Problem 3.1. (a) Show that there are ideals in \( \mathbb{Z}[x] \) which are not generated by a single element.
(b) Show that the element 6 in the ring \( \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \) does not factor uniquely. That is, write 6 as a product \( 6 = ab \) and \( 6 = a'b' \) in two distinct ways such that \( a, b, a', b' \) are irreducible, and both \( a \) and \( b \) are not units times \( a' \) or \( b' \). Conclude that there is an irreducible element of \( \mathbb{Z}[\sqrt{-5}] \) which does not generate a prime ideal.

Problem 3.2. Show \( \mathbb{Q}[x]/(x^2 + x + 1) \cong \{a + b\omega \mid a, b \in \mathbb{Q}\} \), where \( \omega = e^{2\pi i/3} \). Describe explicitly additional, multiplication and division on the right hand side.

Problem 3.3. Prove that \( x^5 - x^2 + 1 \in \mathbb{Q}[x] \) is irreducible. (Hint: consider \( \mathbb{F}_2 \))

Problem 3.4. Eisenstein’s criterion with a twist.
(a) Let \( a \) be any integer. Prove that a polynomial \( f(x) \in \mathbb{Z}[x] \) is irreducible iff \( f(x + a) \in \mathbb{Z}[x] \) is irreducible.
(b) Use this trick to prove that \( x^3 - 3x^2 + 9x - 5 \) is irreducible.
(c) Use this trick to prove that, for any prime \( p \), the polynomial \( x^{p-1} + x^{p-2} + \ldots + x + 1 \) is irreducible.

Problem 3.5. Consider the field extension \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) \).
(a) What is the degree, \([\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}]\), of this field extension?
(b) Prove that this is a primitive field extension; that is, find an element \( \alpha \) such that \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) \).