Tutorial 10.1.
(a) Show that a primitive 14th root of unity satisfies the sextic equation
\[ x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0. \]
(b) Solve the equation from part (1) by radicals.
*Hint:* Substitute \( y = x + 1/x \).

Tutorial 10.2.
(a) Can you give an example of an element \( \alpha \in \mathbb{C} \) which can be expressed by radicals but such that \( \mathbb{Q} \subseteq \mathbb{Q}(\alpha) \) is not a radical field extension (i.e. \( \mathbb{Q}(\alpha) \) is not obtained from \( \mathbb{Q} \) by adjoining an \( n \)th root of an element \( a \in \mathbb{Q} \)).
(b) Can you give an example of an element \( \alpha \in \mathbb{C} \) which can be expressed by radicals but such that there does not exist a sequence of field extensions
\[ \mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r = \mathbb{Q}(\alpha) \]
where each \( K_i \subseteq K_{i+1} \) is a radical field extension.

Tutorial 10.3. Let \( L = \mathbb{Q}(\sqrt{2}, \sqrt{-3}, \sqrt{5}) \).
(a) Prove that \( \mathbb{Q} \subseteq L \) is a Galois extension with Galois group
\[ \text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2 \times S_3. \]
(b) Give the complete correspondence between intermediate field extensions \( \mathbb{Q} \subseteq L' \subseteq L \) and subgroups \( H \subseteq \text{Gal}(L/\mathbb{Q}) \).

Tutorial 10.4. Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible quartic.
(a) Show that the discriminant of \( f \) is equal to the discriminant of the resolvent cubic of \( f \).
(b) Show that if \( \text{Gal}(f) = \mathbb{Z}/4\mathbb{Z} \) then the discriminant of \( f \) is positive.

Tutorial 10.5. Determine the Galois group of the splitting fields over \( \mathbb{Q} \) of the following polynomials:
(a) \( f(x) = x^3 + 3x + 1 \).
(b) \( f(x) = x^3 - 3x + 1 \).
(c) \( f(x) = x^4 + 4x^2 - 2 \).
(d) \( f(x) = x^4 + x^2 + 1 \).
(e) \( f(x) = x^4 + 36x + 63 \)

*Hint:* You may want to use the following formulae:

- the discriminant of \( f(x) = x^3 + ax + b \) is \(-4a^3 - 27b^2\).
- the discriminant of \( f(x) = x^4 + ax + b \) is \(-27a^4 + 256b^3\).
- the resolvent cubic of \( f(x) = x^4 + ax + b \) is \(x^3 - 4bx + a^2\).
- the discriminant of \( f(x) = x^4 + ax^2 + b \) is \(16b(a^2 - 4b)^2\).