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Gradient and oscillation estimates and their applications in geometric PDE

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Abstract. We describe some recent 'oscillation' estimates in geometric PDE, where estimates are produced using the maximum principle applied to functions depending on several points. Applications include sharp short-time regularity results, sharp long-time behaviour which related closely to optimal estimates on eigenvalues, and elegant proofs of several key results on geometric evolution equations.

1. A brief history of gradient estimates

Gradient estimates are bread and butter in PDE theory, but I want to concentrate here on a specific part of that wide picture. The techniques I have in mind are gradient estimates proved using the maximum principle in the spirit of the work of Cheng and Yau [CY] on eigenfunctions and solutions of semilinear equations, and Yau's gradient estimates for harmonic functions [Y1], which in particular gave non-existence of non-constant bounded harmonic functions on complete manifolds of non-negative Ricci curvature. More specifically the topics I will discuss later will have connections with work which used such gradient bounds to prove the first eigenvaue of the Laplacian on a compact manifold, beginning with the work of Peter Li [Li] who gave a lower bound on λ_1 in terms of diameter for manifolds with non-negative Ricci curvature. This was developed by Li and Yau [LY] to a sharper estimate (which I will described briefly below), and culminated in the work of Zhong and Yang [ZY] who gave a sharp lower bound on λ_1 in terms of diameter for manifolds with non-negative Ricci curvature. There is a more general picture which I will discuss later.

Before going on to the main topic (which is not gradient estimates but a somewhat different technique) I want to briefly outline how gradient estimates yield eigenvalue estimates in the work of Li and Yau. Suppose we have an eigenfunction u on a compact manifold M of non-negative Ricci curvature, so that $\Delta u + \lambda u = 0$ everywhere. The eigenfunction can be normalized by scaling so that it takes values in the range $[-a, 1]$ for some $a \in (0, 1]$. Li and Yau proved the following gradient

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estimate:

$$\frac{|\nabla u|^2}{1-u^2} \leq \lambda. \tag{1.1}$$

The motivation for considering the quantity on the left is that it is constant in the case of an eigenfunction in one dimension with this range of values (where we also have $a = 0$): In that case (where the manifold is the circle $\mathbb{R}/(2d\mathbb{Z})$ of intrinsic diameter d) the first eigenfunction is $\sin(\pi x/d)$, with eigenvalue π^2/d^2, and we have $|\nabla u|^2 = \pi^2/d^2 \cos^2(\pi x/d) = \lambda(1-u^2)$. The proof of the bound uses the maximum principle: Define $Q = \frac{|\nabla u|^2}{1-u^2}$. Differentiating the eigenfunction equation and manipulating leads to an equation for ΔQ (here the geometry enters when we commute derivatives producing a curvature term which we can discard under the assumption of non-negative Ricci curvature). Applying this at a maximum point of Q, and using the conditions that $\Delta Q \leq 0$ and $\nabla Q = 0$, and also using the eigenfunction equation, we arrive at the inequality

$$0 \leq -\frac{2u^2}{(n-1)(1-u^2)}(Q-\lambda)^2 - \frac{2Q}{1-u^2}(Q-\lambda).$$

But if $Q > \lambda$ at the maximum point, we have a contradiction since the right-hand side is then negative. In deriving this inequality one has to be extremely careful and make use of every available piece of information. There is a useful guiding idea: To obtain a sharp estimate one should make sure that equality is attained everywhere in the one-dimensional case. Later this same principle will be useful when we look at oscillation estimates.

The gradient estimate (1.1) implies $|\nabla \arcsin u| \leq \sqrt{\lambda}$, and integrating between the points where $u = 1$ and where $u = -a$ gives $\pi/2 + \arctan a \leq \sqrt{\lambda} d$, where d is the diameter of M. This rearranges to give $\lambda \geq (\frac{\pi}{2} + \arctan a)^2 d^{-2}$. Note that if we knew $a = 1$ then we would get the sharp bound $\lambda \geq \frac{\pi^2}{d^2}$, but the best we can say is that $\arctan a > 0$, so that $\lambda > \frac{\pi^2}{4d^2}$ — the lack of sharpness in the estimate comes entirely from the issue of asymmetry of the eigenfunction, i.e. the possibility that we could have $a \neq 1$).

In fact Li and Yau had a slightly smarter argument which improved the bound to $\pi^2/2d^2$. Later Zhong and Yang [ZY] improved this by giving a more delicate estimate to deal with the case $a < 1$, and obtained the sharp result $\lambda \geq \frac{\pi^2}{d^2}$ (equality is attained asymptotically for tori with thickness approaching zero). The argument required is considerably more involved and I will not go into the details here.

2. Oscillation estimates: The Kruzhkov argument

Now I want to introduce the method I will be using as an alternative to gradient estimates, which is essentially to estimate not just the gradient but the entire modulus of continuity. I will start with the simplest situation: Equations in one dimension. Here the idea of using quantities depending on two points was used by Kruzhkov to treat quite general elliptic and parabolic equations. Let us consider parabolic equations of the form $u_t = au_{xx} + bu_x + cu + d$, where a, b, c and d can depend on space and time. The basis of Kruzhkov's method is to consider a function of two spatial variables, defined by

$$v(x, y, t) = u(y, t) - u(x, t).$$

It is a simple observation that the function v satisfies the equation

$$v_t = a(y,t)v_{yy} + a(x,t)v_{xx} + b(y,t)v_y + b(x,t)v_x + F(x,y,t)$$

where $F(x,y,t) = c(y)u(y,t) - c(x)u(x,t) + d(y,t) - d(x,t)$. If we suppose that u is bounded, then F is bounded and v satisfies a relatively well-behaved parabolic equation. Despite the fact that we are doubling the number of spatial variables, there is a crucial advantage to this formulation: The function v vanishes along the diagonal $\{y = x\}$, and a *boundary* gradient estimate for v along this boundary implies a *global* gradient bound for u: The normal derivative at the point (x,x) on this boundary is just a multiple of $u'(x)$. But boundary gradient estimates are easy to produce by simply constructing a barrier, so we get an easy proof of gradient bounds for u.

3. Oscillation estimates in higher dimensions

Now let us consider what happens when we consider this kind of estimate in higher dimensions: At first sight, what we want to do is control $|u(y) - u(x)|$ in terms of the separation $|y-x|$ (and perhaps elapsed time). This amounts to keeping negative a quantity $Z(x,y,t) = u(y,t) - u(x,t) - \psi(|y-x|,t)$ for some suitable function ψ (which we must choose in such a way that Z is initially non-positive). Z is then a function of $2n$ spatial variables — as before we have doubled the number of spatial variables. However the 'boundary' corresponding to the diagonal is no longer so nice: This is the set $\{(x,x) :\ x \in \mathbb{R}^n\}$, which is not of codimension 1 but of codimension n. At this point it looks like the program is sunk, because as we all know there is no hope of finding solutions of a parabolic equation in $\mathbb{R}^{2n}$ with prescribed behaviour on a high codimension submanifold. However, let us look a bit closer at a concrete example: Consider the heat equation on $\mathbb{R}^n$, and for simplicity let us assume the solution is spatially periodic in some lattice, so that we do not have to be concerned about behaviour on the boundary. That is, we have a solution $u:\ \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x^i \partial x^i}, \tag{3.1}$$

such that $u\left(x + \sum_{i=1}^n k_i e_i, t\right) = u(x,t)$ for all $x \in \mathbb{R}^n$, $k_1, \dots, k_n \in \mathbb{Z}$, and $t \geq 0$, where $\{e_1, \dots, e_n\}$ is some basis for $\mathbb{R}^n$. We can compute an evolution equation for Z as follows:

$$\frac{\partial Z}{\partial t} = \Delta u(y,t) - \Delta u(x,t) - \frac{\partial \psi}{\partial t}. \tag{3.2}$$

We want to think of this as a parabolic equation for Z, and we can indeed do this by writing out the second derivatives of Z in terms of second derivatives of u: We have

$$\begin{aligned} \frac{\partial Z}{\partial y^i}\Big|_{(x,y,t)} &= D_i u(y,t) - \psi' \frac{y^i - x^i}{|y-x|}; \\ \frac{\partial Z}{\partial x^i}\Big|_{(x,y,t)} &= -D_i u(x,t) + \psi' \frac{y^i - x^i}{|y-x|}. \end{aligned}$$

A further differentiation of each of these gives the following:

$$\frac{\partial^2 Z}{\partial y^i \partial y^j}\Big|_{(x,y,t)} = D_i D_j u(y,t) - \psi'' \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2} - \frac{\psi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}\right)$$

$$\frac{\partial^2 Z}{\partial x^i \partial x^i}\Big|_{(x,y,t)} = -D_i D_i u(x,t) - \psi'' \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2} - \frac{\psi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}\right).$$

Summing these over i gives the following expression for the evolution of Z:

$$\frac{\partial Z}{\partial t} = \sum_i \frac{\partial^2 Z}{\partial y^i \partial y^i} + \sum_i \frac{\partial^2 Z}{\partial x^i \partial x^i} - \frac{\partial \psi}{\partial t} + 2\psi'' + 2(n-1)\frac{\psi'}{|y-x|}. \tag{3.3}$$

Here we see the trouble: We could apply a maximum principle to Z provided the extra terms $-\psi_t + 2\psi'' + 2(n-1)\frac{\psi'}{|y-x|}$ are non-positive, but it is not possible to find any useful solutions of this differential inequality which are zero for $|y-x| = 0$, because of the last term which is highly singular. However we are missing something here: There is more than one way to write Z as the solution of a parabolic equation. We must also compute the *mixed* second partial derivatives:

$$\frac{\partial^2 Z}{\partial y^i \partial x^j} = \psi'' \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2} + \frac{\psi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}\right).$$

Since these terms do not introduce any new second derivatives of u, we can write

$$\begin{aligned}\frac{\partial Z}{\partial t} &= \sum_i \frac{\partial^2 Z}{\partial y^i \partial y^i} + \sum_i \frac{\partial^2 Z}{\partial x^i \partial x^i} + 2\sum_{i,j} a^{ij}\frac{\partial^2 Z}{\partial y^i \partial x^j} \\ &\quad - \frac{\partial \psi}{\partial t} + 2\psi''\left(1 - a^{ij}\frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}\right) \\ &\quad + 2\frac{\psi'}{|y-x|}\left(n - 1 - a^{ij}\left(\delta_{ij} - \frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}\right)\right),\end{aligned} \tag{3.4}$$

for *any* choice $n \times n$ matrix a^{ij}. Note that the first line forms a (weakly) elliptic operator — the condition needed to apply the maximum principle — provided the $2n \times 2n$ matrix

$$\left[\begin{array}{c|c} \delta^{ij} & a^{ij} \\ \hline a^{ji} & \delta^{ij} \end{array}\right]$$

has non-negative eigenvalues. In particular we want to choose a^{ij} satisfying this condition to kill off the bad terms involving $\psi'/|y-x|$. A quick inspection shows that this can be done most effectively by choosing

$$a^{ij} = \delta^{ij} - 2\frac{(y^i - x^i)(y^j - x^j)}{|y-x|^2}.$$

To see this it is best to choose coordinates such that the first direction is a multiple of $y - x$, so the other $n-1$ directions are orthogonal to $y - x$. Then we have

$a^{11} = -1$, $a^{ii} = 1$ for $i > 1$, and $a^{ij} = 0$ for $i \neq j$. From this we can see easily that the required non-negativity of the eigenvalues holds, and we get

$$\frac{\partial Z}{\partial t} = \sum_i \frac{\partial^2 Z}{\partial y^i \partial y^i} + \sum_i \frac{\partial^2 Z}{\partial x^i \partial x^i} + 2\sum_{i,j} a^{ij} \frac{\partial^2 Z}{\partial y^i \partial x^j} - \frac{\partial \psi}{\partial t} + 4\psi''. \tag{3.5}$$

Now we can apply the maximum principle to keep Z negative provided $\psi_t \geq 4\psi''$. This is equivalent to choosing $\psi(z) = 2\varphi(z/2)$, where φ is a supersolution of the one-dimensional heat equation: $\frac{\partial \varphi}{\partial t} = \varphi''$. In particular if $|u| \leq M$ we can choose M to have initial data $\varphi(z, 0) = M$ for $z > 0$, and fix $\varphi(0, t) = 0$ for $t > 0$, yielding an explicit solution involving the error function. There is a nice interpretation of the result we have just proved, which shows in particular that the result is sharp (in a certain sense made precise below): Consider a special solution of the heat equation which is essentially one-dimensional — precisely, which is a function of only one of the component functions on $\mathbb{R}^n$: $u(x, t) = \varphi(x^1, t)$ where $x = (x^1, \dots, x^n)$. Then φ satisfies the one-dimensional heat equation $\varphi_t = \varphi''$. Suppose in addition that $\varphi(., t)$ is odd, and is concave for positive values. Then it is easy to prove that the largest value of $u(y, t) - u(x, t)$ among all y and x with $|y - x|$ fixed occurs for $y = -x = \frac{|y-x|}{2}$:

$$\max\{|u(y, t) - u(x, t)| : \ |y - x| = z\} = 2\varphi\left(\frac{z}{2}, t\right).$$

It follows that in this special example $Z = 0$ on the set $\{(x, y, t) : \ (x + y) \cdot e_1 = 0, \ t \geq 0\}$. So we can interpret the result as saying: If the initial data has modulus of continuity bounded by that of the initial data for the symmetric solution, then this remains true for all positive times.

The fact that equality holds on this special solution tells us something interesting, which is a useful guiding principle when trying to find the 'right' way to combine terms in the evolution equation: According to the strong maximum principle (for instance, in the form given by Hill [H3]), if Z evolves by a weakly parabolic eqution and has an interior zero somewhere, then the zero set is closed under transport along directions where the diffusion ceofficients are non-vanishing. This corresponds nicely to the $D^2 Z$ terms which arise in the evolution equation for Z: We have nontrivial coefficients in the directions (e_j, e_j) for each $j > 1$, and in the direction $(e_1, -e_1)$. The span of these vectors is exactly the tangent space to the zero set in the symmetric example.

To summarize: To make the argument work we must squeeze everything we can out of the full $2n \times 2n$ matrix of second derivatives of Z, and we must produce an evolution equation in which the top order terms arising are only those corresponding to the directions tangent to the zero set of Z in the symmetric case. We will see this picture recurring in several different situations later.

4. Short term regularity: Nonlinear equations

An obvious application of the 'oscillation' estimates I just described is in proving regularity of solutions. For the heat equation this is well known of course, but one of the nice things about the argument is that it can be applied in much greater generality. The initial motivation for developing the technique came from joint work with Julie Clutterbuck, in which we were trying to prove interior gradient estimates for solutions of anisotropic mean curvature flows. In the simplest case of the graphical mean curvature flow one can prove useful gradient estimates directly

— see [ES2] or [CM] — but in anisotropic cases this seems to be possible only in special cases [C]). The method described here gives time-interior gradient estimates for quite general anisotropic flows, via oscillation estimates. This appeared in [AC1] and [AC2]. The method works for quite general equations in which the coefficients depend on the gradient:

$$u_t = a^{ij}(Du,t)D_iD_ju, \tag{4.1}$$

where the coefficients (a^{ij}) depend smoothly on Du and t, and are positive definite for each value of Du and $t \geq 0$. All we need is a nondegeneracy requirement: We suppose that there exists a continuous $\alpha : \mathbb{R}^+ \times [0,T] \to \mathbb{R}_+$ with

$$0 < \alpha(R,t) \leq R^2 \inf_{|p|=R,(v\cdot p)\neq 0} \frac{v^T A(p,t)v}{(v\cdot p)^2}. \tag{4.2}$$

This includes in particular the anisotropic mean curvature flows for arbitrary smooth positive mobility and smooth uniformly convex Wulff shape [TCH].

Define $Z(x,y,t) := u(y,t) - u(x,t) - 2\varphi\left(\frac{|y-x|}{2}, t\right)$ on the domain $\{y \neq x\}$. We will choose φ so that $Z \leq 0$ for $t = 0$, and impose boundary conditions to guarantee that $Z \leq 0$ at the boundary $\{y = x\}$ for any t. To keep the discussion simple I will brush aside issues to do with behaviour on the boundary, and worry only about what happens near interior points. The key observation is that Z is non-increasing at any interior maximum point, if we choose φ in the right way: Given such a point we choose an orthonormal basis for $\mathbb{R}^n$ such that so that $e_1 = \frac{y-x}{|y-x|}$. The first derivatives vanish, giving

$$\begin{aligned}
0 &= \frac{\partial Z}{\partial x^i} = -D_iu(x,t) + \varphi'\frac{y^i - x^i}{|y-x|} \Longrightarrow Du(x,t) = \varphi' e_1;\\
0 &= \frac{\partial Z}{\partial y^i} = D_iu(y,t) - \varphi'\frac{y^i - x^i}{|y-x|} \Longrightarrow Du(y,t) = \varphi' e_1.
\end{aligned} \tag{4.3}$$

The matrix of second derivatives $[D^2Z]$ is also negative semi-definite, with entries given by the expressions in the previous section.

At the maximum point,

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= u_t(y,t) - u_t(x,t) - 2\frac{\partial\varphi}{\partial t}\\
&\leq a^{ij}(D_yu,t)D_iD_ju(y,t) - a^{ij}(D_xu,t)D_iD_ju(x,t) - 2\frac{\partial\varphi}{\partial t}\\
&= \operatorname{trace}\left(\begin{bmatrix} A(\varphi' e_1) & C\\ C^T & A(\varphi' e_1)\end{bmatrix} D^2Z\right) - 2\frac{\partial\varphi}{\partial t}\\
&\quad + \left(a^{11} - c^{11}\right)\varphi'' + 2\frac{\varphi'}{|y-x|}\sum_{i=2}^n\left(a^{ii} - c^{ii}\right),
\end{aligned} \tag{4.4}$$

where we have added and subtracted $2\operatorname{trace}(CZ_{xy})$, for some $n \times n$ matrix C. If we choose C so that the $2n \times 2n$ matrix

$$A' = \begin{bmatrix} A & C\\ C^T & A\end{bmatrix}$$

is positive semi-definite, then the trace term in (4.4) will be non-positive.

To make the coefficient of $\varphi'/|y-x|$ zero, we require $c^{ij} = a^{ij}(\varphi' e_1)$ for $(i,j) \neq (1,1)$. Finally, we choose c^{11} to maximise the coefficient of φ''. The condition

$A' \geq 0$ is equivalent to $0 \leq 2v^T A(\varphi' e_1)v - \left(a^{11} - c^{11}\right)(v^1)^2$, so we choose $c^{11} = a^{11}(\varphi' e_1) - 2\alpha(|\varphi'|)$ where α is defined by (4.2). This gives the following inequality at (x_0, y_0, t_0):

$$\frac{\partial Z}{\partial t} < 2\left(\alpha\left(|\varphi'|\right)\varphi'' - \frac{\partial \varphi}{\partial t}\right),$$

so we have $\frac{\partial Z}{\partial t} \leq 0$ if we choose φ to satisfy $\varphi_t = \alpha(|\varphi'|)\varphi''$. This provides gradient estimates for u at positive times, provided the particular solution of the one-dimensional equation has bounded gradient for positive time (we derive in [AC1] a necessary and sufficient condition for this to occur, which says roughly that $p^2\alpha(p)$ does not approach zero as $p \to \infty$). In particular this gives gradient bounds for bounded solutions of graphical anisotropic mean curvature flows.

5. Invitation to fully nonlinear PDE: Bellmann equations are everywhere

A quick diversion here: The derivations above were aimed at reducing the regularity estimates to scalar equations, by bounding $u(y) - u(x)$ solely in terms of distance $|y - x|$ at each time, and this works nicely in the setting above where the coefficients depend on the gradient. More generally we can consider bounding $u(y) - u(x)$ by a suitable 'barrier' which is a function of $2n$ variables. What kind of equation must the barrier satisfy to make this work? Precisely, consider

$$Z(x, y, t) = u(y, t) - u(x, t) - f(x, y, t).$$

If u evolves by some parabolic equation (let us say for simplicity the heat equation, but the idea goes through much more generally), what conditions must be satisfied by the function f in order for the inequality $Z \leq 0$ to be preserved by the maximum principle? We can compute exactly as before the second derivatives:

$$\begin{aligned}
\frac{\partial^2 Z}{\partial x^i \partial x^j} &= -D_i D_j u(x) - \frac{\partial^2 f}{\partial x^i \partial x^j};\\
\frac{\partial^2 Z}{\partial y^i \partial y^j} &= D_i D_j u(y) - \frac{\partial^2 f}{\partial y^i \partial y^j};\\
\frac{\partial^2 Z}{\partial x^i \partial y^j} &= -\frac{\partial^2 f}{\partial x^i \partial y^j}.
\end{aligned}$$

Thus we can write Z as a solution of a parabolic equation, with quite a large degree of freedom in how we do so:

$$\frac{\partial Z}{\partial t} = \operatorname{trace}\left(MD^2 Z\right) - \frac{\partial f}{\partial t} + \operatorname{trace}\left(MD^2 f\right),$$

for any non-negative definite $2n \times 2n$ matrix M (which can depend on x, y and t) of the form

$$M = \left[\begin{array}{c|c} I_n & B \\ \hline B^T & I_n \end{array}\right].$$

Thus Z satisfies a maximum principle (i.e. can be kept negative) provided f is a solution of the fully nonlinear degenerate Bellmann-type equation

$$\frac{\partial f}{\partial t}(x,y,t) = \inf\left\{\operatorname{trace}\left(MD^2f\right)\ \Big|\ M = \left[\begin{array}{c|c} I & B \\ \hline B^T & I \end{array}\right] \geq 0\right\}$$
$$= \Delta_x f + \Delta_y f - 2\operatorname{trace}\left(\left(\left(D_xD_yf\right)^T\left(D_xD_yf\right)\right)^{1/2}\right).$$

Particular solutions of this equation include those of the form $f = 2\varphi(|y-x|/2, t)$ where φ satisfies the 1D heat equation and is concave in the first argument.

Understanding equations of this kind seems like a useful project with applications to many regularity questions related to those above.

6. Heat equation on a manifold

Next I want to venture into more geometric territory, and look at how the technique I described can be adapted to equations on manifolds. In particular, what changes if we consider the heat equation on a Riemannian manifold? We can try essentially the same idea of bounding differences in values of the solution in terms of the separation of the points, this time measured with respect to the distance function on the manifold induced by the Riemannian metric: That is, we let $Z(x,y,t) = u(y,t) - u(x,t) - 2\varphi\left(\frac{d(x,y)}{2}, t\right)$. Note that d is a smooth function on $M \times M$ away from the diagonal $\{y = x\}$ and the cut locus. The time derivative looks identical to what we had before:

$$Z_t = \Delta u(y) - \Delta u(x) - 2\varphi_t.$$

But the spatial derivatives look a little different: Given local coordinates $\{x^i\}$ neat the point x, and local coordinates $\{y^i\}$ near the point y, we find: Asssuming that (x,y) is not on the cut locus, we have

$$\frac{\partial Z}{\partial x^i} = -D_iu(x) - \varphi' Dd\left(\frac{\partial}{\partial x^i}, 0\right),$$

while

$$\frac{\partial Z}{\partial y^i} = D_iu(y) - \varphi' Dd\left(0, \frac{\partial}{\partial y^i}\right).$$

The picture is like this: Since we are not on the cut locus there is a unique unit speed minimizing geodesic $\gamma : [0,d] \to M$ with $\gamma(0) = x$ and $\gamma(d) = y$, and then $Dd\Big|_{(x,y)}(u,v) = \langle\gamma'(d), v\rangle - \langle\gamma'(0), u\rangle$. So we can simplify the result by choosing the local coordinates as follows: First choose an orthonormal basis $\{e_1, \dots, e_n\}$ at x so that $e_1 = \gamma'(0)$. Then parallel translate along γ to get an orthonormal basis $\{e_i(s)\}$ at $\gamma(s)$, and choose local coordinates in a neighbourhood of γ by

$$(x^1, \dots, x^n) \mapsto \exp_{\gamma(x^1)}\left(\sum_{i=2}^n x^i e_i(x^1)\right).$$

In particular we then have $\frac{\partial}{\partial y^i} = \gamma'(d)$, and $\left\{\frac{\partial}{\partial y^i}\right\}$ is an orthonormal basis at y. The first order conditions then give $D_1u(x) = D_1u(y) = \varphi'$, $D_iu(x) = D_iu(y) = 0$, just as in the Euclidean case.

Now proceed to the second derivatives. There is a nice simplification here (coming from the structure we saw in the Euclidean case) which allows us to compute only certain simple parts of the second derivatives rather than the entire $2n \times 2n$ matrix of second derivatives: We compute

$$0 \geq \frac{d^2}{ds^2} Z(\gamma(-s), \gamma(d+s))\Big|_{s=0} = \nabla_1\nabla_1 u(y) - \nabla_1\nabla_1 u(x) - 2\varphi'',$$

since in this case we have $\frac{d}{ds}d = 2$ and $\frac{d^2}{ds^2}d = 0$. Then for each $i > 1$ we compute

$$0 \geq \frac{d^2}{ds^2} Z(\exp_x(se_i), \exp_y(se_i))\Big|_{s=0} = \nabla_i\nabla_i u(y) - \nabla_i\nabla_i u(x) - \frac{\varphi'}{2} D^2 d\Big|_{(x,y)}(e_i, e_i),$$

since we have $Dd\big|_{(x,y)}(e_i, e_i) = 0$. Adding the first inequality and the sum over $i = 2, \dots, n$ of the second gives

$$\Delta u(y) - \Delta u(x) \leq 2\varphi'' + \frac{\varphi'}{2} \sum_{i=2}^{n} D^2 d\Big|_{(x,y)}(e_i, e_i).$$

Thus everything is identical to the Euclidean case, except for the last term involving second derivatives of d. This is where the geometry of the space comes into the calculation: In contrast to the gradient estimates where curvature enters directly when we commute derivatives to derive an equation for the derivatives of the solution, here it enters through the comparison geometry of the distance function. Precisely, if the Ricci curvature of M has eigenvalues bounded below by $(n-1)K$, then there is a function f_K such that $\sum_{i=2}^{n} D^2 d\big|_{(x,y)}(e_i, e_i) \geq 4f_K(d)$, with equality holding in the case where M is complete simply connected Riemannian manifold $\bar{M}_K$ with constant sectional curvature K. In the case where we hit the cut locus some extra work is required, but essentially the same inequality holds in a generalized sense which is sufficient for the argument. Thus provided $\varphi' \geq 0$ a lower Ricci curvature bound of this type gives us the following inequality at the maximum point:

$$\frac{\partial}{\partial t} Z \leq -2\varphi_t + 2\varphi'' + 2\varphi' f_K.$$

If we choose φ to be a solution of $\varphi_t = \varphi'' + f_K\varphi'$ we deduce that Z remains non-positive, proving the oscillation bound $u(y,t) - u(x,t) \leq 2\varphi(d(x,y)/2, t)$. Furthermore, the function φ has a nice interpretation: It is exactly a symmetric solution of the heat equation on a warped product manifold $N \times \mathbb{R}$ with a metric which has Ricci curvature K in the $\mathbb{R}$ direction equal to $(n-1)K$ and a reflection symmetry $(z, s) \mapsto (z, -s)$ (here by 'symmetric' I mean a solution which depends only on the $\mathbb{R}$ coordinate, so is constant on each slice $N \times \{s\}$). In particular we could take φ to be the solution which is M for $s > 0$ and $-M$ for $s < 0$ at $t = 0$.

To summarize, we have proved a sharp regularity result: The modulus of continuity of a solution of the heat equation on a manifold with Ricci curvature bounded below is controlled by that of a symmetric solution on a model space. This result was proved in joint work with Julie Clutterbuck (not yet published).

7. Harmonic map heat flow

Let me bring in some more geometry and consider the harmonic map heat flow. Here we have two Riemannian manifolds M and N, and consider maps between them (it can be convenient to visualize N as a submanifold of a high-dimensional Euclidean space $\mathbb{R}^k$). The harmonic map heat flow is the equation

$$\frac{\partial}{\partial t} u = \Delta u,$$

where Δu is the 'map Laplacian', which one can think of as the projection onto the tangent space of N of the Laplacian as a map into $\mathbb{R}^k$. Thus the heat equation is a special case (where the target is $\mathbb{R}$), and the general case amounts to imposing a nonlinear constraint on the values of the function. The harmonic map heat flow is in many ways the grandfather of all the geometric flows now in common use such as the Ricci flow: Eells and Sampson [ES1] used it to prove existence of harmonic maps into targets of nonpositive sectional curvature, and it was part of the inspiration which led Hamilton to the Ricci flow.

I will be as naïve as it is possible to be: As before, we want to bound the 'difference in values', i.e the difference between $u(y,t)$ and $u(x,t)$ for a solution of the harmonic map heat flow, in terms of the separation of the points x and y, which as before we interpret as the distance in M from x to y in the distance function induced by the Riemannian metric on M, which we will denote by $d_M(x,y)$. The only difference here is that the 'difference between $u(y,t)$ and $u(x,t)$' must also be interpreted in this way: These are now points in the target space N, so we interpret their 'difference' as the distance between them as measured in N. Thus we are led to consider the quantity

$$Z(x,y,t) = d_N(u(y,t),u(x,t)) - 2\varphi\left(\frac{d_M(x,y)}{2},t\right).$$

Now we proceed exactly as before. Supposing there is a point (x,y) in $M\times M$ with $x\neq y$ where an interior maximum of Z occurs (and noting that we can assume $u(y,t)\neq u(x,t)$), we set up local coordinates near x and y exactly as before by parallel transport along a minimizing geodesic from x to y (with the same minor modifications if we are on the cut locus). We also let $\sigma:[0,L]\to N$ be a minimizing geodesic from $u(x)$ to $u(y)$ in N. Then we compute the time derivative:

$$\frac{\partial}{\partial t}Z = \langle \Delta u(y),\sigma'(L)\rangle - \langle \Delta u(x),\sigma'(0)\rangle - 2\varphi_t.$$

Next we compute the first derivatives:

$$0=\frac{\partial Z}{\partial x^i} = Dd_N\Big|_{(u(y),u(x))}(\nabla_i u(x),0) - \varphi' Dd_M\Big|_{(x,y)}(e_i,0);$$
$$0=\frac{\partial Z}{\partial y^i} = Dd_N\Big|_{(u(y),u(x))}(0,\nabla_i u(y)) - \varphi' Dd_M\Big|_{(x,y)}(0,e_i).$$

With coordinates as before this gives $\langle\nabla_1 u(x),\sigma'(0)\rangle=\varphi'$ and $\langle\nabla_i u(x),\sigma'(0)\rangle=0$ for $i>1$, while $\langle\nabla_1 u(y),\sigma'(L)\rangle=\varphi'$ and $\langle\nabla_i u(y),\sigma'(L)\rangle=0$ for $i>1$. We can compute second derivatives also, choosing the same variations as before:

$$\begin{aligned}0\geq \frac{d^2}{ds^2}Z(\gamma(-s),\gamma(d+s))\Big|_{s=0} &= -\langle\nabla_1\nabla_1 u(x),\sigma'(0)\rangle + \langle\nabla_1\nabla_1 u(y),\sigma'(L)\rangle\\ &\quad + D^2 d_N\left((\nabla_1 u\big|_x,\nabla_1 u\big|_y),(\nabla_1 u\big|_x,\nabla_1 u\big|_y)\right) - 2\varphi'';\end{aligned}$$

and for $i>1$

$$\begin{aligned}0\geq \frac{d^2}{ds^2}Z(\exp_x(se_i),\exp_y(se_i))\Big|_{s=0} &= -\langle\nabla_i\nabla_i u(x),\sigma'(0)\rangle + \langle\nabla_i\nabla_i u(y),\sigma'(L)\rangle\\ &\quad + D^2 d_N\left((\nabla_i u\big|_x,\nabla_i u\big|_y),(\nabla_i u\big|_x,\nabla_i u\big|_y)\right)\\ &\quad - \frac{\varphi'}{2}D^2 d\Big|_{(x,y)}(e_i,e_i).\end{aligned}$$

Now we must assume something about the target manifold: If the sectional curvatures of N are non-positive, and N is simply connected, then the terms involving $D^2 d_N$ are non-negative. Thus if we also assume a lower Ricci curvature bound for M, then we find exactly as before that

$$\frac{\partial Z}{\partial t} \leq -2(\varphi_t - \varphi'' - f_K\varphi'),$$

so the modulus of continuity of a solution of the harmonic map heat flow in to a negatively curved target is controlled by that of a symmetric solution of the heat equation on a model space. In the case where N is not simply connected, we must first lift to a map from the universal cover of M to the universal cover of N, and there is a little more work to do in constructing a suitable solution of the differential inequality, but we can still arrive at bounds on the energy density for any positive time.

This is quite a nice result: The derivation of an energy density bound in the work of Eells and Sampson (and most later work) in somewhat nontrivial, and goes through the derivation of an evolution equation for the energy density, which with the curvature assumption amounts to a parabolic differential inequality. Some work is required to convert this into an upper bound on energy density, usually by bootstrapping from the assumption that the total energy is bounded. Here we can avoid altogether the derivation of an evolution equation for energy density, and we make no assumption on boundedness of initial energy, but derive a pointwise energy density bound which depends only on the modulus of continuity of the initial data (and perhaps also injectivity radii and diameters of source and target manifolds in the case where the target is not simply connected). This allows us to prove existence of a smooth solution of the harmonic map heat flow starting from an arbitrary continuous map from a compact manifold to a non-positively curved (compact or well-behaved at infinity) manifold.

8. A sharp eigenvalue estimate

So far we have used the oscillation estimates to prove short-time regularity for solutions of various kinds of parabolic equations, and this was indeed the initial motivation for developing the technique. However, the sharpness of the estimates is such that they tell us something very strong even for large times, and in particular we can deduce some very interesting consequences from this.

Let M be a compact Riemannian manifold, and consider any solution u of the heat equation. Our estimate above shows that we can estimate the modulus of continuity of u in terms of the solution of a one-dimensional equation depending on a lower bound for Ricci curvature:

$$|u(y,t) - u(x,t)| \leq 2\varphi\left(\frac{d_M(x,y)}{2}, t\right)$$

where $\varphi_t = \varphi'' + f_K\varphi'$ (and we also required $\varphi' \geq 0$). It is important also to note that if the diameter of M is d, then we need consider only solutions $\varphi(z,t)$ defined for $|z| \leq d/2$. In particular we could choose the particular solution with initial data M for $z > 0$ and $-M$ for $z < 0$, with $\varphi'(\pm d/2, t) = 0$. This solution will be odd and increasing for each t. In particular this implies

$$\operatorname{osc} u(.,t) \leq 2\varphi(d/2, t).$$

But now since φ is a solution of the heat equation on a model space, we know that the limiting behaviour for large times is given by the first eigenfunction: We have $\varphi(d/2,t) \sim C\mathrm{e}^{-\lambda_0 t}$, where λ_0 is the eigenvalue of the first symmetric eigenfunction of the warped product model space. In particular, we can apply this estimate to the particular solution given by the first nontrivial eigenfunction on M, yielding

$$\operatorname{osc}\left(u(.,0)\mathrm{e}^{-\lambda t}\right) \leq C\mathrm{e}^{-\lambda_0 t},$$

where λ is the first nontrivial eigenvalue of M. But this can be true for large times only if $\lambda \geq \lambda_0$. This gives a very clean conclusion: The first nontrivial eigenvalue on a compact manifold with diameter d and Ricci curvature bounded below by $(n-1)K$ is bounded below by that of the warped product model space corresponding to K with length d. This is the optimal lower bound on the first eigenfunction in terms of diameter and lower Ricci curvature bound, since it is attained in the limit of very 'thin' warped products (i.e. where the diameter of the slices $N \times \{t\}$ in $N \times \mathbb{R}$ is sent to zero).

There are several special cases worth mentioning: In the case $K = 0$, the model space is a cylinder and the corresponding eigenfunction is $\sin(\pi x/d)$, with eigenvalue π^2/d^2. Thus a compact manifold with diameter d and non-negative Ricci curvature has first eigenvalue at least π^2/d^2, which is the sharp result first proved by Jia Qing Zhong and Hong Cang Yang [ZY], using a refinement of the gradient estimate techniques of Li and Yau [LY] who had earlier proved the estimate $\lambda \geq \pi^2/2d^2$. Another special case is where $K = 1$, in which case Myers' theorem implies the diameter bound $d \leq \pi$, and the corresponding model space is the standard sphere with eigenvalue n, so we obtain the Lichnerowicz estimate $\lambda \geq n$, which is normally proved using integral estimates and the Bochner-Weitzenbock formula. Thus our result includes these two and all other possible choices of diameter and Ricci curvature bounds. This general result was first proved by Kröger [K], and later by Bakry and Qian in a more general context [BQ].

The Zhong-Yang argument (as well as those by Kröger and Bakry-Qian) required quite a delicate analysis of the effects of asymmetry of the first eigenfunction (it is precisely this point which led to the extra factor 1/2 in the Li-Yau estimate), but in our argument the sharp result falls out in the general case with almost no effort.

9. Log-concavity and the fundamental gap conjecture

Now I move on to the fundamental gap problem, where one wishes to bound the difference $\lambda_2 - \lambda_1$ for Dirichlet problems (perhaps with potential) on a convex domain. This difference in eigenvalues represents the excitation energy for the corresponding quantum system. In the paper by Singer, Wong, Yau and Yau [SWYY] this problem was approached using gradient estimates as follows: The idea is to adapt the gradient estimate methods which were successful in dealing with the eigenvalue problem. In this case we have a Schrödinger operator with potential, so the corresponding eigenfunctions are solutions of the equation

$$\begin{aligned} \Delta u - Vu + \lambda u = 0 &\quad \text{in } \Omega; \\ u = 0 &\quad \text{in } \partial\Omega. \end{aligned}$$

Suppose we have two such eigenfunctions u_1 (which is the first eigenfunction and hence positive on the interior), and u_2. Then we can consider the ratio $v = u_2/u_1$, which satisfies the following equation:

$$\begin{aligned} \Delta v + 2\nabla \log u_1 \cdot \nabla c + (\lambda_2 - \lambda_1) v &= 0 \quad \text{in } \Omega; \\ D_\nu v &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

This looks a lot like the equation for a Neumann eigenfunction (without any potential), except for one extra gradient term involving the derivative of $\log u_1$. In fact, if we carry through the gradient estimate exactly as before, only a single non-trivial extra term arises: If $Q = \frac{|\nabla v|^2}{1-v^2}$, then in computing ΔQ we have to differentiate the equation for Δv, giving a term $\nabla_i \nabla_j \log u_1 \nabla_i v \nabla_j v$. But it was proved by Brascamp and Lieb [BL] (and also in [SWYY]) that $\log u_1$ is concave, provided the potential V is convex, in which case the extra term has a favourable sign and so does not affect the result of the computations. The log-concavity is a very natural statement: If we recall that in heat flow -$\nabla \log u$ is the flux vector, then log-concavity says that the flux vector has positive divergence – i.e. heat is tending to separate. One can prove by the maximum principle (in a manner similar to Yau's argument in [SWYY]) that log-concavity of a solution of the heat equation is preserved.

Thus exactly as in [LY] one obtains the result

$$\frac{|\nabla v|^2}{1-v^2} \leq (\lambda_2 - \lambda_1),$$

which as before yields $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2}$ (again, a slight modification improves this to $\frac{\pi^2}{2d^2}$. Again as before, we lose something due to the possible asymmetry of v, but Yu and Zhong [YZ] showed that the same refinements as in [ZY] can improve this to $\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2}$. However now we have a crucial difference: The result is still not sharp, because we have thrown away a term involving $\nabla^2 \log u_1$ which would be *strictly negative* in the one-dimensional case which we expect should be sharp. Indeed in the one-dimensional case (with zero potential) the first eigenfunction on $[-d/2, d/2]$ is $\sin(\pi x/d)$ with eigenvalue π^2/d^2, and the second is $\sin(2\pi x/d)$ with eigenvalue $4\pi^2/d^2$. Thus we should expect $\lambda_2 - \lambda_1 \geq 3\pi^2/d^2$ if this is the sharp case. This was conjectured by Yau [Y2], and independently by van den Berg [vdB] and by Ashbaugh and Benguria [AB]. But now it seems very hard to get any sharp result using the gradient estimate method, since this would seem to need some sharp control on how concave $\log u_1$ is. A negative upper bound on the eigenvalues of $\nabla^2 \log u_1$ cannot be expect to give a sharp result, and it is hard to see how to formulate a strict log-concavity result any other way. Indeed the conjecture has remained open up to now, although it has been proved in the one-dimensional case [La], and in certain symmetric situations [D, BK].

Here the oscillation estimates turn out to be a good technique: Suppose we have two solutions to the Dirichlet heat equation with potential, say w_2 and w_1, so that

$$\begin{aligned} \frac{\partial}{\partial t} w_i &= \Delta w_i - V w_i \quad \text{on } \Omega \times [0, \infty); \\ w_i &= 0 \quad \text{on } \partial\Omega \times [0, \infty). \end{aligned}$$

Then if $w_1 > 0$, the ratio $v = w_2/w_1$ satisfies a Neumann heat equation with drift:

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v - 2\nabla \log w_1 \cdot \nabla v &= 0 \quad \text{on } \Omega \times [0, \infty); \\ D_\nu v &= 0 \quad \text{on } \partial\Omega \times [0, \infty). \end{aligned} \tag{9.1}$$

We don't have the difference $\lambda_2-\lambda_1$ coming out as obviously as in the eigenfunction setting we mentioned above, but this difference is still present in the asymptotic behaviour of solutions: The positive solution w_1 will approach $C_0\mathrm{e}^{-\lambda_1 t}u_1$ for some $C_0>0$, where u_1 is the first eigenfunction. The second solution w_2 will be asymptotic to $C_1\mathrm{e}^{-\lambda_1 t}u_1+C_2\mathrm{e}^{-\lambda_2 t}u_2$ for some constants C_1 and C_2. Thus v approaches $C_1/C_0+C_2/C_0\mathrm{e}^{-(\lambda_2-\lambda_1)t}u_2/u_1$, and we can hope to obtain a lower bound on $\lambda_2-\lambda_1$ by bounding below the exponential rate of convergence of $\operatorname{osc} v$ to zero.

Now define as before $Z(x,y,t)=v(y,t)-v(x,t)-2\varphi\left(\frac{|y-x|}{2},t\right)$. Then the computations of the first and second derivatives of Z at an interior maximum point are exactly as carried out in Section 3 (or using the simplified argument presented in Section 6), and the only difference comes in the computation of the time derivative:

$$\frac{\partial Z}{\partial t}=\Delta u(y)+2D\log u_1(y)\cdot Dv(y)-2D\log u_1(x)\cdot Dv(x)-2\varphi_t,$$

so we have two extra terms involving $D\log u_1$. From the first derivative conditions we have at the maximum point $Dv(y)=Dv(x)=\varphi'\frac{y-x}{|y-x|}$, so the extra terms simplify to give

$$2\varphi'(D\log u_1(y)-D\log u_1(x))\cdot\frac{y-x}{|y-x|}.$$

This is clearly very closely related to the log-concavity of u_1 — in particular if $\log u_1$ is concave then the extra term is non-positive, and indeed the same argument as before shows that $\operatorname{osc} v\le V\mathrm{e}^{-\pi^2/d^2 t}$, so that $\lambda_2-\lambda_1\ge\pi^2/d^2$, which is the Yu-Zhong result. But importantly the extra term is now one which we could hope to prove is bounded above by the corresponding quantity for the one-dimensional case. In this case we have $u_1(x)=\cos(\pi x/d)$, so $D\log u_1(x)=-\frac{\pi}{d}\tan\left(\frac{\pi x}{d}\right)$. In this case we clearly have

$$(D\log u_1(y)-D\log u_1(x))\cdot\frac{y-x}{|y-x|}\le-\frac{2\pi}{d}\tan\left(\frac{\pi|y-x|}{2d}\right),$$

with equality when $y=-x=|y-x|/2$. This led us to conjecture the following sharp log-concavity result:

Theorem 9.1. *Let V be a convex potential on a convex domain Ω in $\mathbb{R}^n$, and let u_1 be the first eigenfunction of the corresponding Schrödinger operator. Then for all $y\ne x\in\Omega$ we have*

$$(D\log u_1(y)-D\log u_1(x))\cdot\frac{y-x}{|y-x|}\le-\frac{2\pi}{d}\tan\left(\frac{\pi|y-x|}{2d}\right).$$

Once we have identified the right quantity, the proof is simply a maximum principle argument rather similar to that described above (and again guided by the principle that everything should give exact equality in the model case). This estimate now allows us to finish the optimal gap estimate:

Theorem 9.2. *If V is a convex potential on a convex domain Ω in $\mathbb{R}^n$, then the fundamental gap $\lambda_2-\lambda_1$ satisfies*

$$\lambda_2-\lambda_1\ge\frac{3\pi^2}{d^2}.$$

It is clear that this works now, since every term gives equality in the one-dimensional model case. This work was carried out with Julie Clutterbuck, and a more detailed presentation can be found in [AC3].

This finishes the story for the fundamental gap. I should mention that there are more general results when the potential is not convex (i.e. stronger results if we assume stronger convexity, and weaker (but still sharp) results for non-convex cases — we get some estimate provided only that the potential is semiconvex. The same argument also goes through for convex domains in constant curvature spaces, if we use a suitable comparison with a corresponding one-dimensional model problem.

There are still many interesting questions concerning the fundamental gap: What can be said for Neumann or Robin problems? What happens in a manifold of variable curvature? And how can one formulate a useful estimate for non-convex domains?

10. Non-collapsing in the mean curvature flow

Now I want to mention briefly some results proved using similar ideas (in particular, using estimates for functions which depend on more than one point). One which works particularly nicely is an estimate for mean curvature flow, in particular for the evolution of embedded hypersurfaces with positive mean curvature. This work was inspired by some recent work of Xujia Wang and Weimin Sheng [SW] where they prove many results about mean-convex solutions of the mean curvature flow, including a precise non-collapsing result which can be stated as follows: There is a constant $\kappa > 0$ depending only on the initial hypersurface, such that for any point (x, t) in the evolving hypersurface up to the first singular time, there is a ball in the enclosed region of radius $\kappa/H(x, t)$ which touches the hypersurface at (x, t) (here H is the mean curvature).

This result is proved somewhat indirectly in [SW]. Here I want to describe a direct proof of a similar result: The idea is to simply write down a quantity depending on two points, the positivity of which reflects exactly the noncollapsing statement I just described: Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of mean curvature flow. Then we want to say that for any x in M, any other point $X(y, t)$ is further than distance $\kappa/H(x, t)$ from the point $X(x, t) - \kappa/H(x, t)\nu(x, t)$, where $\nu(x, t)$ is the outward-pointing unit normal. That is, the quantity

$$Z(x, y, t) = \left\| X(y, t) - \left(X(x, t) - \frac{\kappa}{H(x, t)} \nu(x, t) \right) \right\|^2 - \frac{\kappa^2}{H(x, t)^2}$$

is non-negative for any $y \neq x$. This simplifies a little: It is the same as

$$Z(x, y, t) = d^2 - \frac{2d\kappa}{H(x, t)} \langle w, \nu(x, t) \rangle,$$

where $d = \|X(y, t) - X(x, t)\|$ and $w = \frac{X(y,t) - X(x,t)}{d}$. We proceed in a similar way to what we did previously, by deriving an evolution for Z, computing the $2n \times 2n$ matrix of second derivatives, and carefully choosing the cross-derivative terms to get a maximum principle to keep Z non-negative. I will not present the full details here, but the result is very nice: For any κ, we can prove that Z remains non-negative if it is initially so (giving a very precise noncollapsing result of the kind mentioned above); we can do this also with negative κ, which amount to an exterior non-collapsing result as well as the interior one; and we can also prove that Z remains non-positive if it is initially so (this applies only in the case of convex hypersurfaces, and shows that at each point there is a circumscribing sphere touching with radius comparable to the reciprocal of the mean curvature. This gives a very quick proof of Huisken's convergence theorem for convex hypersurfaces [H4].

11. Isoperimetric estimates for the Ricci flow and curve shortening flow

The last results I want to mention are joint work with Paul Bryan. In these papers [AB1, AB2, AB3] we modified the arguments of Huisken [H5] and Hamilton [H1, H2] which gave isoperimetric bounds to rule out type II singularity formation in curve shortening flow and in the Ricci flow on the 2-sphere. By using estimates of a very similar form to those described above (and in particular, guided by the principle that we should have equality in certain symmetric model cases) we proved sharp bounds on the isoperimetric profiles in both of these situations. In fact the isoperimetric bounds are strong enough to automatically give sharp curvature bounds which make the convergence of the flows a very easy exercise.

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