Lecture 12. Tensors

In this lecture we define tensors on a manifold, and the associated bundles, and operations on tensors.

12.1 Basic definitions

We have already seen several examples of the idea we are about to introduce, namely linear (or multilinear) operators acting on vectors on M.

For example, the metric is a bilinear operator which takes two vectors to give a real number, i.e. $g_x : T_xM \times T_xM \to \mathbb{R}$ for each x is defined by $u, v \mapsto g_x(u, v)$.

The difference between two connections $\nabla^{(1)}$ and $\nabla^{(2)}$ is a bilinear operator which takes two vectors and gives a vector, i.e. a bilinear operator $S_x : T_xM \times T_xM \to T_xM$ for each $x \in M$. Similarly, the torsion of a connection has this form.

Definition 12.1.1 A **covector** ω at $x \in M$ is a linear map from T_xM to $\mathbb{R}$. The set of covectors at x forms an n-dimensional vector space, which we denote T_x^*M. A **tensor** of type (k, l) at x is a multilinear map which takes k vectors and l covectors and gives a real number

$$T_x : \underbrace{T_xM \times \ldots \times T_xM}_{k \text{ times}} \times \underbrace{T_x^*M \times \ldots \times T_x^*M}_{l \text{ times}} \to \mathbb{R}.$$

Note that a covector is just a tensor of type $(1, 0)$, and a vector is a tensor of type $(0, 1)$, since a vector v acts linearly on a covector ω by $v(\omega) := \omega(v)$.

Multilinearity means that

$$T\left(\sum_{i_1} c^{i_1} v_{i_1}, \ldots, \sum_{i_k} c^{i_k} v_{i_k}, \sum_{j_1} a_{j_1}\omega^{j_1}, \ldots, \sum_{j_l} a_{j_l}\omega^{j_l}\right)$$
$$= \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} c^{i_1}\ldots c^{i_k} a_{j_1}\ldots a_{j_l} T(v_{i_1}, \ldots, v_{i_k}, \omega^{j_1}, \ldots, \omega^{j_l}).$$

To elucidate this definition, let us work in a chart, so that we have a basis of coordinate tangent vector fields $\partial_1, \dots, \partial_n$ at each point. Then we can define a convenient basis for the cotangent space T_x^*M by defining

$$dx^i : \; T_xM \to \mathbb{R}$$

by

$$dx^i(\partial_j) = \delta_j^i.$$

The notation here comes from the fact that dx^i is the differential of the smooth function x^i, in other words

$$dx^i(v) = v(x^i).$$

Similarly, for any $f \in C^\infty(M)$ we have a covector $d_xf \in T_x^*M$ defined by

$$df(v) = v(f).$$

12.2 Tensor products

Definition 12.2.1 Let T and S be two tensors at x of types (k,l) and (p,q) respectively. Then the **tensor product** $T \otimes S$ is the tensor at x of type $(k+p, l+q)$ defined by

$$\begin{aligned} T \otimes S(v_1, \dots, v_{k+p}, \omega_1, \dots, \omega_{l+q}) = {} & T(v_1, \dots, v_k, \omega_1, \dots, \omega_l) \\ & \times S(v_{k+1}, \dots, v_{k+p}, \omega_{l+1}, \dots, \omega_{l+q}) \end{aligned}$$

for all vectors $v_1, \dots, v_{k+p} \in T_xM$ and all covectors $\omega_1, \dots, \omega_{l+q} \in T_x^*M$.

Proposition 12.2.2 *The set of tensors of type (k,l) at x is a vector space of dimension n^{k+l}, with a basis given by*

$$\{dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \dots \otimes \partial_{j_l} : \; 1 \le i_1, \dots, i_k, j_1, \dots, j_l \le n.\}$$

Proof. The vector space structure of the space of (k,l)-tensors is clear.

Note that a tensor T is completely determined by its action on the basis vectors and covectors:

$$\begin{aligned} & T(v_1^i\partial_i, \dots, v_k^j\partial_j, \omega_a^1 dx^a, \dots, \omega_b^l dx^b) \\ & \quad = v_1^i \dots v_k^j \dots \omega_a^1 \dots \omega_b^l T(\partial_i, \dots, \partial_j, dx^a, \dots, dx^b), \end{aligned}$$

by multilinearity.

Then observe that this allows T to be written as a linear combination of the proposed basis elements: Define

$$T_{i_1 \dots i_k}{}^{j_1 \dots j_l} = T(\partial_{i_1}, \dots, \partial_{i_k}, dx^{j_1}, \dots, dx^{j_l})$$

for each $i_1, \ldots, i_k, j_1, \ldots, j_l \in \{1, \ldots, n\}$. Then define $\hat{T}$ by

$$\hat{T} = \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l=1}^{n} T_{i_1\ldots i_k}{}^{j_1\ldots j_l} dx^{i_1} \otimes \ldots dx^{i_k} \otimes \partial_{j_1} \otimes \ldots \partial_{j_l}.$$

We claim that $\hat{T} = T$. To check this, we compute the result of acting with $\hat{T}$ on k elementary tangent vectors and l elementary covectors:

$$\begin{aligned}
&\hat{T}(\partial_{i_1}, \ldots, \partial_{i_k}, dx^{j_1}, \ldots, dx^{j_l}) \\
&= \sum_{a_1,\ldots,a_k,b_1,\ldots,b_l=1}^{n} T_{a_1\ldots a_k}{}^{b_1\ldots b_l} dx^{a_1} \otimes \ldots dx^{a_k} \otimes \partial_{b_1} \otimes \ldots \partial_{b_l} \left(\partial_{i_1}, \ldots, \partial_{i_k}, dx^{j_1}, \ldots, dx^{j_l}\right) \\
&= \sum_{a_1,\ldots,a_k,b_1,\ldots,b_l=1}^{n} T_{a_1\ldots a_k}{}^{b_1\ldots b_l} dx^{a_1}(\partial_{i_1}) \ldots dx^{a_k}(\partial_{i_k}) \partial_{b_1}(dx^{j_1}) \ldots \partial_{b_l}(dx^{j_l}) \\
&= T_{i_1\ldots i_k}{}^{j_1\ldots j_l} \\
&= T\left(\partial_{i_1}, \ldots, \partial_{i_k}, dx^{j_1}, \ldots, dx^{j_l}\right).
\end{aligned}$$

Therefore the proposed basis does generate the entire space.

It remains to show that the basis elements are linearly independent. Suppose we have some linear combination of them which gives zero:

$$A_{i_1\ldots i_k}{}^{j_1\ldots j_l} dx^{i_1} \otimes \ldots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \ldots \otimes \partial_{j_l} = 0.$$

Then in particular the value of this linear combination on any set of vectors $\partial_{a_1}, \ldots, \partial_{a_k}$ and covectors $dx^{b_1}, \ldots, dx^{j_l}$ must give zero. But this value is exactly the coefficient:

$$A_{a_1\ldots a_k}{}^{b_1\ldots b_l} = 0.$$

This establishes the linear independence, and show that we do indeed have a basis. □

For example, we can write the metric as

$$g = g_{ij} dx^i \otimes dx^j,$$

and the torsion T of a connection as

$$T = T_{ij}{}^k dx^i \otimes dx^j \otimes \partial_k.$$

We denote the space of (k, l)-tensors at x by

$$\bigotimes^k T_x M \otimes \bigotimes^l T_x^* M.$$

12.3 Contractions

Let T be a tensor of type (k,l) at x, with k and l at least 1. Then T has components $T_{i_1,\ldots,i_k}{}^{j_1,\cdots j_l}$ as before. Then there is a tensor of type $(k-1,l-1)$ which has components

$$\sum_{a=1}^{n} T_{i_1\ldots i_{k-1}a}{}^{j_1\cdots j_{l-1}a}.$$

This tensor is called a *contraction* of T (If k and l are large then there will be many such contractions, depending on which indices we choose to sum over).

A special case is where T is a tensor of type $(1,1)$. This takes a vector and gives another vector, and so is nothing other than a linear operator on TM. There is a unique contraction in this case, which is just the *trace* of the operator:

$$\mathrm{tr}A = \sum_{i=1}^{n} A_i{}^i.$$

12.4 Musical isomorphisms

If we have an inner product on TM, then there are natural isomorphisms between tensors of types (k,l) and (m,n) whenever $k+l=m+n$. In particular, there is an isomorphism between the tangent space and the dual tangent space, given by sending a vector v to the covector ω which acts on any vector u to give the number $\langle u,v\rangle$.

It is useful to have an explicit expression for this isomorphism in terms of the standard bases for T_xM and T_x^*M: A vector ∂_i gets sent to the covector ω_i which acts on the vector ∂_j to give the number $g(\partial_i,\partial_j)=g_{ij}$. Therefore $\omega_i = \sum_{j=1}^n g_{ij}dx^j$. More generally, a vector $v^i\partial_i$ is sent to the covector $v^i g_{ij} dx^j$.

More generally, the isomorphism from (k,l) tensors to $(k+l,0)$-tensors is given as follows: A (k,l)-tensor T with coefficients $T_{i_1\ldots i_k}{}^{j_1\cdots j_l}$ becomes the $(k+l,0)$-tensor $\tilde{T}$ with coefficients

$$\tilde{T}_{i_1\ldots i_k i_{k+1}\ldots i_{k+l}} = T_{i_1\ldots i_k}{}^{j_1\cdots j_l} g_{j_1 i_{k+1}}\ldots g_{j_l i_{k+l}}.$$

Thus the metric acts to 'lower' an index by a tensor product followed by a contraction. The inverse of this operation ('rasing indices') is given by multiplying by the inverse matrix of the metric, $(g^{-1})^{ij}$, which defines a tensor of type $(0,2)$. For convenience we will denote this by g^{ij}, suppressing the inversion. This is consistent with the fact that is we use the metric to raise its own indices, then we get the tensor

$$g^{ij} = g_{kl}(g^{-1})^{ik}(g^{-1})^{jl} = (g^{-1})^{ij}.$$

Example 12.4.1 If T is a tensor of type $(k,1)$, then it takes k vectors and gives another one, which we denote by $T(v_1,\ldots,v_k)$. The index-lowering map should produce from this a map which takes $(k+1)$ vectors and gives a number, which we denote by $T(v_1,\dots,v_{k+1})$. Following the procedure above, we see that this isomorphism is just given by

$$T(v_1,\dots,v_{k+1}) = g(T(v_1,\dots,v_k),v_{k+1}).$$

12.5 Tensor fields

A *tensor field* T of type (k,l) on M is a smooth choice of a tensor T_x of type (k,l) at x for each $x \in M$. In particular a tensor field of type $(0,1)$ is just a vector field, and a tensor field ω of type $(1,0)$ is given by a covector ω_x at each point. In this case smoothness is interpreted in the sense that for every smooth vector field X on M, the function $x \mapsto \omega_x(X_x)$ is smooth. A smooth tensor field of type $(1,0)$ is also called a 1-form. The space of 1-forms is denoted $\Omega(M)$.

More generally, smoothness of a tensor field T of type (k,l) is to be interpreted in the sense that the function obtained by acting with T on k smooth vector fields and l 1-forms is a smooth function on M.

Equivalently, a (k,l)-tensor field T is smooth if the component functions $T_{i_1\dots i_k}{}^{j_1\dots j_l}$ with respect to any chart are smooth functions.

It is straightforward to check that tensor products, contractions, and index raising and powering operators go across from tensors to smooth tensor fields.

12.6 Tensor bundles

In analogy with the definition of the tangent bundle, we can define (k,l)-tensor bundles:

$$\bigotimes^k TM \otimes \bigotimes^l T^*M = \bigcup_{x\in M} \bigotimes^k T_xM \otimes \bigotimes^l T_x^*M.$$

This is made into a manifold as follows: Given a chart φ for a region U of M, we define a chart $\tilde\varphi$ for the region

$$\bigcup_{x\in U} \bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$$

by taking for $p \in U$ and T a (k,l)-tensor at p,

$$\tilde\varphi(p,T) = (\varphi^1,\dots,\varphi^n, T_{1\dots 1}{}^{1\dots 1},\dots,T_{n\dots n}{}^{n\dots n}) \in \mathbb{R}^{n+n^{k+l}}.$$

If we have two such charts φ and η, the transition maps between them are given by $\eta\circ\varphi^{-1}$ on the first n components, and by the map

$$T_{i_1\dots i_k}{}^{j_1\dots j_l} \mapsto \Lambda^{a_1}_{i_1}\dots\Lambda^{a_k}_{i_k}(\Lambda^{-1})^{j_1}_{b_1}\dots(\Lambda^{-1})^{j_l}_{b_l}T_{a_1\dots a_k}{}^{b_1\dots b_l}$$

in the remaining components, where $\Lambda = D(\eta\circ\varphi^{-1})$ is the derivative matrix of the transition map on M. In particular, these transition maps are smooth, so the tensor bundle is a smooth manifold.

It is common to denote the space of smooth (k,l)-tensor fields on M by $\Gamma\left(\bigotimes^k TM\otimes\bigotimes^l T^*M\right)$, read as "sections of the (k,l)-tensor bundle".

12.7 A test for tensoriality

Often the tensors which we work with will arise in the following way: We are given an operator T which takes k vector fields and l 1-forms, and gives a smooth function. Suppose that the operator is also multilinear (over $\mathbb{R}$), so that multiplying any of the vector fields or 1-forms by a constant just changes the result by the same factor, and taking one of the vector fields or 1-forms to be given by a sum of two such, gives the sum of the results applied to the individual vector fields or 1-forms.

Proposition 12.7.1 *The operator T is a tensor field on M if and only if it is C^∞-linear in each argument:*

$$T(f_1X_1,\dots,f_kX_k,g_1\omega^1,\dots,g_l\omega^l) = f_1\dots f_kg_1\dots g_lT(X_1,\dots X_k,\omega^1,\dots,\omega^l)$$

for any smooth functions $f_1,\dots f_k,g_1,\dots,g_l$, vector fields $X_1,\dots,X_k$, and 1-forms $\omega^1,\dots,\omega^l$.

We have already seen this principle in action, in our discussion of the difference of two connections and of the torsion of a connection.

Proof. One direction is immediate: A (k,l)-tensor field is always C^∞-linear in each argument.

Conversely, suppose $T:\mathcal{X}(M)^k\times\Omega(M)^l\to C^\infty(M)$ is multilinear over $C^\infty(M)$. Choose a chart φ in a neighbourhood of a point $x\in M$. Now compute the action of T on vector fields $X_{(j)} = X^i_{(j)}\partial_i$, $j=1,\dots,k$, and 1-forms $\omega^{j)} = \omega^{(j)}_i dx^i$, $j=1,\dots,l$, evaluated at the point x:

$$\begin{aligned}
&\Big(T(X_{(1)},\dots,X_{(k)},\omega^{(1)},\dots,\omega^{(l)})\Big)_x\\
&\qquad = X^{i_1}_{(1)}(x)\dots X^{i_k}_{(k)}(x)\omega^{(1)}_{j_1}(x)\dots\omega^{(l)}_{j_l}(x)\left(T(\partial_{i_1},\dots,\partial_{i_k},dx^{j_1},\dots,dx^{j_l}\right)_x\\
&\qquad = T_x\left(X_{(1)}(x),\dots,X_{(k)}(x),\omega^{(1)}(x),\dots,\omega^{(l)}(x)\right)
\end{aligned}$$

where T_x is the (k,l)-tensor at x with coefficients given by

$$(T_x)_{i_1\dots i_k}{}^{j_1\dots j_l} = \big(T(\partial_{i_1},\dots,\partial_{i_k},dx^{j_1},\dots,dx^{j_l})\big)_x .$$

□

Example 12.7.2 The Lie bracket of two vector fields takes two vector fields and gives another vector field. This is not a tensor, because

$$[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X \neq fg[X,Y]$$

in general. Similarly, the connection ∇ takes two vector fields and gives a vector field, but is not a tensor, because

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \neq f\nabla_X Y$$

in general.

12.8 Metrics on tensor bundles

A Riemannian metric on M is given by an inner product on each tangent space T_xM. This induces a natural inner product on each of the tensor spaces at x. This is uniquely defined by the requirement that tensor products of basis elements of TM are orthonormal, and that the metric by invariant under the raising and lowering operators.

Explicitly, the inner product of two tensors S and T of type (k,l) at x is given by

$$\langle S,T\rangle = g^{a_1b_1}\dots g^{a_kb_k} g_{i_1j_1}\dots g_{i_lj_l} S_{a_1\dots a_k}{}^{i_1\dots i_l} T_{b_1\dots b_k}{}^{j_1\dots j_l}.$$

12.9 Differentiation of tensors

We can also extend the connection (defined to give derivatives of vector fields) to give a connection on each tensor bundle – i.e. to allow the definition of derivatives for any tensor field on M.

We want the following properties for the connection: ∇ should take a tensor field T of type (k,l), and give a tensor field ∇T of type $(k+1,l)$, such that

(1). The Leibniz rule holds for tensor products: If S and T are tensor fields of type (k,l) and type (p,q) respectively, then

$$\begin{aligned}&\nabla(S\otimes T)(X_0,X_1,\dots,X_{k+p},\omega_1,\dots,\omega_{l+q})\\ &= \nabla S(X_0,X_1,\dots,X_k,\omega_1,\dots,\omega_l)T(X_{k+1},\dots,X_{k+p},\omega_{l+1},\dots,\omega_{l+q})\\ &\quad + S(X_1,\dots,X_k,\omega_1,\dots,\omega_l)\nabla T(X_0,X_{k+1},\dots,X_{k+p},\omega_{l+1},\dots,\omega_{l+q})\end{aligned}$$

for any vector fields $X_0, \ldots, X_{k+p}$ and 1-forms $\omega_1, \ldots, \omega_{l+q}$.

(2). ∇ applied to a contraction of a tensor T is just the contraction of ∇T: More precisely, if T is the tensor with components $T_{i_1 \ldots i_k}{}^{j_1 \ldots j_l}$, and CT is the contracted tensor with components $CT_{i_1 \ldots i_{k-1}}{}^{j_1 \ldots j_{l-1}} = \sum_{i=1}^n T_{i_1 \ldots i_{k-1} i}{}^{j_1 \ldots j_{l-1} i}$, then

$$C\left(\nabla T\right) = \nabla\left(CT\right).$$

(3). If T is a tensor of type $(0,1)$ (i.e. a vector field) then ∇ is the same as out previous definition:

$$\nabla T(X, \omega) = (\nabla_X T)(\omega)$$

for any vector field X and 1-form ω.

(4). If f is a tensor of type $(0,0)$ (i.e. a smooth function) then ∇ is just the usual derivative:

$$\nabla f(X) = X(f).$$

Let us investigate what this means for the derivative of a 1-form: If X is a vector field and ω a 1-form, then the contraction of $X \otimes \omega$ is just the function $\omega(X)$. By conditions 1, 2, and 4, we then have

$$\nabla_v(\omega(X)) = C\left((\nabla_v \omega) \otimes X + \omega \otimes \nabla_v X\right).$$

Taking the special case $X = \partial_k$ and $v = \partial_i$, we get $\omega(X) = \omega_k$, where $\omega = \omega_k dx^k$, and so

$$\partial_i \omega_k = (\nabla_i \omega_k) + \Gamma_{ik}{}^j \omega_j$$

where I used condition (3) to get $\nabla_i \partial_k = \Gamma_{ik}{}^j \partial_j$. This gives

$$\nabla_i \omega_j = \partial_i \omega_j - \Gamma_{ij}{}^k \omega_k.$$

We know already that the derivative of a vector field is given by

$$\nabla_i X^j = \partial_i X^j + \Gamma_{ik}{}^j X^k.$$

Using these two rules and condition (1), and the fact that any tensor can be expressed as a linear combination of tensor products of vector fields and 1-forms, we get for a tensor T of type (k,l),

$$\begin{aligned}
\nabla_i T_{i_1 \ldots i_k}{}^{j_1 \ldots j_l} = \partial_i T_{i_1 \ldots i_k}{}^{j_1 \ldots j_l} \\
&- \Gamma_{i i_1}{}^p T_{p i_2 \ldots i_k}{}^{j_1 \ldots j_l} - \ldots - \Gamma_{i i_k}{}^p T_{i_1 \ldots i_{k-1} p}{}^{j_1 \ldots j_l} \\
&+ \Gamma_{ip}{}^{j_1} T_{i_1 \ldots i_k}{}^{p j_2 \ldots j_l} + \ldots + \Gamma_{ip}{}^{j_l} T_{i_1 \ldots i_k}{}^{j_1 \ldots j_{l-1} p}.
\end{aligned}$$

In particular, we note that the compatibility of the connection with the metric implies that the derivative of the connection is always zero:

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma_{ij}{}^p g_{pk} - \Gamma_{ik}{}^p g_{jp} = \Gamma_{ijk} + \Gamma_{ikj} - \Gamma_{ijk} - \Gamma_{ikj} = 0.$$

This implies in particular that the covarant derivative commutes with the index-raising and lowering operators.

12.10 Parallel transport

As with the case of vector fields, the connection allows us to define derivatives of tensors aloong curves, and parallel transport of tensors along curves (by solving the first order linear ODE system corresponding to $\nabla_t T = 0$. As in the case of vector fields, parallel transport preserves inner products between tensors.

12.11 Computing derivatives of tensors

Since the covariant derivative of a tensor is again a tensor, it is often convenient to use the fact that it is determined by its components with respect to any coordinate tangent basis. In particular, to compute the covariant derivative of a tensor at a point x, one can work in local coordinates in which the connection coefficients ${\Gamma_{ij}}^k$ vanish at the point x (for example, use exponential normal coordinates from x).

Then we have

$$\left(\nabla_i {T_{i_1\dots i_k}}^{j_1\dots j_l}\right)_x = \partial_i {T_{i_1\dots i_k}}^{j_1\dots j_l}(x).$$

One must take care, however, in computing second derivatives of tensors: The expression obtained using geodesic coordinates at x is simple at the point x itself, but not at neighbouring points; thus it cannot be used to get simple expressions for second derivatives.