Lecture 17. Extrinsic curvature of submanifolds

In this lecture we define the extrinsic curvature of submanifolds in Euclidean space.

17.1 Immersed submanifolds

By an *immersed submanifold* of Euclidean space $\mathbb{R}^N$ I will mean a differentiable manifold M together with an immersion $X : M \to \mathbb{R}^N$. Note that for any $x \in M$ there is a neighbourhood U of x such that $X|_U$ is an embedding. A particular case of an immersed submanifold is an embedded submanifold.

The inner product $\langle .,. \rangle$ on $\mathbb{R}^N$ induces a metric g and corresponding Levi-Civita connection ∇ on M, defined by

$$g(u,v) = \langle DX(u), DX(v) \rangle$$

and

$$\nabla_u v = \pi_{TM} \left(D_u(DX(v)) \right).$$

A particular case of this is an *immersed hypersurface*, which is the case where M is of dimension $N-1$. We will develop the theory of extrinsic curvature first for the simpler case of hypersurfaces, and then extend this to the more general case of immersed submanifolds.

17.2 The Gauss map of an immersed hypersurface

Let M^n be an oriented immersed hypersurface in $\mathbb{R}^{n+1}$. Then for each point $x \in M$ there is a well-defined unit normal $\mathbf{n}$ to M (more precisely, to $X(M)$) at x. This is defined by the requirements $\langle \mathbf{n}, \mathbf{n} \rangle = 1$, $\langle n, DX(u) \rangle = 0$ for all $u \in T_xM$, and if $e_1, \ldots, e_n$ are an oriented basis for T_xM then $DX(e_1), \ldots, DX(e_n), \mathbf{n}$ is an oriented basis for $\mathbb{R}^{n+1}$.

This defines a smooth map $\mathbf{n} : M \to S^n \subset \mathbb{R}^{n+1}$, called the Gauss map of M.

17.3 The second fundamental form of a hypersurface

Having defined the Gauss map of an oriented immersed hypersurface, we can define a tensor as follows:

$$h(u, v) = \langle D_u \mathbf{n}, DX(v)\rangle.$$

This is called the second fundamental form on M, and is a tensor of type $(2, 0)$.

The second fundamental form has an alternative expression, which we can deduce as follows: Let U and V be smooth vector fields on M. Since $\langle \mathbf{n}, DX(V) = 0$, we have

$$\begin{aligned} 0 &= U\langle \mathbf{n}, DX(V)\rangle \\ &= \langle D_u \mathbf{n}, DX(V)\rangle + \langle \mathbf{n}, D_U DX(V)\rangle \\ &= h(U, V) + \langle \mathbf{n}, D_U DX(V)\rangle \end{aligned}$$

and therefore

$$h(U, V) = -\langle D_U D_V X, \mathbf{n}\rangle.$$

From this we can deduce a useful symmetry:

$$h(U, V) = -\langle D_U D_V X, \mathbf{n}\rangle = -\langle D_V D_U X + D_{[U,V]}X, \mathbf{n}\rangle = -h(V, U)$$

since $D_{[U,V]}X = DX([U, V])$ is tangential to M, hence orthogonal to $\mathbf{n}$. Therefore the second fundamental form is a symmetric bilinear form on the tangent space T_xM at each point.

Since h is symmetric, it can be diagonalized with respect to the metric g — that is, we can find a basis $e_1, \ldots, e_n$ for T_xM and real numbers $\lambda_1, \ldots, \lambda_n$ such that $h(e_i, u) = \lambda_i g(e_i, u)$ for all vectors $u \in T_xM$. The numbers $\lambda_1, \ldots, \lambda_n$ are called the *principal curvatures* of M at x.

The *mean curvature* H is the trace of h with respect to g: $H = g^{ij}h_{ij}$. This can also be expressed in terms of the principal curvatures: $H = \lambda_1 + \ldots + \lambda_n$.

The *Gauss curvature* K is the determinant of h with respect to g, which is therefore also equal to $\prod_{i=1}^n \lambda_i$.

In the case where M is not orientable, it is not possible to choose a unit normal vector continuously on M, and so $\mathbf{n}$, and hence h and the principal curvatures λ_i are defined only up to sign.

Remark. It is very easy to get a geometric understanding of the second fundamental form of a hypersurface: Fix $z \in M$. Assume that the origin of $\mathbb{R}^{n+1}$ is at $X(z)$ and choose an orthonormal basis $e_1, \ldots, e_{n+1}$ for $\mathbb{R}^{n+1}$ such that $DX(T_zM) = \operatorname{span}\{e_1, \ldots, e_n\}$. By the implicit function theorem, $X(M)$ can be written locally in the form $\{x^i e_i : \ x^{n+1} = u(x^1, \ldots, x^n)\}$. Then near z we have in the coordinates $x^1, \ldots, x^n$

$$\partial_i X = e_i + \frac{\partial u}{\partial x^i} e_{n+1}$$

and

$$n(z) = e_{n+1}.$$

Therefore

$$\partial_i \partial_j X = \frac{\partial^2 u}{\partial x^i \partial x^j} e_{n+1}$$

and so at z,

$$h_{ij} = -\frac{\partial^2 u}{\partial x^i \partial x^j}.$$

To put this another way, we have

$$u(y) = -\frac{1}{2} h_{ij}(z) y^i y^j + O(y^3)$$

as $y \to 0$. This says that the second fundamental form gives the best approximation of the hypersurface by a paraboloid defined over its tangent plane.

17.4 The normal bundle of an immersed submanifold

Now we go on to the general case of an immersed submanifold M^n in $\mathbb{R}^N$. Then at each point of M, rather than having a single unit normal vector, we have a normal subspace $N_x M = \{v \in \mathbb{R}^N : v \perp DX(T_x M)\}$. This defines the *normal bundle* NM of M: $NM = \{(p, v) : p \in M, v \perp DX(T_p M)\}$. This is a differentiable manifold of dimension N.

17.5 Vector Bundles

The normal bundle (and indeed the tangent bundle and the tensor bundles we have already defined) is an example of a more general object called a *vector bundle*. A vector bundle E of dimension k over M is defined by associating to each $x \in M$ a vector space E_x (often called the *fibre* at x), and taking $E = \{(p, v) : p \in M, v \in E_p\}$. We require that E be a smooth manifold, and that for each $x \in M$ there is a neighbourhood U of x in M such that there are k smooth sections $\phi_1, \ldots, \phi_k$ of E (i.e. smooth maps ϕ_i from M to E such that $\pi \circ \phi_i = \mathrm{id}$) such that $\phi_1(y), \ldots, \phi_k(y)$ form a basis for E_y for each $y \in U$ (it follows that the restriction of the bundle E to U is diffeomorphic to $U \times \mathbb{R}^k$).

We denote the space of smooth sections of E (i.e. smooth maps from M to E which take each $x \in M$ to the fibre E_x at x) by $\Gamma(E)$.

A connection on a vector bundle E is a map which takes a vector $u \in T_x M$ and section $\phi \in \Gamma(E)$ and gives an element $\nabla_u \phi \in E_x$, smoothly in the sense that if $U \in \mathcal{X}(M)$ and $\phi \in \Gamma(E)$ then $\nabla_U \phi \in \Gamma(E)$, which is linear in the first argument and satisfies a Leibniz rule in the second:

$$\nabla_u(f\phi) = f\nabla_u\phi + u(f)\phi$$

for all $f \in C^\infty(M)$, $u \in T_xM$ and $\phi \in \Gamma(E)$.

We can also define tensors which either act on E or take their values in E, to be C^∞-multilinear functions acting on sections of E or its dual E^*, and the connection extends to such tensors.

17.6 Curvature of a vector bundle

If E is a vector bundle over M with a metric $\langle .,. \rangle$ and a connection ∇ which is compatible with the metric:

$$\nabla_u\langle\phi,\psi\rangle = \langle\nabla_u\phi,\psi\rangle + \langle\phi,\nabla_u\psi\rangle.$$

Then we can define the curvature of the bundle E as follows: If $X, Y \in \mathcal{X}(M)$ and ϕ, $\psi \in \Gamma(E)$, then we take

$$R(X,Y,\phi,\psi) = \langle\nabla_Y\nabla_X\phi - \nabla_X\nabla_Y\phi - \nabla_{[Y,X]}\phi,\psi\rangle.$$

This is tensorial in all arguments — that is, the value of the resulting function when evaluated at any point $x \in M$ depends only on the values of X, Y, ϕ and ψ at x. The proof of this is identical to the proof that the curvature of M is a tensor (Lecture 16). This can be considered as an operator which takes $\Lambda^2 T_xM$ to $\Lambda^2 E_x$, since it is antisymmetric in the first two and the last two arguments.

17.7 Connection on the normal bundle

We can define a connection on the normal bundle as follows: If V is a section of the normal bundle, and U is a smooth vector field on M, then we define

$$\nabla_U V\big|_x = \pi_{N_xM}\left(D_U V\right).$$

This is a connection: For any $f \in C^\infty(M)$, we have

$$\begin{aligned}
\nabla_U(fV) &= \pi_{NM}\left((Uf)V + fD_UV\right)\\
&= U(f)\pi_{NM}V + f\pi_{NM}D_UV\\
&= U(f)V + f\nabla_UV
\end{aligned}$$

so the Leibniz rule holds. This connection is compatible with the metric induced on NM by the inner product on $\mathbb{R}^N$. By the argument above, this defines a curvature tensor acting on $\Lambda^2 TM \otimes \Lambda^2 E$, which we denote by $R^\perp$ and call the *normal curvature* of M.

17.8 Second fundamental form of a submanifold

The second fundamental form is defined in an analogous way to that for the hypersurface case: Given $U, V \in \mathcal{X}(M)$ define

$$h(U,V) = -\pi_{N_xM}(D_U D_V X) = -D_U D_V X + DX(\nabla_U V).$$

This does in fact define a tensor field, since

$$\begin{aligned} h(fU, gV) &= -\pi_{N_xM}(D_{fU} D_{gV} X) \\ &= -\pi_{N_xM}(fg D_U D_V X + f(Ug) D_V X) \\ &= fgh(U,V) \end{aligned}$$

since $D_V X \perp N_xM$. h therefore defines at each $x \in M$ a bilinear map from $T_xM \times T_xM$ to N_xM.

We can also define an operator $\mathcal{W}$ from $T_xM \times N_xM$ to T_xM as follows:

$$\mathcal{W}(u,\phi) = \pi_{T_xM}(D_u\phi)$$

for $u \in T_xM$ and $\phi \in \Gamma(NM)$. This is again tensorial, since

$$\mathcal{W}(u, f\phi) = \pi_{T_xM}(D_u f\phi) = \pi_{T_xM}(fD_u\phi + (uf)\phi) = f\mathcal{W}(u,\phi).$$

This is related to the second fundamental form as follows:

$$0 = v\langle\phi, D_u X\rangle = \langle D_v\phi, D_u X\rangle + \langle\phi, D_v D_u X\rangle = \langle\mathcal{W}(v,\phi), D_u X\rangle - \langle h(v,u),\phi\rangle$$

and so $\langle\mathcal{W}(v,\phi), D_u X\rangle = \langle h(v,u),\phi\rangle$ for any u and v in T_xM and ϕ in N_xM.

The second fundamental form of a submanifold can be interpreted in a similar way to the hypersurface case: If we fix $z \in M$, then $X(M)$ can be written locally as the graph of a smooth function from T_xM to N_xM — that is, if we choose a basis $e_1, \dots, e_N$ such that $DX(T_zM) = \mathrm{span}\{e_1, \dots, e_n\}$ and $N_zM = \mathrm{span}\{e_{n+1}, \dots, e_N\}$, then for some open set U containing z,

$$X(M) = \{X(z) + x^i e_i :\ x^j = f^j(x^1, \dots, x^n),\ j = n+1, \dots, N\}.$$

Then we find

$$f^j(x^1, \dots, x^n) = -\frac{1}{2}\langle h_{kl}(z), e_j\rangle x^k x^l + O(x^3)$$

as $x \to 0$. Thus the second fundamental form at z defines the best approximation to $X(M)$ as the graph of a quadratic function over $DX(T_zM)$.