Problem 10.1. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial of degree 3. Let \( \mathbb{Q} \subseteq L \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \).
(a) Show that if \( f(x) \) has only one real root, then \( \text{Gal}(L/\mathbb{Q}) \cong S_3 \).
(b) Recall that the discriminant \( \Delta \) is defined as
\[
\Delta = (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2
\]
where \( \alpha_1, \alpha_2, \alpha_3 \) are the roots of \( f(x) \). Also recall from HW1 that if \( f(x) = x^3 + ax + b \), then the discriminant
\[
\Delta = -4a^3 - 27b^2.
\]
Show that \( \text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_3 \) if \( \Delta \) is a square of a rational number and is \( S_3 \) otherwise.
(c) Does there exist a cubic polynomial \( f(x) \in \mathbb{Q}[x] \) with three real roots such that \( \text{Gal}(L/\mathbb{Q}) \cong S_3 \).

Problem 10.2. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial of degree \( p \) where \( p \) is prime. Let \( \mathbb{Q} \subseteq L \) be the splitting field of \( f(x) \) over \( \mathbb{Q} \). Show that if \( f(x) \) has precisely \( p - 2 \) real roots, then \( \text{Gal}(L/\mathbb{Q}) \cong S_p \).

Hint: Use the lemma proved in class regarding when subgroups of \( S_p \) are the entire group.

Problem 10.3.
(1) Show that the polynomial \( x^5 - 4x^2 + 2 \in \mathbb{Q}[x] \) is not solvable by radicals.
(2) Show that the polynomial \( x^7 - 10x^5 + 15x + 5 \) is not solvable by radicals.