j=1}^{n} e_j \lrcorner (\partial_j F)(x),$$

$$\mathbf{D}F(x) := \nabla \vartriangle F(x) = \sum_{j=1}^{n} e_j \vartriangle (\partial_j F)(x) = dF(x) + \delta F(x).$$

In the same spirit we also denote the full differential of F by $\nabla \otimes F(x) = \sum e_j \otimes (\partial_j F)(x) \in \mathbf{R}^n \otimes \wedge$. Here the formal adjoint of the *exterior derivative* operator d is the negative of the *interior derivative* δ; this differs from the standard convention. Sometimes we refer to d as (generalised) curl and to δ as (generalised) divergence. The (elliptic) *Dirac operator* $\mathbf{D} = d + \delta$ is formally skew-adjoint. Here $\mathbf{D}$ is a square root of the Hodge–Laplace operator $\Delta = d\delta + \delta d$.

The most important property of the differential operators d and δ is that they commute with a change of variables if we change the direction of the field in an appropriate way.

Definition 2.5. Let $\rho : U \to V$ be a diffeomorphism between two open sets U and $V \subset \mathbf{R}^n$. Denote by $\underline{\rho}_x$ the Jacobian matrix of ρ at $x \in U$ and extend this linear map $T_x\mathbf{R}^n \to T_{\rho(x)}\mathbf{R}^n$ to a $\wedge$-isomorphism $\underline{\rho}_x : \wedge \to \wedge$ such that $\underline{\rho}_x(e_0) = e_0$ and

$$\underline{\rho}_x(e_{i_1} \wedge \ldots \wedge e_{i_k}) = (\underline{\rho}_x e_{i_1}) \wedge \ldots \wedge (\underline{\rho}_x e_{i_k}),$$

if $\{i_1, \ldots, i_k\} \subset \{0, 1, \ldots, n\}$. To a field $F : V \to \wedge$ we associate the *pullback* $\rho^* F : U \to \wedge$ and *push forward* $\rho_*^{-1} F : U \to \wedge$ of F as follows.

$$(\rho^* F)(x) := (\underline{\rho}_x)^*(F(\rho(x))), \qquad (\rho_*^{-1} F)(x) := (\underline{\rho}_x)^{-1}(F(\rho(x))).$$

For convenience, we also define the *reduced push forward*

$$\tilde{\rho}_*^{-1} F := J(\rho) \rho_*^{-1} F : U \longrightarrow \wedge,$$

where $J(\rho)(x)(e_1 \wedge \ldots \wedge e_n) = \underline{\rho}_x(e_1 \wedge \ldots \wedge e_n)$ denotes the Jacobian determinant.

Proposition 2.6. *If ρ and F are as in Definition 2.5, then we have commutation properties*

$$d(\rho^* F) = \rho^*(dF), \qquad \delta(\tilde{\rho}_*^{-1} F) = \tilde{\rho}_*^{-1}(\delta F), \tag{10}$$

and homomorphism properties

$$\rho^*(F \wedge G) = \rho^* F \wedge \rho^* G, \qquad \rho_*^{-1}(F \wedge G) = \rho_*^{-1} F \wedge \rho_*^{-1} G, \tag{11}$$

$$\rho^*(F \lrcorner G) = \rho_*^{-1} F \lrcorner \rho^* G, \qquad \rho_*^{-1}(F \lrcorner G) = \rho^* F \lrcorner \rho_*^{-1} G. \tag{12}$$

In particular, if $F^\perp := F \lrcorner e_{01\ldots n}$ denotes the complement of F, then $\rho^*(F^\perp) = (\tilde{\rho}_*^{-1} F)^\perp$.

Proof. Note that we have the two pairs of adjoint operators

$$\underline{\rho}_x : T_x \mathbf{R}^n \longrightarrow T_{\rho(x)} \mathbf{R}^n, \qquad (\underline{\rho}_x)^* : T_{\rho(x)} \mathbf{R}^n \longrightarrow T_x \mathbf{R}^n,$$

and

$$\tilde{\rho}_* : L_2(U; \wedge) \longrightarrow L_2(V; \wedge), \qquad \rho^* : L_2(V; \wedge) \longrightarrow L_2(U; \wedge),$$

if $\nabla \otimes \rho$, $\nabla \otimes \rho^{-1} \in L_\infty$. The identities $d(\rho^* F) = \rho^*(dF)$, $\rho^*(F \wedge G) = \rho^* F \wedge \rho^* G$ and $\rho_*^{-1}(F \wedge G) = \rho_*^{-1} F \wedge \rho_*^{-1} G$ are well known facts from the theory of differential forms. The remaining identities follow by duality. □

In order to treat Stokes' type theorems in a unified way, we record the following theorem, here referred to as the *boundary theorem.*

Theorem 2.7. *Let V be a finite dimensional linear space and let $F : \overline{\Omega} \to V$ be a function in Ω smooth up to $\Sigma = \partial\Omega$ with boundary trace $f := F|_\Sigma$. Then we have*

$$\int_\Sigma \nu(y) \otimes f(y)\, d\sigma(y) = \int_\Omega \nabla \otimes F(x)\, dx,$$

where the integrand is $\mathbf{R}^n \otimes V$ valued, ν is the outward pointing normal and $d\sigma$ is the scalar surface measure.

Remark 2.8. (i) Note that, via a limiting argument, the boundary theorem can be extended to less regular functions.

(ii) Recall that this theorem is universal in the sense that for any given finite dimensional linear space W and bilinear form $L : \mathbf{R}^n \times V \to W$, L can be lifted to a linear map $L : \mathbf{R}^n \otimes V \to W$. Applying this to the formula in the boundary theorem gives the special case $\int_\Sigma L(\nu(y), f(y))\, d\sigma(y) = \int_\Omega L(\nabla, F(x))\, dx$.

We end this section with a discussion, preliminary to Section 4, about natural boundary conditions for d and δ. Let $F : \Omega \to \wedge$ be a multivector field in Ω, smooth up to Σ, and extend it by zero to a field F_z on $\mathbf{R}^n$. If σ denotes the surface measure on Σ, it follows that in distribution sense we have

$$d(F_z) = dF|_\Omega - (\nu \wedge f)\sigma \quad \text{and} \quad \delta(F_z) = \delta F|_\Omega - (\nu \lrcorner f)\sigma.$$

For example, the first identity follows from the boundary theorem, using $V = \wedge \otimes \wedge$ and the linear map $L : \mathbf{R}^n \otimes (\wedge \otimes \wedge) \to \mathbf{C} : a \otimes (F \otimes G) \mapsto (a \wedge F, G)$, since

$$(\nabla \wedge F_z, \Phi) = -\int_\Omega \langle F, \nabla \lrcorner \Phi\rangle = \int_\Omega \langle \nabla \wedge F, \Phi\rangle - \int_\Sigma \langle \nu \wedge f, \phi\rangle,$$

for any $\Phi \in C_0^\infty(\mathbf{R}^n; \wedge)$. Thus, requiring that $d(F_z) \in L_2(\mathbf{R}^n; \wedge)$ means that $dF \in L_2(\Omega; \wedge)$ and that $\nu \wedge f = 0$, i.e. the field F is *normal* to Σ. Similarly, requiring that $\delta(F_z) \in L_2(\mathbf{R}^n; \wedge)$ means that $\delta F \in L_2(\Omega; \wedge)$ and that $\nu \lrcorner f = 0$, i.e. the field F is *tangential* to Σ. We note that each boundary condition refers to half of the components (in the full exterior algebra $\wedge$) of the field vanishing on Σ.

When $F \in L_2(\Omega; \wedge)$ and $d(F_z) \in L_2(\mathbf{R}^n; \wedge)$, although the field F is normal to Σ, it does not necessarily have a well defined normal component $\nu \lrcorner f$ on Σ. To see this, consider the vector field

$$F(x) := \begin{cases} e_n, & 1/(2j+1) < x_n \leq 1/(2j), \\ 0, & 1/(2j) < x_n \leq 1/(2j-1), \end{cases}$$

locally around $x = (x', x_n) = 0$. Then $F_z \in L_{2,\text{loc}}(\mathbf{R}^n; \wedge)$ and $d(F_z) = 0$, but clearly F does not have a well defined trace.

Similarly, control of F and $\delta(F_z)$ is not enough for defining the tangential part of the trace.

3. Nilpotent operators in Hilbert spaces

In this section we develop the operator theory for a nilpotent operator Γ. This is then applied to the d and δ operators in Section 4.

Recall the following basic spaces associated with a linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces $\mathcal{H}_i$.

- Domain $\mathsf{D}(A) := \{x \in \mathcal{H}_1 \,;\, Ax \text{ is defined}\}$
- Null space $\mathsf{N}(A) := \{x \in \mathsf{D}(A) \,;\, Ax = 0\}$
- Range $\mathsf{R}(A) := \{Ax \,;\, x \in \mathsf{D}(A)\}$
- Graph $\mathsf{G}(A) := \{(x, Ax)^t \in \mathcal{H}_1 \oplus \mathcal{H}_2 \,;\, x \in \mathsf{D}(A)\}$

If A_1 and A_2 are two linear operators, then we write $A_1 \subset A_2$ if $\mathsf{G}(A_1) \subset \mathsf{G}(A_2)$.

Definition 3.1. An operator $\Gamma : \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is said to be *nilpotent* if it is closed (i.e. $\mathsf{G}(\Gamma)$ is closed), densely defined (i.e. $\mathsf{D}(\Gamma)$ is dense in $\mathcal{H}$) and if $\mathsf{R}(\Gamma) \subset \mathsf{N}(\Gamma)$. In particular, $\Gamma^2 \subset 0$.

Recall that $\mathsf{N}(A)$ always is closed in $\mathcal{H}_1$ for a closed operator. For a nilpotent operator Γ in $\mathcal{H}$, we have inclusions

$$\mathsf{R}(\Gamma) \subset \overline{\mathsf{R}(\Gamma)} \subset \mathsf{N}(\Gamma) \subset \mathsf{D}(\Gamma) \subset \mathcal{H}. \tag{13}$$

Note carefully that $\mathsf{R}(\Gamma)$ may not be closed. Our main work will be to prove that $\mathsf{R}(\Gamma)$ is closed when Γ is one of the d and δ operators in Ω.

From (13) we also see that Γ acts as a bounded nilpotent operator $\Gamma : \mathsf{D}(\Gamma) \to \mathsf{D}(\Gamma)$, where $\mathsf{D}(\Gamma)$ is a Hilbert space with the graph norm $\|x\|^2_{\mathsf{D}(\Gamma)} := \|x\|^2 + \|\Gamma x\|^2$.

Definition 3.2. Let $A_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $A_2 : \mathcal{H}_2 \to \mathcal{H}_1$ be two linear operators. We say that A_1 and A_2 are (maximal) *adjoint* operators if

$$\mathsf{G}(A_1) = \{(x, A_1 x)^t \in \mathcal{H}_1 \oplus \mathcal{H}_2 \,;\, x \in \mathsf{D}(A_1)\}$$
$$I\mathsf{G}(A_2) = \{(-A_2 y, y)^t \in \mathcal{H}_1 \oplus \mathcal{H}_2 \,;\, y \in \mathsf{D}(A_2)\}$$

are orthogonal complements in $\mathcal{H}_1 \oplus \mathcal{H}_2$, where $I(x,y)^t := (-y,x)^t$. In particular both A_1 and A_2 are closed, densely defined operators, and if $x \in \mathsf{D}(A_1)$ and $y \in \mathsf{D}(A_2)$, then $(x, A_2 y) = (A_1 x, y)$.

Given a closed, densely defined operator A in $\mathcal{H}$, we define $\mathsf{G}(A^*) := (I\mathsf{G}(A))^\perp$. Since A is densely defined, $\mathsf{G}(A^*)$ is the graph of a closed linear operator A^*, and since A is closed it follows that A^* is densely defined. We say that A^* is the (maximal) adjoint operator of A.

A fundamental result for adjoint operators is the following, which for example can be found in Kato [13].

Theorem 3.3. *Let A and A^* be adjoint closed, densely defined Hilbert space operators. Then $\mathsf{R}(A)^\perp = \mathsf{N}(A^*)$ and $\mathsf{R}(A^*)^\perp = \mathsf{N}(A)$. Moreover, $\mathsf{R}(A)$ is closed if and only if $\mathsf{R}(A^*)$ is closed.*

Corollary 3.4. *If Γ is a nilpotent operator, then so is Γ^*.*

Note that a nilpotent operator acts

$$\Gamma : \mathsf{N}(\Gamma)^\perp = \overline{\mathsf{R}(\Gamma^*)} \longrightarrow \mathsf{R}(\Gamma) \subset \mathsf{N}(\Gamma),$$

where the restriction of Γ is injective. Thus $\mathsf{N}(\Gamma)$ is at least "half" of $\mathcal{H}$.

Proposition 3.5. *Let Γ be a nilpotent operator in a Hilbert space $\mathcal{H}$, with adjoint Γ^*. For each $\alpha \in \mathbf{C}$ with $|\alpha| = 1$, let the corresponding* swapping operator *be $\Pi_\alpha := \Gamma + \alpha\Gamma^*$ with domain $\mathsf{D}(\Pi_\alpha) := \mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma^*)$. Then we have $\mathcal{H}$-orthogonal decompositions*

$$\mathcal{H} = \overline{\mathsf{R}(\Gamma^*)} \oplus \big(\mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma)\big) \oplus \overline{\mathsf{R}(\Gamma)}, \tag{14}$$

$$\mathsf{D}(\Gamma) = \big(\mathsf{D}(\Gamma) \cap \overline{\mathsf{R}(\Gamma^*)}\big) \oplus \big(\mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma)\big) \oplus \overline{\mathsf{R}(\Gamma)}, \tag{15}$$

$$\mathsf{D}(\Gamma^*) = \overline{\mathsf{R}(\Gamma^*)} \oplus \big(\mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma)\big) \oplus \big(\mathsf{D}(\Gamma^*) \cap \overline{\mathsf{R}(\Gamma)}\big), \tag{16}$$

$$\mathsf{D}(\Pi_\alpha) = \big(\mathsf{D}(\Gamma) \cap \overline{\mathsf{R}(\Gamma^*)}\big) \oplus \big(\mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma)\big) \oplus \big(\mathsf{D}(\Gamma^*) \cap \overline{\mathsf{R}(\Gamma)}\big). \tag{17}$$

The swapping operator is a closed, densely defined operator in $\mathcal{H}$ with null space $\mathsf{N}(\Pi_\alpha) = \mathsf{N}(\Gamma) \cap \mathsf{N}(\Gamma^*)$ and range $\mathsf{R}(\Pi_\alpha) = \mathsf{R}(\Gamma) \oplus \mathsf{R}(\Gamma^*)$. The adjoint of Π_α is $\alpha^c \Pi_\alpha$. Thus Π_1 is a self adjoint operator and Π_{-1} is a skew adjoint operator.

The swapping operator Π_α is unitary equivalent to both $\sqrt{\alpha}\Pi_1$ and $-\Pi_\alpha$. In particular, the spectrum $\sigma(\Pi_\alpha)$ is contained in the line $\sqrt{\alpha}\mathbf{R}$ and it is symmetric with respect to 0.

Proof. From Theorem 3.3 we obtain the two orthogonal splittings

$$\mathcal{H} = \overline{\mathsf{R}(\Gamma^*)} \oplus \mathsf{N}(\Gamma) = \mathsf{N}(\Gamma^*) \oplus \overline{\mathsf{R}(\Gamma)}.$$

Using (13), we get inclusions $\overline{\mathsf{R}(\Gamma^*)} \subset \mathsf{N}(\Gamma^*)$ and $\overline{\mathsf{R}(\Gamma)} \subset \mathsf{N}(\Gamma)$. Therefore taking the intersection of the two splittings gives (14). Now write

$$\begin{aligned}
\mathcal{H}_1 &:= \overline{\mathsf{R}(\Gamma^*)} = \mathsf{N}(\Gamma)^\perp \approx \mathcal{H}/\mathsf{N}(\Gamma), \\
\mathcal{H}_0 &:= \mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma) \approx \mathsf{N}(\Gamma)/\overline{\mathsf{R}(\Gamma)} \approx \mathsf{N}(\Gamma^*)/\overline{\mathsf{R}(\Gamma^*)}, \\
\mathcal{H}_2 &:= \overline{\mathsf{R}(\Gamma)} = \mathsf{N}(\Gamma^*)^\perp \approx \mathcal{H}/\mathsf{N}(\Gamma^*),
\end{aligned}$$

and let P_i denote the orthogonal projection onto $\mathcal{H}_i$. To prove the decomposition (15), note that the inclusion $\supset$ is trivial. For the opposite inclusion, decompose $x \in \mathsf{D}(\Gamma)$ with (14) as $x = x_1 + x_0 + x_2$, where $x_i \in \mathcal{H}_i$. Since $x_2 \in \overline{\mathsf{R}(\Gamma)} \subset \mathsf{D}(\Gamma)$ and $x_0 \in \mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma) \subset \mathsf{D}(\Gamma)$ we deduce that $x_1 = x - x_0 - x_2 \in \mathsf{D}(\Gamma)$.

The decomposition of $\mathsf{D}(\Gamma^*)$ follows similarly, and taking the intersection of (15) and (16) yields (17).

To determine $\mathsf{N}(\Pi_\alpha)$, note that the inclusion $\supset$ is trivial and $\subset$ follows since $\mathsf{R}(\Gamma)$ and $\mathsf{R}(\Gamma^*)$ are orthogonal. For $\mathsf{R}(\Pi_\alpha)$, the inclusion $\subset$ is trivial. On the other hand if $y = \Gamma x_1 + \alpha\Gamma^* x_2$, then $y = \Pi_\alpha(P_1 x_1 + P_2 x_2)$ where $P_1 x_1 + P_2 x_2 \in \mathsf{D}(\Gamma) \cap \mathsf{D}(\Gamma^*)$.

We now show that Π_α and $\alpha^c\Pi_\alpha$ are maximal adjoint operators. First note that $(\Pi_\alpha x, y) = (x, \alpha^c\Pi_\alpha y)$ if $x, y \in \mathsf{D}(\Pi_\alpha)$, i.e. $\mathsf{G}(\Pi_\alpha)$ and $I\mathsf{G}(\alpha^c\Pi_\alpha)$ are orthogonal. To prove that $\mathsf{G}(\Pi_\alpha)^\perp \subset I\mathsf{G}(\alpha^c\Pi_\alpha)$, let $(-z, y)^t \in \mathsf{G}(\Pi_\alpha)^\perp$ and decompose $y = y_1 + y_0 + y_2$ with (14). We see that $y_1 \in \overline{\mathsf{R}(\Gamma^*)} \subset \mathsf{D}(\Gamma^*)$, $y_0 \in \mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma) \subset \mathsf{D}(\Gamma^*) \cap \mathsf{D}(\Gamma)$ and $y_2 \in \overline{\mathsf{R}(\Gamma)} \subset \mathsf{D}(\Gamma)$. To verify that $y_2 \in \mathsf{D}(\Gamma^*)$, let $x \in \mathsf{D}(\Gamma)$ and calculate

$$(\Gamma x, y_2) = (\Pi_\alpha P_1 x, y_2) = (\Pi_\alpha P_1 x, y) = (x, P_1 z).$$

This proves that $(y_2, P_1 z)^t \in (I\mathsf{G}(\Gamma))^\perp = \mathsf{G}(\Gamma^*)$. Similarly it follows that $y_1 \in \mathsf{D}(\Gamma)$ and thus $y \in \mathsf{D}(\Pi_\alpha)$.

That Π_α is closed and densely defined now follows from the adjointness of Π_α and $\alpha^c\Pi_\alpha$ (or can be verified directly). Furthermore, note that for any $\beta \in \mathbf{C}$, $|\beta| = 1$ we have

$$\Pi_\alpha(P_1 + P_0 + \beta^c P_2) = (P_1 + P_0 + \beta^c P_2)\beta\Pi_{(\beta^c)^2\alpha}.$$

The case $\beta = -1$ show that Π_α and $-\Pi_\alpha$ are unitary equivalent, and the case $\beta = \sqrt{\alpha}$ shows that Π_α and $\sqrt{\alpha}\Pi_1$ are unitary equivalent. ☐

Remark 3.6. We have chosen the name "swapping operator" since we have the following mapping diagram

$$\begin{array}{ccccccc} \mathcal{H} & = & \overline{\mathsf{R}(\Gamma^*)} & \oplus & \mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma) & \oplus & \overline{\mathsf{R}(\Gamma)} \\ & & \uparrow{\scriptstyle \Gamma^*} & & & & \downarrow{\scriptstyle \Gamma} \\ \mathcal{H} & = & \overline{\mathsf{R}(\Gamma^*)} & \oplus & \mathsf{N}(\Gamma^*) \cap \mathsf{N}(\Gamma) & \oplus & \overline{\mathsf{R}(\Gamma)}, \end{array}$$

in which Π_α swaps the subspaces $\overline{\mathsf{R}(\Gamma^*)}$ and $\overline{\mathsf{R}(\Gamma)}$.

We now investigate when a nilpotent operator is maximal in the sense that it is "half elliptic". More precisely we make the following definitions.

Definition 3.7. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a closed, densely defined operator between Hilbert spaces. We say that A is a Fredholm operator if the null space $\mathsf{N}(A)$ and the cokernel $\mathcal{H}_2/\mathsf{R}(A)$ are finite dimensional and the range $\mathsf{R}(A)$ is closed (which follows from $\dim(\mathcal{H}_2/\mathsf{R}(A)) < \infty$).

Proposition 3.8. *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a closed, densely defined operator between Hilbert spaces. Then A is a Fredholm operator if and only if there exist bounded operators T_1, $T_2 : \mathcal{H}_2 \to \mathcal{H}_1$ and compact operators $K_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $K_2 : \mathcal{H}_2 \to \mathcal{H}_2$ such that $\mathsf{R}(T_2) \subset \mathsf{D}(A)$ and*

$$\begin{aligned} T_1 A &= I + K_1 \qquad \text{on } \mathsf{D}(A) \subset \mathcal{H}_1, \\ A T_2 &= I + K_2 \qquad \text{on } \mathcal{H}_2. \end{aligned}$$

In this case, the following are equivalent.

- *The embedding $\mathsf{D}(A) \hookrightarrow \mathcal{H}_1$ is compact.*
- *The left inverse T_1 is compact.*
- *The right inverse T_2 is compact.*

The Fredholm inverses T_1 and T_2 satisfies $T_1 + T_1 K_2 = T_2 + K_1 T_2$.

Two references on Fredholm operator theory are Schechter [22] and Kato [13].

Definition 3.9. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a Fredholm operator between Hilbert spaces. We say that A is *diffuse* if its domain $\mathsf{D}(A)$ is compact in $\mathcal{H}_1$, or equivalently if it has a compact Fredholm inverse.

Definition 3.10. Let Γ be a nilpotent operator in a Hilbert space $\mathcal{H}$. We say that Γ is a *Fredholm-nilpotent operator* if the reduced operator

$$\widetilde{\Gamma} : \mathcal{H}/\mathsf{N}(\Gamma) \longrightarrow \mathsf{N}(\Gamma)$$

with domain $\mathsf{D}(\widetilde{\Gamma}) := \mathsf{D}(\Gamma)/\mathsf{N}(\Gamma)$ is a Fredholm operator. If $\widetilde{\Gamma}$ is a diffuse Fredholm operator, then Γ is said to be a *diffuse Fredholm-nilpotent operator*.

Proposition 3.11. *Let Γ and Π_α be as in Proposition 3.5. Then the following are equivalent.*

(i) Γ *is a* Fredholm-nilpotent operator*.*
(i') Γ^* *is a* Fredholm-nilpotent operator*.*

(ii) Π_α *is a* Fredholm operator.

When this holds, Γ *induces a* Hodge type decomposition (or splitting) *of* $\mathcal{H}$*, i.e.*

$$\mathcal{H} = R(\Gamma^*) \oplus (N(\Gamma^*) \cap N(\Gamma)) \oplus R(\Gamma),$$

where the ranges $R(\Gamma^*)$ *and* $R(\Gamma)$ *are closed and* $N(\Gamma^*) \cap N(\Gamma)$ *is finite dimensional. If in addition* $N(\Gamma^*) \cap N(\Gamma) = \{0\}$*, then the splitting is said to be exact.*

The equivalence of (i), (i′) and (ii) remains true if "Fredholm(-nilpotent) operator" is replaced by "diffuse Fredholm(-nilpotent) operator". In this case, we also have the following.

(iii) *The spectrum* $\sigma(\Pi_\alpha)$ *is a discrete set consisting of eigenvalues only.*
(iv) *If* Γ_0 *is a bounded, nilpotent operator such that* $\Gamma\Gamma_0 + \Gamma_0\Gamma = 0$ *on* $D(\Gamma) = D(\Gamma + \Gamma_0)$*, then the perturbed operator* $\Gamma + \Gamma_0$ *is also a diffuse Fredholm-nilpotent operator.*

Proof. Split $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2$ as in the proof of Proposition 3.5.

Theorem 3.3 shows that (i) and (i') are equivalent since

$$\mathcal{H}_0 = \mathsf{N}(\Gamma) \cap \mathsf{R}(\Gamma)^\perp = \mathsf{N}(\Gamma^*) \cap \mathsf{R}(\Gamma^*)^\perp.$$

Furthermore, since $\Gamma : \mathcal{H}_1 \to \mathcal{H}_2$ and $\Gamma^* : \mathcal{H}_2 \to \mathcal{H}_1$ are adjoint operators, it follows that $(\Gamma^*)^{-1} = (\Gamma^{-1})^* : \mathcal{H}_1 \to \mathcal{H}_2$ is compact if and only if $\Gamma^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ is. Therefore Γ is a diffuse Fredholm-nilpotent operator if and only if Γ^* is.

Note that since $\mathsf{R}(\Gamma)$ and $\mathsf{R}(\Gamma^*)$ are orthogonal, they are both closed if and only if $\mathsf{R}(\Pi_\alpha) = \mathsf{R}(\Gamma^*) \oplus \mathsf{R}(\Gamma)$ is closed. As $\mathsf{N}(\Pi_\alpha) = \mathcal{H}_0 = \mathcal{H}/\overline{\mathsf{R}(\Pi_\alpha)}$ it follows that (i) and (i') are equivalent with (ii). Moreover $\mathsf{D}(\Pi_\alpha) = \mathsf{D}(\widetilde{\Gamma^*}) \oplus \mathcal{H}_0 \oplus \mathsf{D}(\widetilde{\Gamma})$, so Π_α is a diffuse Fredholm operator if and only if both Γ and Γ^* are diffuse Fredholm-nilpotent operators.

The discreteness result (iii) follows from the identity

$$(\sqrt{-\alpha} - \lambda)^{-1} - (\sqrt{-\alpha} - \Pi_\alpha)^{-1} = (\sqrt{-\alpha} - \lambda)^{-1}(\lambda - \Pi_\alpha)(\sqrt{-\alpha} - \Pi_\alpha)^{-1}$$

which shows that $(\lambda - \Pi_\alpha)$ fails to be invertible if and only if $(\sqrt{-\alpha} - \lambda)^{-1} \in \sigma(K)$. But since $K := (\sqrt{-\alpha} - \Pi_\alpha)^{-1}$ is compact, its spectrum is discrete.

To prove (iv), let Π_α and Π'_α be swapping operators corresponding to Γ and $\Gamma + \Gamma_0$. Then $\mathsf{D}(\Pi'_\alpha) = \mathsf{D}(\Pi_\alpha)$ is compactly embedded in $\mathcal{H}$. Lemma 3.14 below now shows that $\Gamma + \Gamma_0$ is a diffuse Fredholm-nilpotent operator. □

Remark 3.12. An important observation here is that the statement (i) is independent of which Hilbert norm on $\mathcal{H}$ we are using (as long as it induces the same topology), whereas in (i') and (ii), the adjoint operator Γ^* and Π_α depends on the scalar product.

We finish this section with two techniques to establish Fredholm-nilpotence of a given nilpotent operator Γ.

Lemma 3.13. *Let $\mathcal{H}$ and $\mathcal{H}_0$ be two Hilbert spaces and consider the diagram*

$$\begin{array}{ccc} \mathcal{H} & \overset{T}{\underset{S}{\rightleftarrows}} & \mathcal{H}_0 \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma_0} \\ \mathcal{H} & \overset{T}{\underset{S}{\rightleftarrows}} & \mathcal{H}_0, \end{array}$$

where Γ and Γ_0 are closed, densely defined operators in $\mathcal{H}$ and $\mathcal{H}_0$ respectively and T and S are bounded maps such that $TS = I_{\mathcal{H}}$. If $T\Gamma_0 \subset \Gamma T$ and $S\Gamma \subset \Gamma_0 S$, then we have the following.

(i) *If Γ_0 is a nilpotent operator, then so is Γ.*
(ii) *If Γ_0 is a Fredholm-nilpotent operator, then so is Γ.*
(ii) *If Γ_0 is a diffuse Fredholm-nilpotent operator, then so is Γ.*

The proof of this intertwining lemma is straightforward and we omit it.

Lemma 3.14. *Let Π_α be a swapping operator as in Proposition 3.5. If the embedding $\mathsf{D}(\Pi_\alpha) \hookrightarrow \mathcal{H}$ is compact, then Π_α is a diffuse Fredholm operator with index zero.*

Proof. Consider the operators

$$\lambda I - \Pi_\alpha : \mathsf{D}(\Pi_\alpha) \longrightarrow \mathcal{H}. \tag{18}$$

Since $\sigma(\Pi_\alpha) \subset \sqrt{\alpha}\mathbf{R}$ by Proposition 3.5, (18) is an isomorphism when $\lambda \notin \sqrt{\alpha}\mathbf{R}$. Now observe that $\lambda I : \mathsf{D}(\Pi_\alpha) \to \mathcal{H}$ is a compact operator. Thus $\Pi_\alpha : \mathsf{D}(\Pi_\alpha) \to \mathcal{H}$ is a Fredholm operator with index zero. □

4. Hodge decompositions for d and δ

In this section we apply the general theory for nilpotent operators from Section 3 to the following d and δ operators in a bounded weakly Lipschitz domain Ω.

Definition 4.1. (i) Let d_Ω and δ_Ω be the closed, nilpotent d and δ operators (without boundary conditions) in $L_2(\Omega; \wedge)$ with natural domains, i.e.

$$\mathsf{D}(d_\Omega) := \{F \in L_2(\Omega; \wedge) \, ; \, dF \in L_2(\Omega; \wedge)\},$$

and similarly for δ_Ω.

(ii) Let $d_{\overline{\Omega}}$ (d with normal boundary conditions) and $\delta_{\overline{\Omega}}$ (δ with tangential boundary conditions) be the closed, nilpotent d and δ operators in Ω with domains

$$\mathsf{D}(d_{\overline{\Omega}}) := \{F \in L_2(\Omega; \wedge) \, ; \, d(F_z) \in L_2(\mathbf{R}^n; \wedge)\},$$
$$\mathsf{D}(\delta_{\overline{\Omega}}) := \{F \in L_2(\Omega; \wedge) \, ; \, \delta(F_z) \in L_2(\mathbf{R}^n; \wedge)\},$$

where $F_z \in L_2(\mathbf{R}^n; \wedge)$ denotes the zero-extension of F to $\mathbf{R}^n$.

Remark 4.2. If $F \in \mathsf{D}(d_{\overline{\Omega}})$, then F is normal to Σ. The nilpotence of $d_{\overline{\Omega}}$ shows that not only is $d_{\overline{\Omega}}F$ curl free, it is also normal to Σ. Similarly, if $F \in \mathsf{D}(\delta_{\overline{\Omega}})$, then F is tangential to Σ. The nilpotence of $\delta_{\overline{\Omega}}$ shows that not only is $\delta_{\overline{\Omega}}F$ divergence free, it is also tangential to Σ.

Examples of nilpotent operators considered in this paper are, for each wave number $k \in \mathbf{C}$, the four operators

$$d_{k,\Omega} = d_\Omega + ke_0\wedge, \qquad \delta_{k,\Omega} = \delta_\Omega + ke_0 \lrcorner,$$
$$d_{k,\overline{\Omega}} = d_{\overline{\Omega}} + ke_0\wedge, \qquad \delta_{k,\overline{\Omega}} = \delta_{\overline{\Omega}} + ke_0 \lrcorner\,.$$

Obviously we have $d_{k,\overline{\Omega}} \subset d_{k,\Omega}$ and $\delta_{k,\overline{\Omega}} \subset \delta_{k,\Omega}$.

Proposition 4.3. *The operators d_Ω and $d_{\overline{\Omega}}$ have cores (i.e. a subset of the domain which is dense in graph norm)*

$$C_0^\infty(\mathbf{R}^n;\wedge)|_\Omega \subset \mathsf{D}(d_\Omega), \qquad C_0^\infty(\Omega;\wedge) \subset \mathsf{D}(d_{\overline{\Omega}})$$

respectively. In particular, the inclusions

$$d(C_0^\infty(\mathbf{R}^n;\wedge)|_\Omega) \subset \mathsf{R}(d_\Omega), \qquad d(C_0^\infty(\Omega;\wedge)) \subset \mathsf{R}(d_{\overline{\Omega}})$$

are dense. We also have dense subspaces

$$\{F|_\Omega\ ;\ F \in C_0^\infty(\mathbf{R}^n;\wedge),\ supp\, dF \subset\subset \Omega^-\} \subset \mathsf{N}(d_\Omega),$$
$$\{F \in C_0^\infty(\Omega;\wedge)\ ;\ dF = 0\} \subset \mathsf{N}(d_{\overline{\Omega}}).$$

The same holds true when d is replaced by δ.

Before giving the proof, we note the following important corollary.

Corollary 4.4. *The two operators $d_{\overline{\Omega}}$ and $-\delta_\Omega$ are adjoint in the sense of Definition 3.2 and so are d_Ω and $-\delta_{\overline{\Omega}}$. For the zero order perturbations we have $d^*_{k,\overline{\Omega}} = -\delta_{-k^c,\Omega}$ and $d^*_{k,\Omega} = -\delta_{-k^c,\overline{\Omega}}$.*

Proof. Consider the first pair. By Definition 3.2 we need to prove that $\mathsf{G}(d^*_{\overline{\Omega}}) = \mathsf{G}(-\delta_\Omega)$, where $\mathsf{G}(d^*_{\overline{\Omega}}) = I\mathsf{G}(d_{\overline{\Omega}})^\perp$.

To show $d^*_{\overline{\Omega}} \subset -\delta_\Omega$, let $(U,F)^t \in \mathsf{G}(d^*_{\overline{\Omega}})$. Then in particular

$$\int_\Omega \langle U, d\Phi^c\rangle = \int_\Omega \langle F, \Phi^c\rangle \quad \text{for all } \Phi \in C_0^\infty(\Omega) \subset \mathsf{D}(d_{\overline{\Omega}}).$$

Thus $-\delta_\Omega U = F \in L_2(\Omega;\wedge)$ in distribution sense, which proves $(U,F)^t \in \mathsf{G}(-\delta_\Omega)$.

Conversely, to show $d^*_{\overline{\Omega}} \supset -\delta_\Omega$, by Proposition 4.3 it suffices to prove that $I\mathsf{G}(d_{\overline{\Omega}})$ and $\{(\Phi, -\delta_\Omega\Phi)^t\ ;\ \Phi \in C_0^\infty(\mathbf{R}^n;\wedge)|_\Omega\}$ are orthogonal. If $U \in \mathsf{D}(d_{\overline{\Omega}})$ and $\Phi \in C_0^\infty(\mathbf{R}^n;\wedge)|_\Omega$, we get

$$\int_\Omega \langle \Phi, d_{\overline{\Omega}}U^c\rangle = \int_{\mathbf{R}^n} \langle \Phi, dU^c_z\rangle = \int_{\mathbf{R}^n} \langle -\delta\Phi, U^c_z\rangle = \int_\Omega \langle -\delta_\Omega\Phi, U^c\rangle.$$

This shows that $d_{\overline{\Omega}}$ and $-\delta_\Omega$ are adjoint. The adjointness of d_Ω and $-\delta_{\overline{\Omega}}$ follows similarly. Moreover, since $(A+T)^* = A^* + T^*$ whenever A is a closed, densely defined operator and T is a bounded operator, this proves the rest of the corollary. $\square$

To prove Proposition 4.3, we use *Lie flows* $t \mapsto \alpha_t^*$ and $t \mapsto \tilde{\alpha}_{t*}^{-1}$ constructed as follows.

Lemma 4.5. *There exists a family $\alpha_t : \mathbf{R}^n \to \mathbf{R}^n$ of bilipschitz maps (all being identity outside a compact set), $|t| < T$, with the following properties.*

$$\begin{aligned} \overline{\alpha_t(\Omega)} &\subset \Omega, && 0 < t < T, \\ \alpha_t(\Omega) &\supset \overline{\Omega}, && -T < t < 0, \\ \|\alpha_t^* F - F\|_{L_2(\mathbf{R}^n;\wedge)} &\longrightarrow 0, && t \longrightarrow 0, \quad F \in L_2(\mathbf{R}^n;\wedge), \\ \|\tilde{\alpha}_{t*}^{-1} F - F\|_{L_2(\mathbf{R}^n;\wedge)} &\longrightarrow 0, && t \longrightarrow 0, \quad F \in L_2(\mathbf{R}^n;\wedge). \end{aligned}$$

Proof. Let $\overline{\Omega} \subset \cup_{j=0}^N V_j$ and $\rho_j : \mathbf{R}^n \to \mathbf{R}^n$, $j = 1 \ldots N$, be the local bilipschitz parametrisations from Definition 2.1 and let $V_0 \subset\subset \Omega$ be contained in the interior. Let $B_1 := B(0,1) \subset B_2 := B(0,2)$ be concentric balls. We may assume that $\Sigma \subset \cup_1^N \rho_j B_1$ and that $\rho_j B_2 \subset\subset V_j$, $j = 1 \ldots N$. Let $\eta \in C_0^\infty(\mathbf{R}^n)$ be such that $0 \le \eta \le 1$, $\eta|_{B_1} = 1$ and $\eta|_{\mathbf{R}^n \setminus B_2} = 0$. Define for $|t| \le T$ the translation map

$$\beta_t : \mathbf{R}^n \longrightarrow \mathbf{R}^n : x \longmapsto x + \eta(|x|) t e_n,$$

let $\alpha_t^j := \rho_j \circ \beta_t \circ \rho_j^{-1}$ and extend to identity outside V_j and let $\alpha_t := \alpha_t^N \circ \ldots \circ \alpha_t^1$. Obviously the required mapping properties for α_t holds.

For the Lie pullback flow, if $\alpha_t^* F = (\alpha_t^1)^* G$ where $G := (\alpha_t^2)^* \ldots (\alpha_t^N)^* F$ then

$$\begin{aligned} \|\alpha_t^* F - F\|_{L_2(\mathbf{R}^n;\wedge)} &\le \|(\alpha_t^1)^* G - F\|_{L_2(V_1;\wedge)} + \|(\alpha_t^1)^* G - F\|_{L_2(V_1^c;\wedge)} \\ &\le \|(\alpha_t^1)^* G - G\|_{L_2(V_1;\wedge)} + \|G - F\|_{L_2(V_1;\wedge)} + \|G - F\|_{L_2(V_1^c;\wedge)}. \end{aligned}$$

Thus it suffices to show that $\|(\alpha_t^j)^* F - F\|_{L_2(V_j;\wedge)} \to 0$. But (essentially) since translation is L_2 continuous, this follows from

$$\|(\alpha_t^j)^* F - F\|_{L_2(V_j;\wedge)} \lesssim \|(\beta_t)^*(\rho_j^* F) - (\rho_j^* F)\|_{L_2(U_j;\wedge)} \longrightarrow 0.$$

The proof of the L_2-continuity of the reduced push forward flow is similar. $\square$

Proof of Proposition 4.3. Let $\eta_s(x) := s^{-n}\eta(x/s)$ be a mollifier. For $F \in \mathsf{D}(d_\Omega)$, let $0 < s << t$ and define approximating fields

$$F_{s,t} := \eta_s * (\alpha_t^* F_z).$$

Then $\|F_{s,t} - F\|_{\mathsf{D}(d_\Omega)} \approx \|F_{s,t} - F\|_{L_2} + \|(dF)_{s,t} - dF\|_{L_2} \to 0$ as $s, t \to 0$. Furthermore, if $d_\Omega F = 0$ and $0 < s << t$, then $dF_{s,t} = 0$ in a neighbourhood of $\overline{\Omega}$, due to Proposition 2.6.

On the other hand for $F \in \mathsf{D}(d_{\overline{\Omega}})$, let $0 < s << -t$ and define approximating fields $F_{s,t}$ as above. Then $F_{s,t} \in C_0^\infty(\Omega;\wedge)$ if $0 < s << -t$ and $\|F_{s,t} - F\|_{\mathsf{D}(d_{\overline{\Omega}})} \to 0$ as $s, t \to 0$. Furthermore, if $d_{\overline{\Omega}} F = 0$ then $d_{\overline{\Omega}} F_{s,t} = 0$. $\square$

Example 4.6. Using Proposition 3.5, we form the corresponding swapping operators Π_{-1}.

(i) If $\Gamma = d_{\overline{\Omega}}$, then the (Hodge–)Dirac operator on Ω with *normal boundary conditions* is

$$\mathbf{D}_{\Omega^\perp} := d_{\overline{\Omega}} + \delta_\Omega.$$

Note that $\mathbf{D}^2_{\Omega^\perp} = d_{\overline{\Omega}}\delta_\Omega + \delta_\Omega d_{\overline{\Omega}}$ is the Hodge–Laplace operator with relative (generalised Dirichlet) boundary conditions. For a scalar function $U : \Omega \to \wedge^0$, we have $\mathbf{D}^2_{\Omega^\perp} U = \delta_\Omega d_{\overline{\Omega}} U$, and $U \in \mathsf{D}(d_{\overline{\Omega}})$ incorporates the boundary condition $U|_\Sigma = 0$ since all scalars are tangential.

(ii) If $\Gamma = d_\Omega$, then the (Hodge–)Dirac operator on Ω with *tangential boundary conditions* is

$$\mathbf{D}_{\Omega^\parallel} := d_\Omega + \delta_{\overline{\Omega}}.$$

Here the Hodge–Laplace operator with absolute (generalised Neumann) boundary conditions is $\mathbf{D}^2_{\Omega^\parallel} = d_\Omega\delta_{\overline{\Omega}} + \delta_{\overline{\Omega}} d_\Omega$. Note that for a scalar function $U : \Omega \to \wedge^0$, we have $\mathbf{D}^2_{\Omega^\perp} U = \delta_{\overline{\Omega}} d_\Omega U$, and $d_\Omega U \in \mathsf{D}(\delta_{\overline{\Omega}})$ incorporates the boundary condition $\frac{\partial U}{\partial \nu} = \langle \nu, \nabla U|_\Sigma \rangle = 0$.

(iii) As is well known, the null spaces

$$\mathsf{N}(\mathbf{D}_{\Omega^\perp}) = \mathsf{N}(d_{\overline{\Omega}}) \cap \mathsf{N}(\delta_\Omega) = \mathsf{N}(\mathbf{D}^2_{\Omega^\perp}),$$
$$\mathsf{N}(\mathbf{D}_{\Omega^\parallel}) = \mathsf{N}(d_\Omega) \cap \mathsf{N}(\delta_{\overline{\Omega}}) = \mathsf{N}(\mathbf{D}^2_{\Omega^\parallel})$$

of the Dirac operators $\mathbf{D}_{\Omega^\perp}$ and $\mathbf{D}_{\Omega^\parallel}$ can be identified with the de Rham cohomology spaces of Ω with normal (relative) and tangential (absolute) boundary conditions, and are thus determined by the global topology of Ω. However, we are here mainly interested in how the local regularity of the boundary Σ influences the Fredholm properties of the Dirac operators $\mathbf{D}_{\Omega^\perp}$ and $\mathbf{D}_{\Omega^\parallel}$.

We now turn to the proof of Theorem 1.3 and show how to reduce to the case $\Omega = B$, where B is the unit ball in $\mathbf{R}^n$. The proof that $\mathbf{D}_{B^\perp}$ and $\mathbf{D}_{\Omega^\parallel}$ are diffuse Fredholm operators is deferred to the end of this section. It follows from either Theorem 4.10, Theorem 4.13 (combined with Lemma 3.14) or Theorem 4.17 (combined with Proposition 3.11).

Proof of Theorem 1.3. (i) We first consider the unperturbed case $k_1 = k_2 = 0$. By Proposition 3.11 it suffices to show that $\mathbf{D}_{\Omega^\perp}$ is a diffuse Fredholm operator. Using Definition 2.1 we see that there exist bilipschitz maps $\rho_j : B \to \Omega_j$, $j = 1, \ldots, N$, where B denotes the open unit ball in $\mathbf{R}^n$, such that $\Omega = \bigcup_{j=1}^N \Omega_j$. Furthermore we may assume that ρ_j extends to a bilipschitz map between slightly larger open sets. Choose a smooth partition of unity $\{\eta_j\}$ such that $\operatorname{supp}\eta_j \subset\subset \mathbf{R}^n \setminus \overline{(\Omega \setminus \Omega_j)}$ and $\sum \eta_j^2 = 1$ on Ω.

Assuming that $\mathbf{D}_{B^\perp}$ is a diffuse Fredholm operator, it follows from Proposition 3.11 that $d_{\overline{B}}$ is a diffuse Fredholm-nilpotent operator. We may now apply Lemma 3.13 with $\mathcal{H} = L_2(\Omega_j; \wedge)$, $\mathcal{H}_0 = L_2(B; \wedge)$, $A = d_{\overline{\Omega_j}}$, $A_0 = d_{\overline{B}}$, $T = (\rho_j^{-1})^*$

and $S = \rho_j^* = T^{-1}$, since Proposition 2.6 proves that T and S intertwine A and A_0. This shows that $d_{\overline{\Omega}_j}$ is a diffuse Fredholm-nilpotent operator.

Applying Proposition 3.11 again with $\Gamma = d_{\overline{\Omega}_j}$ shows that $\mathbf{D}_{\Omega_j^\perp}$ is a diffuse Fredholm operator. Localising, we can now prove that the Dirac operator $\mathbf{D}_{\Omega^\perp}$ is a diffuse Fredholm operator. Indeed, if T_j are compact Fredholm inverses to $\mathbf{D}_{\Omega_j^\perp}$ respectively as in Proposition 3.8, then a compact Fredholm inverse to $\mathbf{D}_{\Omega^\perp}$ is

$$T(F) := \sum_j \eta_j T_j(\eta_j F).$$

Similarly one can show that the Dirac operator $\mathbf{D}_{\Omega^\parallel}$ is a diffuse Fredholm operator.

(ii) To prove that the map (2) is a diffuse Fredholm map for general k_1 and k_2, note that (i) above and Proposition 3.11(iv) with $\Gamma + \Gamma_0 = d_{-k_2^c,\overline{\Omega}}$ and $\Gamma^* + \Gamma_0^* = -\delta_{k_2,\Omega}$ shows that we have a Hodge decomposition

$$L_2(\Omega; \wedge) = \mathsf{R}(\delta_{k_2,\Omega}) \oplus \big(\mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{-k_2^c,\overline{\Omega}})\big) \oplus \mathsf{R}(d_{-k_2^c,\overline{\Omega}})$$

and that $\delta_{k_2,\Omega} : \mathsf{N}(d_{-k_2^c,\overline{\Omega}}) \to \mathsf{N}(\delta_{k_2,\Omega})$ is a diffuse Fredholm map. In particular $\mathsf{N}(\delta_{k_2,\Omega})/\mathsf{R}(\delta_{k_2,\Omega})$ is finite dimensional, and similarly $\mathsf{N}(d_{k_1,\overline{\Omega}})/\mathsf{R}(d_{k_1,\overline{\Omega}})$ is finite dimensional. Thus it suffices to prove that

$$\delta_{k_2,\Omega} : \mathsf{R}(d_{k_1,\overline{\Omega}}) \longrightarrow \mathsf{R}(\delta_{k_2,\Omega})$$

is a diffuse Fredholm map. Consider the following diagram

$$\begin{array}{ccccc}
\mathsf{R}(d_{k_1,\overline{\Omega}}) & \xrightarrow{P_1} & \mathsf{R}(d_{-k_2^c,\overline{\Omega}}) & \xrightarrow{\delta_{k_2,\Omega}} & \mathsf{R}(\delta_{k_2,\Omega}) \\
& \searrow^{P_2} & \oplus & & \\
& & \mathsf{N}(\delta_{k_2,\Omega}) & \xrightarrow{\delta_{k_2,\Omega}} & 0,
\end{array}$$

where P_1 and P_2 denotes the associated orthogonal projections. To show that $\delta_{k_2,\Omega} P_1|_{\mathsf{R}(d_{k_1,\overline{\Omega}})} = \delta_{k_2,\Omega}|_{\mathsf{R}(d_{k_1,\overline{\Omega}})}$ is a diffuse Fredholm map, we first prove a priori estimates for $P_1|_{\mathsf{R}(d_{k_1,\overline{\Omega}})}$. Note that (i) above and Proposition 3.11(iv) show that any $F \in \mathsf{R}(d_{k_1,\overline{\Omega}})$ has a potential $F = d_{k_1,\overline{\Omega}} U$ where the map $F \mapsto U$ is compact. This gives

$$\|F\|^2 = \int_\Omega \langle P_1 F, F^c\rangle + \langle P_2 F, (d_{k_1,\overline{\Omega}} U)^c\rangle = \int_\Omega \langle P_1 F, F^c\rangle - (k_1^c + k_2)\langle e_0 \lrcorner F, U^c\rangle.$$

Dividing by $\|F\|$ gives the a priori estimate $\|F\| \lesssim \|P_1 F\| + \|U\|$. This shows that $\delta_{k_2,\Omega}(\mathsf{N}(d_{k_1,\overline{\Omega}}))$ is closed and that $\mathsf{N}(\delta_{k_2,\Omega}|_{\mathsf{N}(d_{k_1,\overline{\Omega}})})$ is finite dimensional. Now Lemma 4.7 below shows that the cokernel $\mathsf{N}(\delta_{k_2,\Omega}) \ominus \delta_{k_2,\Omega}\mathsf{N}(d_{k_1,\overline{\Omega}})$ is finite dimensional, which completes the proof. □

Lemma 4.7. *The deficiency indices of the maps (2) and (3) are*

$$\alpha(\delta_{k_2,\Omega}|_{N(d_{k_1,\overline{\Omega}})}) = \dim(N(\delta_{k_2,\Omega}) \cap N(d_{k_1,\overline{\Omega}}))$$
$$\beta(\delta_{k_2,\Omega}|_{N(d_{k_1,\overline{\Omega}})}) = \dim(N(\delta_{k_2,\Omega}) \cap N(d_{-k_2^c,\overline{\Omega}})) + \dim(R(\delta_{-k_1^c,\Omega}) \cap R(d_{-k_2^c,\overline{\Omega}}))$$
$$\alpha(d_{k_1,\overline{\Omega}}|_{N(\delta_{k_2,\Omega})}) = \dim(N(\delta_{k_2,\Omega}) \cap N(d_{k_1,\overline{\Omega}}))$$
$$\beta(d_{k_1,\overline{\Omega}}|_{N(\delta_{k_2,\Omega})}) = \dim(N(\delta_{-k_2^c,\Omega}) \cap N(d_{k_1,\overline{\Omega}})) + \dim(R(\delta_{-k_1^c,\Omega}) \cap R(d_{-k_2^c,\overline{\Omega}})).$$

For any k_1 and k_2 these indices are finite. Moreover, if the wave numbers are non zero and $\arg(k_1) + \arg(k_2) \neq 0$ *mod* 2π*, then* $N(\delta_{k_2,\Omega}) \cap N(d_{k_1,\overline{\Omega}}) = \{0\}$.

Proof. (i) Using Theorem 3.3 we get identities

$$\begin{aligned} \mathsf{N}(\delta_{k_2,\Omega}|_{\mathsf{N}(d_{k_1,\overline{\Omega}})}) &= \mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}}) \\ \mathsf{N}(\delta_{k_2,\Omega}) \ominus \delta_{k_2,\Omega}\mathsf{N}(d_{k_1,\overline{\Omega}}) &= \mathsf{N}(\delta_{k_2,\Omega}) \cap d^{-1}_{-k_2^c,\overline{\Omega}}\mathsf{R}(\delta_{-k_1^c,\Omega}) \\ &= \mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{-k_2^c,\overline{\Omega}}) \oplus \mathsf{R}(\delta_{k_2,\Omega}) \cap d^{-1}_{-k_2^c,\overline{\Omega}}\mathsf{R}(\delta_{-k_1^c,\Omega}), \end{aligned}$$

which gives the deficiency indices for $\delta_{k_2,\Omega}|_{\mathsf{N}(d_{k_1,\overline{\Omega}})}$. Similarly for $d_{k_1,\overline{\Omega}}|_{\mathsf{N}(\delta_{k_2,\Omega})}$.

(ii) The a priori estimate in (ii) in the proof of Theorem 1.3 above shows that $\dim(\mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}})) < \infty$ for all $k_1, k_2 \in \mathbf{C}$. To prove that the space $\mathsf{N}(\delta_{k_2,\Omega}) \cap \mathsf{N}(d_{k_1,\overline{\Omega}})$ vanishes unless k_1 and k_2^c have the same direction, write $\Gamma = d_{\overline{\Omega}}$, $\Gamma^* = -\delta_\Omega$, $\Gamma_0 = e_0 \wedge (\cdot)$ and $\Gamma_0^* = e_0 \lrcorner (\cdot)$. The algebraic property we use here is that not only is $\Gamma\Gamma_0 + \Gamma_0\Gamma = 0$ but also $\Gamma^*\Gamma_0 + \Gamma_0\Gamma^* = 0$, which follows from the derivation property (9). Assuming $(\Gamma + k_1\Gamma_0)F = (-\Gamma^* + k_2\Gamma_0^*)F = 0$, we calculate

$$\begin{aligned} 0 = (F, (\Gamma^*\Gamma_0 + \Gamma_0\Gamma^*)F) &= (\Gamma F, \Gamma_0 F) + (\Gamma_0^* F, \Gamma^* F) \\ &= -k_1\|\Gamma_0 F\|^2 + k_2^c\|\Gamma_0^* F\|^2. \end{aligned}$$

This shows that $F = 0$ under the assumptions on k_1 and k_2 since Γ_0 induces an exact Hodge decomposition. □

Another way of reducing to the case of a smooth domain with Lemma 3.13 is to use the extension maps constructed in Proposition 4.8 below. Let $B \subset \mathbf{R}^n$ be a ball containing Ω, let $\chi_\Omega : L_2(B; \wedge) \to L_2(\Omega; \wedge)$ be the restriction map and pick a δ-extension map $\varepsilon_\Omega : L_2(\Omega; \wedge) \to L_2(B; \wedge)$ as in Proposition 4.8 below. Then, modulo a partition of unity, Lemma 3.13 applies with $\mathcal{H} = L_2(\Omega; \wedge)$, $\mathcal{H}_0 = L_2(B; \wedge)$, $A = \delta_\Omega$, $A_0 = \delta_B$, $T = \chi_\Omega$ and $S = \varepsilon_\Omega$.

Proposition 4.8. *Let $\chi_\Omega : L_2(\mathbf{R}^n; \wedge) \to L_2(\Omega; \wedge)$ be the restriction operator and let $K \supset \Omega$ be a compact set. Then there exists a bounded extension operator $\varepsilon_\Omega : L_2(\Omega; \wedge) \to L_2(\mathbf{R}^n; \wedge)$ such that*

(i) $\chi_\Omega\varepsilon_\Omega =$ *identity on* $L_2(\Omega; \wedge)$.
(ii) *supp* $(\varepsilon_\Omega F) \subset K$ *for all* $F \in L_2(\Omega; \wedge)$.
(iii) $d_{\mathbf{R}^n}\varepsilon_\Omega - \varepsilon_\Omega d_\Omega$ *extends to an* $L_2(\Omega; \wedge) \to L_2(\mathbf{R}^n; \wedge)$ *bounded map. In particular,* ε_Ω *restricts to a bounded map*

$$\varepsilon_\Omega : D(d_\Omega) \to D(d_{\mathbf{R}^n}).$$

(iv) $\varepsilon_\Omega(F_1 + e_0 \wedge F_2) = \varepsilon_\Omega F_1 + e_0 \wedge \varepsilon_\Omega F_2$.

The same holds true when d is replaced by δ.

Proof. Let $\overline{\Omega} \subset \cup_{j=0}^N V_j$, $\eta_j \in C_0^\infty(V_j)$, $\sum_0^N \eta_j|_\Omega = 1$ and let $\rho_j : \mathbf{R}^n \to \mathbf{R}^n$, $j = 1 \dots N$, be the local bilipschitz parametrisations from Definition 2.1 and $V_0 \subset\subset \Omega$ be contained in the interior.

(i) We first note that it suffices to construct an extension map $\varepsilon : L_2(\mathbf{R}^n_+; \wedge) \to L_2(\mathbf{R}^n; \wedge)$ acting on fields supported in $(\rho_j^{-1}\text{supp}\,\eta_j) \cap \overline{\mathbf{R}^n_+}$. Indeed, this gives local extension maps $\varepsilon_j := (\rho_j^{-1})^* \varepsilon \rho_j^* : L_2(V_j \cap \Omega; \wedge) \to L_2(V_j; \wedge)$, extending fields supported in $\text{supp}\,\eta_j \cap \overline{\Omega}$ to fields compactly supported in V_j, $j = 1, \dots, N$. Then we can construct ε_Ω as

$$\varepsilon_\Omega F := \eta_0 F + \sum_{j=1}^N \varepsilon_j(\eta_j F). \tag{19}$$

Moreover, from the construction of ε below and Proposition 2.6, d commutes with ε_j and thus

$$(d_{\mathbf{R}^n}\varepsilon_\Omega - \varepsilon_\Omega d_\Omega)F = (d\eta_0) \wedge F + \sum_{j=1}^N \varepsilon_j((d\eta_j) \wedge F). \tag{20}$$

Clearly, both (19) and (20) define L_2-bounded operators.

(ii) To construct the extension map $\varepsilon : L_2(\mathbf{R}^n_+; \wedge) \to L_2(\mathbf{R}^n; \wedge)$, consider the stretched reflections

$$r_k : (x', -x_n) \longmapsto (x', kx_n).$$

By Proposition 4.3, it suffices to consider $G \in C_0^\infty(\mathbf{R}^n; \wedge)|_{\mathbf{R}^n_+}$. If we decompose $G(x) = G_1(x) + e_n \wedge G_2(x)$, $e_n \lrcorner G_i = 0$, into parts tangential and normal to $\mathbf{R}^{n-1}$, then the pullbacks are given by $r_k^* G(x', -x_n) = G_1(x', kx_n) - ke_n \wedge G_2(x', kx_n)$, and we see that both tangential and normal parts of the field

$$\widetilde{G} := \begin{cases} G, & x_n > 0, \\ 3r_1^*G - 2r_2^*G, & x_n < 0, \end{cases}$$

are continuous across Σ. We can assume that $\text{supp}\,\eta_j$ is small enough so that $\text{supp}\,\widetilde{G} \subset \rho_j^{-1}V_j$ if $\text{supp}\,G \subset (\rho_j^{-1}\text{supp}\,\eta_j) \cap \overline{\mathbf{R}^n_+}$. Now define $\varepsilon := 3r_1^* - 2r_2^*$.

The proof for δ is analogous. We here use the reduced pushforwards $(\tilde{r}_{k*}^{-1})F = -kF_1 + e_n \wedge F_2$. □

Remark 4.9. (i) We see that in a natural way $\mathsf{D}(d_{\overline{\Omega}}) \subset \mathsf{D}(d_{\mathbf{R}^n})$ and

$$\mathsf{D}(d_\Omega) = \mathsf{D}(d_{\mathbf{R}^n})/\mathsf{N}(\chi_\Omega).$$

Proposition 4.8 shows that $\chi_\Omega : \mathsf{D}(d_{\mathbf{R}^n}) \to \mathsf{D}(d_\Omega)$ is surjective and that $\varepsilon_\Omega : \mathsf{D}(d_\Omega) \to \mathsf{D}(d_{\mathbf{R}^n})$ embeds $\mathsf{D}(d_\Omega)$ as a complement of $\mathsf{N}(\chi_\Omega)$ in $\mathsf{D}(d_{\mathbf{R}^n})$.

(ii) From expressions (19) and (20) we obtain norm estimates

$$\|\varepsilon_\Omega\|_{L_2(\Omega;\wedge)\to L_2(\mathbf{R}^n;\wedge)} \lesssim 1+\sum_{j=1}^N\Big(\sup_{x\in\rho_j^{-1}V_j}\frac{\|\underline{\rho}_j(x)\|_{\text{op}}}{\sqrt{J(\rho_j)(x)}}\Big)\Big(\sup_{y\in V_j}\frac{\|\underline{\rho}_j^{-1}(y)\|_{\text{op}}}{\sqrt{J(\rho_j^{-1})(y)}}\Big),$$

$$\|[d,\varepsilon_\Omega]\|_{L_2(\Omega;\wedge)\to L_2(\mathbf{R}^n;\wedge)} \lesssim \|\nabla\eta_0\|_\infty+$$
$$\sum_{j=1}^N\Big(\sup_{x\in\rho_j^{-1}V_j}\frac{\|\underline{\rho}_j(x)\|_{\text{op}}}{\sqrt{J(\rho_j)(x)}}\Big)\Big(\sup_{y\in V_j}\frac{\|\underline{\rho}_j^{-1}(y)\|_{\text{op}}}{\sqrt{J(\rho_j^{-1})(y)}}\Big)\|\nabla\eta_j\|_\infty,$$

$$\|\varepsilon_\Omega\|_{\mathsf{D}(d_\Omega)\to\mathsf{D}(d_{\mathbf{R}^n})} \le \|\varepsilon_\Omega\|_{L_2(\Omega;\wedge)\to L_2(\mathbf{R}^n;\wedge)} + \|[d,\varepsilon_\Omega]\|_{L_2(\Omega;\wedge)\to L_2(\mathbf{R}^n;\wedge)},$$

where $\underline{\rho}_j(x):\wedge\to\wedge$ denotes the $\wedge$-homomorphism which extends the Jacobian matrix and $J(\rho_j)$ is the Jacobian determinant.

We end with a discussion of various ways to prove that the Dirac operators $\mathbf{D}_{\Omega^\perp}$ and $\mathbf{D}_{\Omega^\parallel}$ are diffuse Fredholm operators under certain additional regularity and topological assumptions on Σ. First we recall the standard proof in the smooth case. Both here and in Theorem 4.13 we use Lemma 3.14, which shows that it suffices to prove that $\mathsf{D}(d_{\overline{\Omega}})\cap\mathsf{D}(\delta_\Omega)$ and $\mathsf{D}(d_\Omega)\cap\mathsf{D}(\delta_{\overline{\Omega}})$ are compactly embedded in $L_2(\Omega;\wedge)$.

Theorem 4.10. *Assume that Ω is a bounded open set with C^2-regular boundary Σ. Then*

$$D(\mathbf{D}_{\Omega^\perp}) = D(d_{\overline{\Omega}})\cap D(\delta_\Omega) = W_2^1(\Omega^\perp;\wedge) := \{F\in W_2^1(\Omega;\wedge)\ ;\ \nu\wedge f=0\},$$
$$D(\mathbf{D}_{\Omega^\parallel}) = D(d_{\Omega})\cap D(\delta_{\overline{\Omega}}) = W_2^1(\Omega^\parallel;\wedge) := \{F\in W_2^1(\Omega;\wedge)\ ;\ \nu\lrcorner f=0\}.$$

In particular, $D(d_{\overline{\Omega}})\cap D(\delta_\Omega)$ and $D(d_\Omega)\cap D(\delta_{\overline{\Omega}})$ are compactly embedded in $L_2(\Omega;\wedge)$.

Moreover, if $\{v_1,\dots,v_{n-1}\}$ is an ON-frame on Σ of directions of principal inward curvatures κ_i, then we have the Weitzenböck formulae

(21)
$$\int_\Omega|\nabla\otimes F(x)|^2 = \int_\Omega|dF(x)|^2+|\delta F(x)|^2-\begin{cases}\sum\int_\Sigma\kappa_i(y)|v_i(y)\wedge f(y)|^2, & F\in D(\mathbf{D}_{\Omega^\perp}),\\ \sum\int_\Sigma\kappa_i(y)|v_i(y)\lrcorner f(y)|^2, & F\in D(\mathbf{D}_{\Omega^\parallel}).\end{cases}$$

Remark 4.11. (i) Note that when Σ is convex, but not necessarily C^2, then $\kappa_i\ge 0$ and we obtain the inequality $\int_\Omega|\nabla\otimes F(x)|^2 \le \int_\Omega|dF(x)|^2+|\delta F(x)|^2$ if either $\nu\wedge f=0$ or $\nu\lrcorner f=0$. See Mitrea [18] for generalisations of this result.

(ii) Consider also the special case of the Laplace equation as explained in Example 1.1. If U is in the domain $\mathsf{D}(\Delta_D)$ of the Dirichlet Laplacian in Ω, then the gradient $F:=\nabla U\in\mathsf{D}(\mathbf{D}_{\Omega^\perp})$. The Weitzenböck formula now reads

$$\int_\Omega|\nabla\otimes\nabla U(x)|^2 = \int_\Omega|\Delta U|^2-(n-1)\int_\Sigma H(y)\big|\tfrac{\partial u}{\partial\nu}(y)\big|^2,$$

where H is the (inward) mean curvature of Σ, since for normal vector fields $|v_i\wedge f| = |v_i||f| = |f|$. This formula is known as Kadlec's formula, see p.341 in Taylor [24].

Proof. We here only give the proof for $\mathbf{D}_{\Omega^\perp}$, since that for $\mathbf{D}_{\Omega^\parallel}$ is similar.

(i) Assume that $F \in W_2^1(\Omega^\perp; \wedge) \subset \mathsf{D}(\mathbf{D}_{\Omega^\perp})$. Using the boundary theorem 2.7, we obtain identities

$$\int_\Omega |\nabla \otimes F|^2 + \langle F, \Delta F^c \rangle = \int_\Sigma \langle f, (\nu, \nabla)\dot{F}^c|_\Sigma \rangle,$$
$$\int_\Omega |dF|^2 + \langle F, \delta d F^c \rangle = \int_\Sigma \langle f, \nu \lrcorner dF^c|_\Sigma \rangle = 0,$$
$$\int_\Omega |\delta F|^2 + \langle F, d\delta F^c \rangle = \int_\Sigma \langle f, \nu \wedge \delta F^c|_\Sigma \rangle,$$

where $\dot{F}$ denotes the function on which the differential operator ∇ is acting. Thus, subtracting the last two equations from the first gives

$$\int_\Omega |\nabla \otimes F|^2 = \int_\Omega |dF|^2 + |\delta F|^2 - \int_\Sigma \langle f, \nu \wedge \delta F^c|_\Sigma \rangle + \langle f, (\nu, \nabla)\dot{F}^c|_\Sigma \rangle.$$

Using the derivation property (9) and that $\nu \wedge f = 0$ and $\partial_{v_i}\nu = \kappa_i v_i$ we rewrite the boundary integral as

$$\int_\Sigma \langle f, \nu \wedge \delta F^c|_\Sigma \rangle - \langle f, (\nu, \nabla)\dot{F}^c|_\Sigma \rangle = -\int_\Sigma \langle f, \nabla \lrcorner (\nu \wedge \dot{F}^c) \rangle$$
$$= -\sum_{i=1}^{n-1} \int_\Sigma \langle v_i \wedge f, \nu \wedge \partial_{v_i} f^c \rangle = \sum_{i=1}^{n-1} \int_\Sigma \langle v_i \wedge f, (\partial_{v_i}\nu) \wedge f^c \rangle = \sum_{i=1}^{n-1} \int_\Sigma \kappa_i |v_i \wedge f|^2.$$

Since Σ is of regularity C^2, κ_i are continuous on Σ and thus the Sobolev trace theorem shows that the inclusion $i_\Omega : W_2^1(\Omega^\perp; \wedge) \hookrightarrow \mathsf{D}(\mathbf{D}_{\Omega^\perp})$ is a bounded semi-Fredholm map.

(ii) What is left to prove is that the inclusion is surjective. Note that since $e_0 \mathbf{D}_{\Omega^\perp}$ is a self-adjoint operator by Proposition 3.5, we have that $\mathbf{D}_{\Omega^\perp} + ie_0 : \mathsf{D}(\mathbf{D}_{\Omega^\perp}) \to L_2(\Omega; \wedge)$ is an isomorphism (for any weakly Lipschitz domain). Thus it suffices to prove that $\mathbf{D}_{\Omega^\perp} + ie_0 : W_2^1(\Omega^\perp; \wedge) \to L_2$ is surjective.

One way to prove this is to perturb the given domain to a domain with an isometric double, e.g. the upper half $T_+^n := \{x \in \mathbf{R}^n \ ; \ 0 < x_n < 1\}/(2\mathbf{Z}+1)^n$ of the flat n-torus $T^n := \mathbf{R}^n/(2\mathbf{Z}+1)^n$ as in Taylor [24]. Since the problem is local, it suffices to prove that if $\rho_t : \Omega = \Omega_0 \to \Omega_t$ is a continuous family of C^2 diffeomorphisms, where Ω_t is a C^2 domain in T^n for $t \in [0,1]$ and $\Omega_1 = T_+^n$, then $\mathbf{D}_{\Omega_0^\perp} + ie_0 : W_2^1(\Omega_0^\perp; \wedge) \to L_2(\Omega_0; \wedge)$ is an isomorphism. From (i) we have a continuous family of semi-Fredholm maps

$$\rho_t^*(\mathbf{D}_{\Omega_t^\perp} + ie_0)(\rho_t^{-1})^* = d_{\overline{\Omega}_0} + \rho_t^* \delta_{\Omega_t} (\rho_t^{-1})^* + ie_0 : W_2^1(\Omega_0^\perp; \wedge) \longrightarrow L_2(\Omega_0; \wedge),$$

since pullbacks preserves normal boundary conditions, and since $[\rho_t^*, \delta] : W_2^1 \to L_2$ depends continuously on t. Perturbation theory [13] now shows that it suffices to prove that $\mathbf{D}_{(T_+^n)^\perp} + ie_0 : W_2^1((T_+^n)^\perp; \wedge) \to L_2(T_+^n; \wedge)$ is surjective. Note that $\mathbf{D}_{T^n} + ie_0 : W_2^1(T^n; \wedge) \to L_2(T^n; \wedge)$ is an isomorphism. We see that, given any $G \in L_2(T^n; \wedge)$ with $\operatorname{supp} G \subset \overline{T_+^n}$, there exists $F \in W_2^1(T^n; \wedge)$ such that $(\mathbf{D}_{T^n} +$

$ie_0)F = G$. Now the anti symmetrised field $F - r^*F$, where $r : T^n_\pm \to T^n_\mp$ is the isometric reflection, belongs to $W^1_2((T^n_+)^\perp; \wedge)$ and $(d + \delta + ie_0)(F - r^*F) = G - r^*G = G$ in T^n_+ since d commutes with r^* and δ commutes with $\tilde{r}_*^{-1} = r^*$ by Proposition 2.6. This finishes the proof. □

For non-smooth Σ, not only the source function $F := \mathbf{D}_{\Omega^\perp} U$ influences the regularity of $U \in \mathsf{D}(\mathbf{D}_{\Omega^\perp})$, but also Σ. A standard example, see e.g. Grisvard [11], is the following.

Example 4.12. Consider a bounded domain $\Omega \subset \mathbf{R}^2$ whose boundary Σ is smooth except at 0 where it coincides with $\overline{\mathbf{R}}_+ \cup e^{i\alpha}\mathbf{R}_+$. Let $U : \mathbf{R}^2 \to \wedge^0$ be a scalar function in Ω, smooth up to the boundary except at 0, such that $U(x) = r^{\frac{\pi}{\alpha}} \sin(\frac{\pi}{\alpha}\theta)$ around 0 and $U|_\Sigma = 0$. Define

$$F(x) := \nabla U(x) = \tfrac{\pi}{\alpha} r^{\frac{\pi}{\alpha}-1}(\sin(\tfrac{\pi}{\alpha}\theta)\hat{r} + \cos(\tfrac{\pi}{\alpha}\theta)\hat{\theta}),$$

for x around 0, where $\hat{r}$ and $\hat{\theta}$ denotes the radial and angular unit vector fields. Then the estimate $|F| \lesssim r^{\frac{\pi}{\alpha}-1}$ shows that $F \in \mathsf{D}(d_{\overline{\Omega}}) \cap \mathsf{D}(\delta_\Omega)$, whereas the estimate $|\frac{\partial F}{\partial r}| \approx r^{\frac{\pi}{\alpha}-2}$ shows that

$$\|F\|^2_{W^1_2(\Omega)} \geq \int_\Omega \left|\frac{\partial F}{\partial r}\right|^2 \gtrsim \int_0^1 r^{2(\frac{\pi}{\alpha}-2)} r\, dr.$$

But in the non-convex case $\alpha > \pi$ the right hand side is infinite so that $F \notin W^1_2(\Omega; \wedge)$. However, one can verify that $\|F\|_{W^{1/2}_2} < \infty$ for any $0 < \alpha < 2\pi$.

For a strongly Lipschitz domain, we use the $L_2(\Sigma; \wedge)$ theory of boundary value problems. This uses the Rellich estimate technique, which was first applied by Verchota [25] to the Laplace equation. This technique was later extended to the full Dirac operator by McIntosh–Mitrea [16] and McIntosh–Mitrea–Mitrea [15].

Theorem 4.13. *Assume that Ω is a bounded, strongly Lipschitz domain. Then we have continuous inclusions*

$$\mathsf{D}(\mathbf{D}_{\Omega^\perp}),\ \mathsf{D}(\mathbf{D}_{\Omega^\parallel}) \subset W^{1/2}_2(\Omega; \wedge).$$

In particular, $\mathsf{D}(d_{\overline{\Omega}}) \cap \mathsf{D}(\delta_\Omega)$ and $\mathsf{D}(d_\Omega) \cap \mathsf{D}(\delta_{\overline{\Omega}})$ are compactly embedded in $L_2(\Omega; \wedge)$.

Proof. Consider the map $\mathbf{D}_{\Omega^\perp} + ie_0 : \mathsf{D}(\mathbf{D}_{\Omega^\perp}) \longrightarrow L_2(\Omega; \wedge)$, which is an isomorphism since $e_0\mathbf{D}_{\Omega^\perp}$ is self-adjoint, and the dense subset

$$S := \{F \in \mathsf{D}(\mathbf{D}_{\Omega^\perp}) \ ; \ (\mathbf{D}_{\Omega^\perp} + ie_0)F \in C^\infty_0(\Omega; \wedge)\} \subset \mathsf{D}(\mathbf{D}_{\Omega^\perp}).$$

It suffices to show that we have a continuous inclusion $S \hookrightarrow W^{1/2}_2(\Omega; \wedge)$. Given $G = (\mathbf{D}_{\Omega^\perp} + ie_0)F \in C^\infty_0(\Omega; \wedge)$, let $F_0 := (\mathbf{D}_{\mathbf{R}^n} + ie_0)^{-1}G \in C^\infty(\mathbf{R}^n; \wedge)$ and form its tangential trace $\nu \wedge f_0 \in L_2(\Sigma; \wedge)$. We now apply the Rellich $L_2(\Sigma; \wedge)$ theory of boundary value problems on strongly Lipschitz domains, see [1] for more details, which shows the existence of a field $F_1 : \Omega \to \wedge$ such that

$$\begin{aligned} (\mathbf{D} + ie_0)F_1 &= 0 \qquad \text{in } \Omega \\ \nu \wedge f_1 &= \nu \wedge f_0 \qquad \text{on } \Sigma \\ \|F_1\|_{W_2^{1/2}(\Omega;\wedge)} &\approx \|\nu \wedge f_0\|_{L_2(\Sigma;\wedge)} \\ dF_1, \delta F_1 &\in W_2^{1/2}(\Omega;\wedge) \subset L_2(\Omega;\wedge). \end{aligned}$$

Now let $F' := F_0 - F_1$. We see that

$$\begin{aligned} \|F_0\|_{W_2^{1/2}(\Omega;\wedge)} &\lesssim \|F_0\|_{W_2^1(\Omega;\wedge)} \lesssim \|G\|_{L_2(\Omega;\wedge)} \approx \|F\|_{\mathsf{D}(\mathbf{D}_{\Omega^\perp})} \\ \|F_1\|_{W_2^{1/2}(\Omega;\wedge)} &\approx \|\nu \wedge f_0\|_{L_2(\Sigma;\wedge)} \lesssim \|F_0\|_{W_2^1(\Omega;\wedge)} \lesssim \|F\|_{\mathsf{D}(\mathbf{D}_{\Omega^\perp})}. \end{aligned}$$

Moreover dF', $\delta F' \in L_2(\Omega;\wedge)$ and $\nu \wedge f' = \nu \wedge f_0 - \nu \wedge f_1 = 0$. Thus $F' \in \mathsf{D}(\mathbf{D}_{\Omega^\perp}) \cap W_2^{1/2}(\Omega;\wedge)$ and $(\mathbf{D}_{\Omega^\perp} + ie_0)F' = G = (\mathbf{D}_{\Omega^\perp} + ie_0)F$. Thus $F = F' \in W_2^{1/2}(\Omega;\wedge)$ with $\|F\|_{W_2^{1/2}(\Omega;\wedge)} \lesssim \|F\|_{\mathsf{D}(\mathbf{D}_{\Omega^\perp})}$. □

Remark 4.14. In the proof above we used the fact that

$$\|F_1\|_{W_2^{1/2}(\Omega;\wedge)} \approx \|f_1\|_{L_2(\Sigma;\wedge)}$$

when $(\mathbf{D} + ie_0)F_1 = 0$ in Ω. This result is presented in Fabes [7] with an incomplete proof which is corrected in Mitrea–Mitrea–Pipher [19]. See also [3] for an alternative correction.

Note that Theorem 4.13 is more constructive than Theorem 4.10 in the sense that F' is found by solving the boundary equation $\nu \wedge f_1 = \nu \wedge f_0$. However, if one is only interested in solving for example $d_\Omega U = F$, where $F \in \mathsf{N}(d_\Omega)$, with a "good inverse" in the sense that $F \mapsto U$ is an L_2 compact map, then this can be done much more explicitly using path integrals as we now explain.

Lemma 4.15. *Let $\Omega \subset \mathbf{R}^n$ be an open set with a smooth retraction $\mathcal{F}_t : \Omega \to \mathcal{F}_t(\Omega) \subset \Omega$ to $p \in \Omega$ such that $\mathcal{F}_1 = I$, $\mathcal{F}_t\mathcal{F}_s = \mathcal{F}_{ts}$ for $0 \le t, s \le 1$ and $\mathcal{F}_0 = p$. If $\theta = d\mathcal{F}_t/dt|_{t=1}$ is the vector field with flow $\mathcal{F}_t$, then for smooth fields F in Ω we have the path integral formulae*

$$F(x) = \nabla \wedge \Big(\int_0^1 \theta(x) \lrcorner \mathcal{F}_t^* F(x) \, \tfrac{dt}{t} \Big), \qquad \text{if } \nabla \wedge F = 0 \text{ and } F|_{\wedge^0} = 0,$$

$$F(x) = \nabla \lrcorner \Big(\int_0^1 \theta(x) \wedge \widetilde{\mathcal{F}}_{t*}^{-1} F(x) \, \tfrac{dt}{t} \Big), \qquad \text{if } \nabla \lrcorner F = 0 \text{ and } F|_{\wedge^n} = 0.$$

One can prove this lemma by using *Cartan's formula*

$$\mathcal{L}_\theta F = \tfrac{d}{dt}(\mathcal{F}_t^* F(x))|_{t=1} = \nabla \wedge (\theta \lrcorner F) + \theta \lrcorner (\nabla \wedge F)$$

for the Lie derivative of the differential form F and using the homomorphism formulae from Proposition 2.6. For more details, see for example Taylor [24].

Example 4.16. Let Ω be star shaped with respect to 0 and $\mathcal{F}_t(x) := tx$. Then for a smooth j-vector field $F : \Omega \to \wedge^j$ we have the path integrals

$$F(x) = \nabla \wedge \Big(\int_0^1 t^{j-1} x \lrcorner F(tx)\, dt \Big), \qquad \text{if } \nabla \wedge F = 0 \text{ and } j \geq 1,$$

$$F(x) = \nabla \lrcorner \Big(\int_0^1 t^{n-j-1} x \wedge F(tx)\, dt \Big), \qquad \text{if } \nabla \lrcorner F = 0 \text{ and } j \leq n-1.$$

Indeed, using the derivation formula (9) and that $\sum_i e_i \wedge (e_i \lrcorner F(tx)) = jF(tx)$, one can directly verify that $\nabla \wedge (x \lrcorner F(tx)) = jF(tx) + t\frac{d}{dt}(F(tx))$.

In particular, a curl free vector field F has a scalar potential given by the path integral $U(x) = \int_0^1 (x, F(tx))\, dt$ and a divergence free vector field F has a bivector potential $U(x) := \int_0^1 t^{n-2} x \wedge F(tx)\, dt \in \wedge^2$. In classical notation in $\mathbf{R}^3$, the latter translates to $F = -\nabla \times U^\perp$ if $U(x)^\perp := \int_0^1 tx \times F(tx)\, dt$.

A third way to prove that the Dirac operators are diffuse Fredholm operators uses an L_2 version of the classical Poincaré lemma. We here only consider fields with values in $\wedge \mathbf{R}^n = \wedge^0 \oplus \ldots \oplus \wedge^n$. The extension to spacetime setting is straightforward.

Theorem 4.17. *Let $\Omega \subset \mathbf{R}^n$ be a star shaped strongly Lipschitz domain. Let $0 < \epsilon < R < \infty$ be such that $B(0,\epsilon) \subset \Omega \subset B(0,R)$ and such that Ω is star shaped with respect to each $p \in B(0,\epsilon)$. For $1 \leq j \leq n$, let $T^{(j)}$ denote the integral operator*

$$T^{(j)} F(x) := \int_\Omega (x-y) \lrcorner F(y)\, k_j(x,y) dy, \qquad F \in L_2(\Omega; \wedge^j), \tag{22}$$

where k_j denotes the kernel

$$k_j(x,y) := \int_0^1 \eta\Big(x + \frac{y-x}{1-t} \Big) \frac{t^{j-1} dt}{(1-t)^{n+1}}$$

for some fixed $\eta \in C_0^\infty(B(0,\epsilon))$ with $\int \eta = 1$. In particular

$$\operatorname{supp} k_j \subset \{(x,y)\,;\, y \in \overline{\operatorname{conv}}\,(B(0,\epsilon), x)\},$$

where $\overline{\operatorname{conv}}$ denotes the closed convex hull, k_j is smooth off the diagonal $\{x = y\}$ with estimates

$$|k_j(x,y)| \leq \tfrac{1}{n} \|\eta\|_\infty (R+\epsilon)^n \frac{1}{|x-y|^n}$$

and $T^{(j)}$ defines a compact operator $L_2(\Omega; \wedge^j) \to L_2(\Omega; \wedge^{j-1})$. Then $T := 0 \oplus T^{(1)} \oplus \ldots \oplus T^{(n)} : \mathsf{N}(d_\Omega) \to L_2(\Omega; \wedge \mathbf{R}^n)/\mathsf{N}(d_\Omega)$ is a compact Fredholm inverse to d_Ω. Thus d_Ω is a diffuse Fredholm-nilpotent operator.

The corresponding result for δ_Ω holds true as well.

Proof. Assume that $F \in C_0^\infty(\mathbf{R}^n; \wedge^j)$ and $\operatorname{supp}(\nabla \wedge F) \cap \overline{\Omega} = \emptyset$ as in Proposition 4.3. We define

$$T^{(j)} F(x) = \int \eta(p) dp \Big(\int_0^1 t^{j-1} (x-p) \lrcorner F(p + t(x-p))\, dt \Big),$$

which by using Fubini's theorem and the change of variables $y = p + t(x - p)$ becomes (22). Since $\int \eta = 1$, it follows from Example 4.16 that $d_\Omega T^{(j)} F = F$.

The estimates off $\operatorname{supp} k_j$ and $|k_j(x, y)|$ are straightforward to verify. Since the full kernel for $T^{(j)}$ has the estimate $\lesssim 1/|x-y|^{n-1}$, Schur's lemma shows that $T^{(j)}$ defines a compact operator $L_2(\Omega; \wedge^j) \to L_2(\Omega; \wedge^{j-1})$.

We can now apply Proposition 3.8 with $\mathcal{H}_1 = L_2(\Omega; \wedge)/\mathsf{N}(d_\Omega)$, $\mathcal{H}_2 = \mathsf{N}(d_\Omega)$, $A = d_\Omega$, $T_1 = T_2 = T = 0 \oplus T^{(1)} \oplus \ldots \oplus T^{(n)}$, $K_1 = 0$ and $K_2 =$ orthogonal projection onto scalar constants, which shows that d_Ω is a diffuse Fredholm-nilpotent operator. □

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Centre for Mathematics and its Applications, Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia
E-mail address: `Alan.McIntosh@maths.anu.edu.au`