UNIQUENESS FOR A CLASS OF EMBEDDED WEINGARTEN HYPERSURFACES IN S^{n+1}

BEN ANDREWS, ZHIJIE HUANG, AND HAIZHONG LI

ABSTRACT. We apply the 'non-collapsing' technique, previously applied by Brendle in the proof of the Lawson conjecture and by Andrews and Li in the proof of the Pinkall-Sterling conjecture, to higher dimensional hypersurfaces satisfying a linear relation between the principal curvatures, under the additional assumption that the hypersurface has two distinct principal curvatures at each point. As special cases, the result gives simple new proofs of results of Otsuki for minimal hypersurfaces, and of Li and Wei for hypersurfaces with vanishing mth mean curvature.

1. Introduction

In classical differential geometry, the study of minimal surfaces is one of the most basic subjects. The study of minimal surfaces in space forms, such as the Euclidean space $\mathbb{R}^3$ or the sphere S^3, is of particular interest. In 1970, H.B. Lawson [L2] proved that for any positive integer g, there exists at least one compact embedded minimal surface with genus g in S^3. If $g > 1$ is not a prime, such embedded minimal surface is not unique. He also conjectured in [L3] that for $g = 1$, the Clifford torus is the only compact embedded minimal torus in S^3. This conjecture was proved by Simon Brendle [B2] in 2012 (see also the survey [B3]):

Theorem 1 (Brendle [B2]). *If $F : \Sigma \to S^3$ is an embedded minimal torus, then F is congruent to Clifford torus.*

An important ingredient in Brendle's proof was the first author's work on "non-collapsing" in mean curvature flow [A], in which a maximum principle argument was used to compare the size of balls touching the hypersurface to the mean curvature at each point. By modifying this argument to involve the maximum principal curvature, Brendle made the argument applicable to minimal surfaces, and used it to prove the rigidity statement. Soon afterwards, the first and third authors [AL] extended the argument to embedded constant mean curvature tori and proved that these must be surfaces of rotation, confirming a 1989 conjecture of U. Pinkall and I. Sterling [PS]. The results of Andrews-Li [AL] are:

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Theorem 2 (Andrews-Li [AL]). (1) *Every embedded CMC torus* Σ *in* S^3 *is a surface of rotation: There exists a two-dimensional subspace* Π *of* $\mathbb{R}^4$ *such that* Σ *is invariant under the group* S^1 *of rotations fixing* Π.

(2) *If* Σ *is an embedded CMC torus which is not congruent to a Clifford torus, then there exists a maximal integer* $m \geq 2$ *such that* Σ *has* m*-fold symmetry: Precisely,* Σ *is invariant under the group* $\mathbb{Z}_m$ *generated by the rotation which fixes the orthogonal plane* $\Pi^\perp$ *and rotates* Π *through angle* $2\pi/m$.

(3) *For given* $m \geq 2$*, there exists at most one such CMC torus (up to congruence).*

(4) *For given* $m \geq 2$*, there exists an embedded CMC torus with mean curvature* H *and maximal symmetry* $S^1 \times \mathbb{Z}_m$ *if* $|H|$ *lies strictly between* $\cot\frac{\pi}{m}$ *and* $\frac{m^2-2}{2\sqrt{m^2-1}}$.

(5) *If* $H \in \left\{0, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}$*, then every embedded torus with mean curvature* H *is congruent to the Clifford torus.*

The higher dimensional cases are also interesting, and in particular the study of minimal hypersurfaces in space forms has attracted considerable interest. There seems no prospect for results in high dimensions of a generality comparable to the two-dimensional case: Not only do the methods of proof break down in higher dimensions, but there are many more examples of constant mean curvature hypersurfaces known. Our results in this paper are partly motivated by those of T. Ôtsuki [Ô1, Ô2] who proved that every embedded minimal hypersurface with two distinct principal curvatures in S^{n+1} is congruent to the Clifford torus.

Theorem 3 (Ôtsuki [Ô1, Ô2]). *A compact embedded minimal hypersurface with two distinct principal curvatures in* S^{n+1} *is congruent to product*

$$S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), \qquad 1 \leq m \leq n-1.$$

In 2007, H. Li and G. Wei extended Ôtsuki's result to hypersurfaces Σ with $H_m = 0$, where H_m is the m-th elementary symmetric function of the principal curvatures (also called the m-th mean curvature of Σ):

Theorem 4 (Li-Wei [LW]). *If* $1 \leq m \leq n = 1$*, there are no compact embedded rotational hypersurfaces of* S^{n+1} *with* $H_m = 0$ *other than*

$$S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$$

and the totally geodesic spheres.

In this paper, we prove the following, imposing as in [Ô1, Ô2, LW] an assumption on the number of distinct principal curvatures:

Theorem 5. *Suppose M^n is a compact embedded hypersurface in S^{n+1} with two distinct principal curvatures λ and μ, whose multiplicities are m and $n-m$ respectively. If for some positive number α,*

$$\lambda + \alpha\mu = 0,$$

then λ and μ are both constant and M is congruent to a Clifford torus

$$S^m\left(\sqrt{\frac{1}{\alpha+1}}\right) \times S^{n-m}\left(\sqrt{\frac{\alpha}{\alpha+1}}\right), \quad 1 \le m \le n-1.$$

2. Hypersurfaces with two principal curvatures in S^{n+1}

Let $M^n \subset S^{n+1}$ be a compact hypersurface with second fundamental form h and two principal curvatures λ and μ, whose multiplicities are m and $n-m$ respectively. The maximum and minimum eigenvalues of a symmetric matrix are continuous, so λ and μ are continuous functions on M. Therefore the matrix $\frac{h-\mu g}{\lambda-\mu}$ is continuous since $\lambda \neq \mu$ everywhere, and has trace equal to m. Therefore m is continuous, hence constant.

In a neighbourhood of any point on M, we can choose an orthonormal frame $\{e_i : i = 1, \cdots, n\}$ for TU, and a normal vector field e_0. Let ω_A be the dual forms of e_A on S^{n+1}. The connection forms are given by

$$d\omega_A = \omega_{AB} \wedge \omega_B$$
$$d\omega_{AB} = \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}$$

where $\Omega_{AB} = -\frac{1}{2}\bar{R}_{ABCD}\omega_C \wedge \omega_D$ and $\bar{R}_{ABCD}$ are the components of Riemannian curvature tensor of S^{n+1}. Let θ_A, θ_{AB} be the restriction of ω_A, ω_{AB} to M. Since $\theta_0 = 0$, take its exterior derivative and we have

$$\theta_i \wedge \theta_{i0} = 0.$$

Therefore, $\theta_{i0} = h_{ij}\theta_j$, where h_{ij} are symmetric in i, j, and they are called the components of the second fundamental form of M. We can choose $\{e_i\}$ such that h_{ij} is diagonal, i.e., $h_{ij} = \lambda_i\delta_{ij}$. Moreover, in our case, since M has only two distinct principal curvatures λ, μ with multiplicities m and $n-m$, we can assume

$$\lambda_1 = \cdots = \lambda_m = \lambda, \quad \lambda_{m+1} = \cdots = \lambda_n = \mu.$$

Lemma 6. *Under the assumptions above, if $m \ge 2$, we have*

$$e_i(\lambda) = 0, \quad \forall\ i = 1, \cdots, m.$$

Proof. Let the $h_{ij,k}$ denote the covariant derivative of h_{ij} with respect to e_k. The Codazzi equation gives us

$$h_{ij,k} = h_{ik,j}.$$

Therefore, $h_{ij,k}$ are symmetric in i, j, k. From the definition of covariant derivative, we have

$$h_{ij,k}\theta_k = dh_{ij} + h_{kj}\theta_{ki} + h_{ik}\theta_{kj} = dh_{ij} + (\lambda_i - \lambda_j)\theta_{ij}.$$

Take $j = i$, we have

$$h_{ii,k}\theta_k = dh_{ii} = d\lambda_i = e_k(\lambda_i)\theta_k,$$

that is, $h_{ii,k} = e_k(\lambda_i)$, for any i and k. If $\lambda_i = \lambda_j$ and $i \neq j$ then $h_{ij} = 0$, we have $h_{ij,k} = 0$, for any k. If $m \geq 2$, for any $1 \leq i \leq m$, we can choose $1 \leq j \neq i \leq m$, thus $\lambda_i = \lambda_j = \lambda$,

$$e_i(\lambda) = e_i(\lambda_j) = h_{jj,i} = h_{ij,j} = 0.$$

This completes the proof of the lemma. $\square$

Using Lemma 6 and the equation $\lambda + \alpha\mu = 0$ we deduce the following:

Corollary 7. *If the multiplicities of λ, μ are both greater than 1, then*

$$e_i(\lambda) = e_i(\mu) = 0, \quad \forall 1 \leq i \leq n,$$

and M is congruent to the Clifford torus

$$S^m\left(\sqrt{\frac{1}{\alpha+1}}\right) \times S^{n-m}\left(\sqrt{\frac{\alpha}{\alpha+1}}\right), \quad 2 \leq m \leq n-2.$$

The last conclusion follows since M is a compact isoparametric hypersurface with two constant distinct principal curvatures λ and μ.

From now on, we concentrate on the case when one of the principal curvature, say λ, is simple. We assume M is a compact hypersurface in S^{n+1} with second fundamental form h and two principal curvatures λ and μ of multiplicity 1 and $n-1$ respectively, related by the equation $\lambda + \alpha\mu = 0$ for some $\alpha > 0$.

Lemma 6 implies that $e_j(\mu) = 0$ for $j = 2, \cdots, n$. By the equation we also have $e_j(\lambda) = 0$ for $j = 2, \cdots, n$.

The following lemma provides useful information on the covariant derivatives of the second fundamental form.

Lemma 8. *Suppose $\{e_i\}$ is a local orthonormal frame such that the components of second fundamental form h_{ij} are diagonal, i.e., $h_{ij} = \lambda_i\delta_{ij}$, and $\lambda_1 = \lambda, \lambda_i = \mu (i \geq 2)$. Then $h_{ij,k}$ are symmetric in i, j, k and*

$$h_{11,1} = \lambda_{,1}, \ h_{ii,1} = \mu_{,1}, i = 2, \cdots, n.$$

and $h_{ij,k} = 0$, for i, j, k are distinct and $h_{ii,j} = 0$, for $i \geq 1$ and $j \neq 1$.

Proof. The covariant derivatives of h_{ij} are given by

$$h_{ij,k}\theta_k = dh_{ij} + h_{kj}\theta_{ki} + h_{ik}\theta_{kj} = dh_{ij} + (\lambda_i - \lambda_j)\theta_{ij}.$$

The Codazzi identities imply that $h_{ij,k}$ is totally symmetric. Therefore, to compute $h_{ij,k}$, we need only consider the following three types:

$$h_{ij,k} \quad h_{ii,k} \quad h_{ii,i}$$

If i, j, k are distinct, then $h_{ij,k}$ must be zero: At least two of i, j, k are greater then 1, say i, j. In this case $\lambda_i = \lambda_j = \mu$, $i \neq j$, and we have

$$h_{ij,k}\theta_k = dh_{ij} = 0.$$

that is $h_{ij,k} = 0$ for any k.

To calculate $h_{ii,k}$, we notice that $h_{ii,k}\theta_k = dh_{ii} = d\lambda_i$, and therefore

$$h_{ii,k} = e_k(\lambda_i).$$

Thus, for any $i, j = 2, \cdots, n$, we have

$$\begin{aligned} h_{11,1} &= e_1(\lambda) \quad h_{11,j} = e_j(\lambda) = 0 \\ h_{ii,1} &= e_1(\mu) \quad h_{ii,j} = e_j(\mu) = 0. \end{aligned}$$

This completes the proof. □

Let ∇ be the Levi-Civita connection of M, and $\{e_i : i = 1, \cdots, n\}$ a local orthogonal frame such that e_1 is the principal direction of λ. Then e_1 is a smooth vector field since $m = 1$ and $\lambda \neq \mu$. Define an elliptic operator $\mathcal{L}$ by

$$\mathcal{L} = a^{ij}(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_i}e_j),$$

where $a = e_1 \otimes e_1 + \frac{\alpha}{n-1}\sum_{i=2}^{n} e_i \otimes e_i$. That is, a is diagonal in the basis $\{e_1, \ldots, e_n\}$, and $a^{11} = 1$, $a^{22} = \cdots = a^{nn} = \frac{\alpha}{n-1}$. Note that $\mathcal{L}$ is smoothly defined, since e_1 is.

By the Gauss equation, we have

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}) + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

Thus $R_{ipjp} = (\lambda_i\lambda_p + 1)(\delta_{ij} - \delta_{ip}\delta_{jp})$.

Proposition 9. *A connected complete hypersurface M in S^{n+1} with principal curvatures $\lambda \neq \mu$ of multiplicity 1 and $n-1$ respectively is a rotation hypersurface: There exists a two-dimensional subspace Π of $\mathbb{R}^{n+2}$ such that M is invariant under the group $O(\Pi^\perp)$ of rotations fixing Π.*

Proof. We begin by establishing expressions for the certain covariant derivatives of the vector fields e_i: The two identities

$$0 = g(\nabla_k e_i, e_j) + g(e_i, \nabla_k e_j)$$

and

$$e_k(h(e_i, e_j)) = \nabla_k h_{ij} + h(\nabla_k e_i, e_j) + h(e_i, \nabla_k e_j)$$

yield (since h is diagonal) the expressions for $i \neq j$

$$\nabla_k e_i \cdot e_j(\lambda_i - \lambda_j) = \nabla_k h_{ij}.$$

Choosing $k, j > 1$ and $i = 1$ we deduce that

$$\nabla_k e_j \cdot e_1 = -\nabla_k e_1 \cdot e_j = \frac{\nabla_1 h_{jk}}{\lambda - \mu}.$$

The right hand side vanishes if $j \neq k$, and equals $\frac{e_1\mu}{\lambda-\mu}$ if $j = k$.

Next we establish an identity involving the second derivatives of μ, which can be considered an analogue of a well-known identity for minimal hypersurfaces first proved by Simons [S]:

Lemma 10. *Under the conditions of the proposition,*

$$0=\nabla^2\mu(e_1,e_1)+\frac{e_1\mu\,(2e_1\mu-e_1\lambda)}{\lambda-\mu}+(\lambda-\mu)(1+\lambda\mu); \tag{1}$$

and for $i>1$,

$$0=\nabla^2\mu(e_i,e_i)+\frac{(e_1\mu)^2}{\lambda-\mu}. \tag{2}$$

Proof. Applying the Gauss and Codazzi equations we find the following:

$$\begin{aligned}\nabla_i\nabla_j h_{kl} &= \nabla_i\nabla_k h_{jl}\\ &= \nabla_k\nabla_i h_{jl}+R_{ikj}{}^p h_{pl}+R_{ikl}{}^p h_{jp}\\ &= \nabla_k\nabla_l h_{ij}+h_{ij}h^2_{kl}-h_{jk}h^2_{il}+h_{il}h^2jk-h_{ij}h^2_{kl}\\ &\quad+g_{ij}h_{kl}-g_{jk}h_{il}+g_{il}h_{jk}-g_{kl}h_{ij}.\end{aligned}$$

Substituting $k=l=1$ and $i=j=2$ gives

$$\nabla_2\nabla_2 h_{11}=\nabla_1\nabla_1 h_{22}-(\lambda\mu+1)(\lambda-\mu). \tag{3}$$

We next compute the left hand side: We have

$$\nabla_2 h_{11}=e_2\lambda=0,$$

and so

$$\begin{aligned}\nabla_2\nabla_2 h_{11} &= e_2\left(\nabla_2 h_{11}\right)-\nabla_{\nabla_2 e_2}h_{11}-2\nabla_2 h(e_1,\nabla_2 e_1)\\ &= -\left(\nabla_2 e_2\cdot e_1\right)\nabla_1\lambda-2(\nabla_2 e_1\cdot e_2)\nabla_1\mu\\ &= \frac{e_1\mu e_1\lambda}{\lambda-\mu}-2\frac{(e_1\mu)^2}{\lambda-\mu}. \end{aligned}\tag{4}$$

Finally we compute

$$\begin{aligned}\nabla_1\nabla_1 h_{22} &= e_1\left(\nabla_1 h_{22}\right)-\nabla_{\nabla_1 e_1}h_{22}-2\nabla_1 h(e_2,\nabla_1 e_2)\\ &= e_1e_1\mu\\ &= \nabla^2\mu(e_1,e_1). \end{aligned}\tag{5}$$

The identity (1) follows after substituting the identities (4) and (5) into (3). The identity (2) follows more directly: Since $e_i\mu=0$, we have

$$\nabla^2\mu(e_i,e_i)=e_i(\nabla_i\mu)-\nabla_{\nabla_i e_i}\mu=-\frac{(e_1\mu)^2}{\lambda-\mu}.$$

□

Now we identify at each point of M a plane Π: Let $p=\mu x-\nu$ and $q=(e_1\mu)x-(\lambda-\mu)e_1$, and define Π to be the span of p and q. We compute

$$e_ip=e_iq=0 \quad \text{for i} > 1;$$

and

$$e_1p=q,$$

while

$$e_1 q = -\lambda(\lambda-\mu)p - \frac{2e_1\mu - e_1\lambda}{\lambda-\mu}q + \left[\nabla^2\mu(e_1,e_1) + \frac{e_1\mu(2e_1\mu - e_1\lambda)}{\lambda-\mu} + (\lambda-\mu)(1+\lambda\mu)\right].$$

The term in the last bracket vanishes by Lemma 10, so that derivatives of vectors in Π along M are in Π. Since M is connected, Π is constant.

Finally, we observe that at each point x of M the tangent space to the $(n-1)$-sphere given by the orbit of the action of $O(\Pi^\perp)$ is orthogonal to x, p and q, hence to x, ν and e_1, and so coincides with the span of $\{e_2,\dots,e_n\}$. Therefore the action of $O(\Pi^\perp)$ is tangent to M at each point and preserves M, so M is rotationally symmetric. □

From Lemma 10 we can deduce the following Simons-type identity for the class of hypersurfaces we are considering:

Proposition 11. *Let M be a complete hypersurface in S^{n+1} with two principal curvatures λ,μ of multiplicity 1 and $n-1$ respectively, satisfying the equation $\lambda+\alpha\mu=0$. Then*

$$\mathcal{L}\lambda = \frac{2}{1+\alpha}\frac{|\nabla\lambda|^2}{\lambda} + \frac{1+\alpha}{\alpha}\lambda(\alpha-\lambda^2).$$

Proof. This follows from equations (1) and (2) using the relations $\lambda=-\alpha\mu$ and $e_1\lambda = -\alpha e_1\mu$, since $\mathcal{L}\mu = \nabla^2\mu(e_1,e_1)+\sum_{i=2}^n \frac{\alpha}{n-1}\nabla^2\mu(e_i,e_i)$. □

3. Interior ball curvature

In this section, we use the non-collapsing argument previously employed in [A, B2, AL]. Let M^n be an embedded hypersurface in $S^{n+1}\subset\mathbb{R}^{n+2}$ given by an embedding F, with two principal curvatures λ and μ of multiplicity 1 and $n-1$ respectively, satisfying the equation $\lambda+\alpha\mu=0$ for some $\alpha>0$. Noting that $\lambda\neq\mu$ and therefore $\mu\neq 0$ everywhere, we choose the direction of the unit normal in such a way that $\lambda>0>\mu$ everywhere. The hypersurface bounds a region Ω in S^{n+1}, which we choose in such a way that the unit normal vector ν of M^n is pointing out of Ω. For any point $x\in M^n$, the arguments in [AL] and [ALM] show that the existence of a ball in Ω of boundary curvature Φ which touches at x is equivalent to the inequality

$$k(x,y) := \frac{2\langle\nu(x), x-y\rangle}{|x-y|^2} \le \Phi(x), \quad \text{for all } y\in M.$$

where $\nu(x)$ is the unit normal vector of M at x as a hypersurface of S^{n+1}. In particular, the largest ball in Ω which touches M at x has boundary curvature equal to $\bar{k}(x) := \sup\{k(x,y):\ y\in M\setminus\{x\}\}$. If the supremum is achieved at some $\bar{y}\in M\setminus\{x\}$, then there is a ball of boundary curvature $\bar{k}(x)$ in Ω which touches at both x and $\bar{y}$. Note that this ball is simply the intersection of the ball $B(P,\bar{k}^{-1}(x))$ in $\mathbb{R}^{n+2}$ with S^{n+1}, where $P = F(x)-\bar{k}^{-1}(x)\nu(x)$ and $B(P,r)$ denotes the ball centred at P with radius r.

Proposition 12. *Let M be a compact embedded hypersurface in S^{n+1} which has two principal curvature $\lambda \neq \mu$ of multiplicity 1 and $n-1$ respectively, satisfying $\lambda + \alpha\mu = 0$ for some $\alpha > 0$. Then on $\tilde{M} = \{x : \bar{k}(x) > \lambda(x)\}$, the function $\bar{k}$ is a viscosity subsolution of the equation*

$$\mathcal{L}\bar{k} - 2\frac{(e_1\bar{k})^2}{\bar{k}-\lambda} + \frac{1+\alpha}{\alpha}(\lambda^2 - \alpha)\bar{k} \geq 0.$$

Proof. We explain below the meaning of viscosity solution (see also [ALM]). By Proposition 9, M is a rotational hypersurface, invariant under the action of the group of rotations $O(\Pi^\perp)$ fixing a two-dimensional subspace Π. Choose a three-dimensional subspace $\tilde{\Pi}$ containing Π. Then M is given by the action of $O(\Pi^\perp)$ on a curve $\gamma = M \cap \tilde{\Pi}$ in the two-dimensional sphere $S = S^{n+1} \cap \tilde{\Pi}$. The vector e_1 is tangent to γ at any $x \in \gamma$, and $\tilde{\Pi}$ is the span of x, $\nu(x)$ and $e_1(x)$ at any such point.

We fix any $\bar{x} \in \tilde{M}$ and suppose $\bar{y}$ is chosen to maximize $k(\bar{x}, \bar{y})$, so that $\bar{k}(\bar{x}) = k(\bar{x}, \bar{y})$ (note that the assumption $\bar{x} \in \tilde{M}$ ensures that the supremum is attained at some $\bar{y} \neq \bar{x}$). Since k and λ are invariant under the isometric action of $O(\Pi^\perp)$ on M, we can rotate so that $\bar{x} \in \tilde{\Pi}$. Therefore also $\nu(\bar{x})$ and $e_1(\bar{x})$ are in $\tilde{\Pi}$. The ball $B(P, \bar{k}(\bar{x})^{-1})$ touches M at both $\bar{x}$ and $\bar{y}$. In particular we have $P = \bar{x} - \bar{k}(\bar{x})^{-1}\nu(\bar{x}) = \bar{y} - \Phi(\bar{x})^{-1}\nu(\bar{y}) \in \tilde{\Pi}$. We also know that $p = \mu(\bar{y})\bar{y} - \nu(\bar{y}) \in \Pi \subset \tilde{\Pi}$, and $\bar{k}(\bar{x}) \geq \lambda(\bar{y}) > \mu(\bar{y})$, so it follows that $\bar{y} \in \tilde{\Pi}$ and therefore $\bar{y} \in \gamma$. The identity $P = \bar{x} - \bar{k}(\bar{x})^{-1}\nu(\bar{x}) = \bar{y} - \bar{k}(\bar{x})^{-1}\nu(\bar{y})$ also gives the useful identity

$$\nu(\bar{y}) = \nu(\bar{x}) + \bar{k}(\bar{x})(\bar{y} - \bar{x}). \tag{6}$$

Let $\vec{l} = \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|}, d = |\bar{y} - \bar{x}|$, and let $R_{\vec{l}}$ be the reflection in the hyperplane perpendicular to $\vec{l}$, given by $R_{\vec{l}}(v) = v - 2\langle v, \vec{l}\rangle\vec{l}$. Then by equation (6) we have

$$R_{\vec{l}}(\nu(\bar{x})) = \nu(\bar{x}) - 2\langle \nu(\bar{x}), \vec{l}\rangle\vec{l} = \nu(\bar{x}) + \bar{k}(\bar{x})d\vec{l} = \nu(\bar{y}). \tag{7}$$

We conclude that $R_{\vec{l}}$ maps $T_{\bar{x}}M$ isometrically into $T_{\bar{y}}M$. Furthermore, since $\bar{x}$ and $\bar{y}$ are both in $\tilde{\Pi}$, and $e_1(\bar{x})$ is in $\tilde{\Pi}$, we have that $R_{\vec{l}}(e_1(\bar{x})) \in \tilde{\Pi}$, and therefore $e_1(\bar{y}) := R_{\vec{l}}(e_1(\bar{x}))$ is an eigenvector of $h(\bar{y})$ with eigenvalue $\lambda(\bar{y})$. It follows also that $e_j(\bar{y}) := R_{\vec{l}}(e_j(\bar{x})) = e_j(\bar{x})$ defines an orthonormal basis for the eigenspace of $h(\bar{y})$ with eigenvector $\mu(\bar{y})$.

To prove the proposition we must show the following: If $\varphi : U \to \mathbb{R}$ is a smooth function on an open neighbourhood U of $\bar{x}$ in $\tilde{M}$, and $\varphi \geq \bar{k}$ on U with equality holding at $\bar{x}$, then φ satisfies

$$\left(\mathcal{L}\varphi - 2\frac{(e_1\varphi)^2}{\varphi - \lambda} + \frac{1+\alpha}{\alpha}(\lambda^2 - \alpha)\varphi\right)\Big|_{\bar{x}} \geq 0.$$

Let x_i, y_i be geodesic normal coordinates for M around $\bar{x}$ and $\bar{y}$ respectively, chosen in such a way that $\partial_i\big|_{\bar{x}} = e_i(\bar{x})$ and $\partial_i(\bar{y}) = e_i(\bar{y})$ for $i = 1, \ldots, n$.

We observe that since $\varphi(x) \geq \bar{k}(x) = \sup_y k(x,y)$, we have $\varphi(x) \geq k(x,y)$ for all $x \in U$ and all y, with equality holding at $(\bar{x},\bar{y})$. Since both φ and k are smooth, it follows that $\frac{\partial\varphi}{\partial x^i} = \frac{\partial k}{\partial x^i}$ and $\frac{\partial k}{\partial y^i} = 0$ for each i at $(\bar{x},\bar{y})$, while

$$D^2\varphi\big|_{\bar{x}}(v,v) \geq D^2k\big|_{(\bar{x},\bar{y})}((v,w),(v,w))$$

for any $v \in T_{\bar{x}}M$ and any $w \in T_{\bar{y}}M$. It follows that

$$\mathcal{L}\varphi\big|_{\bar{x}} \geq a^{ij}\left(\partial_j^x + \partial_j^y\right)\left(\partial_i^x + \partial_i^y\right)k\big|_{(\bar{x},\bar{y})},$$

where we denote by ∂_i^x and ∂_i^y the coordinate tangent vectors at $\bar{x}$ and $\bar{y}$ respectively. We now proceed to compute the right hand side: The first derivatives are as follows:

$$\begin{aligned}
(\partial_i^x + \partial_i^y)\,k\big|_{(\bar{x},\bar{y})} &= \frac{2}{d^2}\left((\partial_i^x - \partial_i^y)\cdot\nu_x + (x-y)\cdot(h^x)_i^p\partial_p^x - k(x-y)\cdot(\partial_i^x - \partial_i^y)\right)\\
&= \frac{2}{d^2}\left((\partial_i^x - \partial_i^y)\cdot\left(\nu_x + kd\vec{l}\right) - d(h^x)_i^p\partial_p^x\cdot\vec{l}\right).
\end{aligned}$$

Here the y derivatives vanish, while the x derivatives yield $e_1\varphi = e_1k = \frac{2(k-\lambda)\partial_1^x\cdot\vec{l}}{d}$ and $e_ik = 0$ for $i > 1$. We differentiate again, multiply by a^{ij} and take the sum, yielding

$$\begin{aligned}
a^{ij}\left(\partial_j^x + \partial_j^y\right)\left(\partial_i^x + \partial_i^y\right)k\big|_{(\bar{x},\bar{y})} = \frac{2}{d^2}a^{ij}\Big(&-\nabla^p h_{ij}^x d\vec{l}\cdot\partial_p^x - 2(h^x)_i^p\partial_p^x\cdot(\partial_j^y - \partial_j^x)\\
&+(h^x)_i^p d\vec{l}\cdot(h_{pj}^x\nu_x + g_{pj}x)\\
&+\left(-h_{ij}^x\nu_x - g_{ij}x + h_{ij}^y\nu_y + g_{ij}y\right)\cdot(\nu_x + kd\vec{l})\\
&-k(\partial_i^x - \partial_i^y)\cdot(\partial_j^x - \partial_j^y) + 2\partial_jk(\partial_i^x - \partial_i^y)\cdot d\vec{l}\Big).
\end{aligned}$$

Now we observe that $a^{ij}\nabla^p h_{ij} = \nabla^p(a^{ij}h_{ij}) = 0$ since ∇a^{ij} is off-diagonal in i and j while h_{ij} is diagonal, and $a^{ij}h_{ij} = 0$. We also note that $\nu_x + kd\vec{l} = \nu_x$, and that $\nu_x\cdot\vec{l} = -\frac{kd}{2}$, $\vec{l} = -\frac{d}{2}$, $a^{ij}h_{ik}^x g^{kl}h_{lj}^x = \lambda^2 + \alpha\mu^2 = \frac{1+\alpha}{\alpha}\lambda^2$, $a^{ij}g_{ij} = 1+\alpha$, and $a^{ij}h_{ij}^x = a^{ij}h_{ij}^y = 0$. The last identity uses the fact that both h^x and h^y are diagonal, $h_{11}^x = \lambda(x)$, $h_{11}^y = \lambda(y)$, and $h_{ii}^x = \mu(x)$ and $h_{ii}^y = \mu(y)$ for $i > 1$. Finally we observe that since $\partial_i^y = R_{\vec{l}}(\partial_i^x)$, we have for each i

$$\partial_i^y - \partial_i^x = -2\vec{l}\cdot\partial_i^x\vec{l}$$

Thus $\partial_i^y = \partial_i^x$ for $i = 1,\ldots,n$, while $\partial_1^y - \partial_1^x = -\frac{e_1k}{k-\lambda}d\vec{l}$. From this we find

$$\begin{aligned}
a^{ij}\left(\partial_j^x + \partial_j^y\right)\left(\partial_i^x + \partial_i^y\right)k &= -\frac{1+\alpha}{\alpha}k\lambda^2 + (1+\alpha)k + 2\frac{(e_1k)^2}{k-\lambda}\\
&= \frac{1+\alpha}{\alpha}(\alpha - \lambda^2)\varphi + 2\frac{(e_1\varphi)^2}{\varphi-\lambda}.
\end{aligned}$$

This completes the proof. ∎

4. Proof of Theorem 3

We can now prove Theorem 5, which follows directly from Corollary 7 and the following theorem.

Theorem 13. *Let $F : M^n \to S^{n+1}(n \geq 3)$ be a compact embedded hypersurface with two distinct principal curvatures λ, μ, whose multiplicities are 1 and $n-1$ respectively. If for some positive number α,*

$$\lambda + \alpha\mu = 0,$$

then λ, μ are all constants, and M is congruent to the Clifford torus

$$S^1\left(\sqrt{\frac{1}{\alpha+1}}\right) \times S^{n-1}\left(\sqrt{\frac{\alpha}{\alpha+1}}\right).$$

Proof. By combining Proposition 11 and Proposition 12 we arrive at the following:

Lemma 14. *The function $f = \bar{k}/\lambda$ is a viscosity solution of the equation*

$$\mathcal{L}f - \frac{2}{(f-1)}(e_1 f)^2 - 2\frac{(f+1)}{\lambda(f-1)}e_1\lambda\, e_1 f - \frac{2f(\alpha f+1)}{\lambda^2(1+\alpha)(f-1)}(e_1\lambda)^2 \geq 0$$

on the set $U = \{f > 1\}$ in M.

Proof. If ψ is a smooth function which satisfies $\psi \geq f$ with equality at some point $\bar{x}$, then $\varphi = \lambda\psi$ is a smooth function satisfying $\varphi \geq \bar{k}$ with equality at $\bar{x}$. Proposition 12 therefore implies that

$$\mathcal{L}\varphi - 2\frac{(e_1\varphi)^2}{\varphi - \lambda} + \frac{1+\alpha}{\alpha}(\lambda^2 - \alpha)\varphi \geq 0$$

at the point $\bar{x}$. Expanding this in terms of derivatives of ψ and using the result of Proposition 11 we arrive at

$$\mathcal{L}\psi - \frac{2}{(\psi-1)}(e_1\psi)^2 - 2\frac{(\psi+1)}{\lambda(\psi-1)}e_1\lambda\, e_1\psi - \frac{2\psi(\alpha\psi+1)}{\lambda^2(1+\alpha)(\psi-1)}(e_1\lambda)^2 \geq 0$$

at the point $\bar{x}$, which proves the Lemma. □

Corollary 15. *$\bar{k} = \lambda$ everywhere on M.*

Proof. The inequality $\bar{k} \geq \lambda$ holds everywhere (compare [ALM]). If there exists a point with $\bar{k} > \lambda$ then the set U is a non-empty open set on which $f > 1$, and on the boundary of U (if non-empty) we have $f = 1$. In particular f attains an interior maximum on U. Lemma 14 and the strong maximum principle for viscosity solutions of uniformly elliptic equations (see for example [CC]) implies that f is a constant larger than 1. But at the point of largest maximum principle curvature on M we have $\bar{k} = \lambda$, hence $f = 1$. This is a contradiction, so $\bar{k} = \lambda$ everywhere on M. □

We now complete the proof, by showing that $e_1\lambda$ is identically zero, using essentially the argument from [B2]: For any $x \in \gamma$ we have $\bar{k} = \lambda$, and hence the inequality $k(x, \gamma(t)) \le \lambda(x)$ holds for all s and t (with respect to a local arc-length parametrisation of γ, which we choose to have $\gamma(0) = x$). This may be written as

$$Z(t) = \lambda(x)|x - \gamma(t)|^2 - 2\nu(x) \cdot (x - \gamma(t)) \ge 0.$$

We have $Z(0) = 0$, and differentiation gives

$$Z'(t) = 2\left(\nu(x) - \lambda(x)(x - \gamma(t)\right) \cdot \gamma'(t),$$

so that $Z'(0) = 0$. Differentiating again gives

$$Z''(t) = -2\left(\nu(x) - \lambda(x)(x - \gamma(t)\right) \cdot \left(\lambda_{\gamma(t)}\nu_{\gamma(t)} + \gamma(t)\right) + 2\lambda(x)|\gamma'(t)|^2,$$

so that $Z''(0) = -2\lambda(x) + 2\lambda(x) = 0$. It follows that $Z'''(0) = 0$ since otherwise positivity of Z if violated. This gives

$$0 = Z'''(0) = -2e_1\lambda(x).$$

Since $\bar{x} \in \gamma$ is arbitrary and M is rotationally symmetric, λ is constant. By the equation μ is also constant, and M is a Clifford product. □

Remark 1. *If $m = 1$, choose $\alpha = n - 1$, then our Theorem 3 implies that any compact embedded rotational minimal hypersurface is either totally geodesic, i.e. S^n, or the Clifford torus*

$$S^1\left(\sqrt{\frac{1}{n}}\right) \times S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right),$$

which was proved by T. Ôtsuki [Ô1, Ô2].

Remark 2. *In* [LW]*, H. Li and G. Wei extended Ôtsuki's theorem to the case when M is rotational and $H_m = 0$, where H_m is the normalized m-th symmetric function of the principal curvatures. In fact, in this case, they get a relation between λ and μ:*

$$\mu^{m-1}\big(m\lambda + (n - m)\mu\big) = 0.$$

Our theorem here gives a simple proof of their result.

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Mathematical Sciences Institute, Australia National University; and Mathematical Sciences Center, Tsinghua University.

E-mail address: Ben.Andrews@anu.edu.au

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China

E-mail address: huangzj07@mails.tsinghua.edu.cn

Department of Mathematical Sciences and Mathematical Sciences Center, Tsinghua University, 100084, Beijing, P. R. China

E-mail address: hli@math.tsinghua.edu.cn