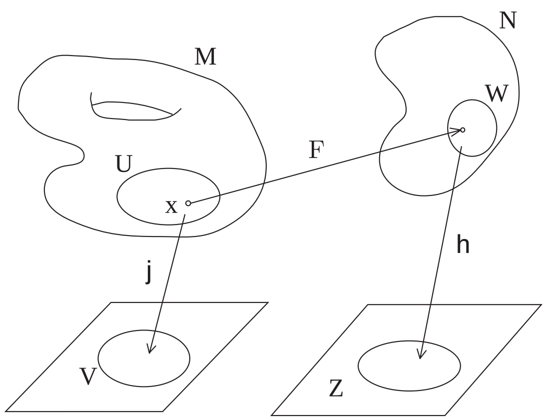
Lecture 2. Smooth functions and maps

2.1 Definition of smooth maps

Given a differentiable manifold, all questions of differentiability are to be reduced to questions about functions between Euclidean spaces, by using charts compatible with the differentiable structure. This principle applies in particular when we decide which maps between manifolds are smooth:

Definition 2.1.1 Let M^n be a differentiable manifold, with an atlas $\mathcal{A}$ representing the differentiable structure on M. A function $f : M \to \mathbb{R}$ is **smooth** if for every chart $\varphi : U \to V$ in $\mathcal{A}$, $f \circ \varphi^{-1}$ is a smooth function on $V \subset \mathbb{R}^n$. Let N^k be another differentiable manifold, with atlas $\mathcal{B}$. Let F be a map from M to N. F is **smooth** if for every $x \in M$ and all charts $\varphi : U \to V$ in $\mathcal{A}$ with $x \in U$ and $\eta : W \to Z$ in $\mathcal{B}$ with $F(x) \in W$, $\eta \circ F \circ \varphi^{-1}$ is a smooth map from $\varphi(F^{-1}(W) \cap U) \subseteq \mathbb{R}^n$ to $Z \subseteq \mathbb{R}^k$.



Remark. Although the definition requires that $f \circ \varphi^{-1}$ be smooth for *every* chart, it is enough to show that this holds for at least one chart around each point: If $\varphi : U \to V$ is a chart with $f \circ \varphi^{-1}$ smooth, and $\eta : W \to Z$ is another chart with W overlapping U, then $f \circ \eta^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \eta^{-1})$ is smooth. A similar argument applies for checking that a map between manifolds is smooth.

Exercise 2.1.1 Show that a map χ between smooth manifolds M and N is smooth if and only if $f \circ \chi$ is a smooth function on M whenever f is a smooth function on N.

Exercise 2.1.2 Show that the map $x \mapsto [x]$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}P^n$ is smooth.

Example 2.1.1 The group $GL(n, \mathbb{R})$. On $GL(n, \mathbb{R})$ we have a natural family of maps: Fix some $\mathsf{M} \in GL(n, \mathbb{R})$, and define $\rho_{\mathsf{M}} : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ by $\rho_{\mathsf{M}}(\mathsf{A}) = \mathsf{MA}$. The trivial chart ι given by inclusion of $GL(n, \mathbb{R})$ in $\mathbb{R}^{n^2}$ is the map which takes a matrix A to its components $(\mathsf{A}_{11}, \ldots, \mathsf{A}_{1n}, \ldots, \mathsf{A}_{nn})$. To check that ρ_{M} is smooth, we need to check that $\iota \circ \rho_{\mathsf{M}} \circ \iota^{-1}$ is smooth. But this is just the map which takes the components of A to the components of MA, which is

$$\iota^{-1} \circ \rho_{\mathsf{M}} \circ \iota\,(\mathsf{A}_{11}, \mathsf{A}_{12}, \ldots, \mathsf{A}_{nn}) = \left(\sum_{j=1}^{n} \mathsf{M}_{1j}\mathsf{A}_{j1}, \sum_{j=1}^{n} \mathsf{M}_{1j}\mathsf{A}_{j2}, \ldots, \sum_{j=1}^{n} \mathsf{M}_{nj}\mathsf{A}_{jn} \right)$$

Each component of this map is linear in the components of A, hence smooth.

Exercise 2.1.3 Show that the multiplication map $\rho : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $\rho(\mathsf{A}, \mathsf{B}) = \mathsf{AB}$ is smooth.

Definition 2.1.2 A **Lie group** G is a group which is also a differentiable manifold such that the multiplication map from $G \times G \to G$ is smooth and the inversion $g \mapsto g^{-1}$ is smooth.

2.2 Further classification of maps

Definition 2.2.1 A map $F : M \to N$ is a **diffeomorphism** if it is smooth and has a smooth inverse. F is a **local diffeomorphism** if for every $x \in M$ there exists a neighbourhood U of x in M such that the restriction of F to U is a diffeomorphism to an open set of N.

A smooth map $F : M \to N$ is an **embedding** if F is a homeomorphism onto its image (with the subspace topology), and for any charts φ and η for M and N respectively, $\eta \circ F \circ \varphi^{-1}$ has derivative of full rank. F is an **immersion** if for every x in M there exist charts $\varphi : U \to V$ for M and $\eta : W \to Z$ for N with $x \in U$ and $F(x) \in Z$, such that the map $\eta \circ F \circ \varphi^{-1}$ has derivative which is injective at $\varphi(x)$. F is a **submersion** if for each $x \in M$ there are charts φ and η such that the derivative of $\eta \circ F \circ \varphi^{-1}$ at $\varphi(x)$ is surjective.

Example 2.2.1 Let G be a Lie group, and $\rho : G \times G \to G$ the multiplication map. For fixed $g \in G$, define $\rho_g : G \to G$ by $\rho_g(h) = \rho(g, h)$. This is a diffeomorphism of G, since ρ_g is smooth and has smooth inverse $\rho_{g^{-1}}$.

Exercise 2.2.1 Show that if M is a manifold, and φ a chart for M, then φ^{-1} is a local diffeomorphism.

Exercise 2.2.2 Consider the map $\pi : S^n \to \mathbb{R}P^n$ defined by $\pi(x) = [x]$ for all $x \in S^n \subset \mathbb{R}^{n+1}$. Show that π is a local diffeomorphism.

Remark. The definitions above reflect the fact that the **rank** of the derivative of a smooth map between manifolds is well-defined: If we change coordinates from φ to $\tilde{\varphi}$ on M and from η to $\tilde{\eta}$ on N, then the chain rule gives:

$$D_{\tilde{\varphi}(x)}\left(\tilde{\eta} \circ F \circ \tilde{\varphi}^{-1}\right) = D_{\eta(x)}\left(\tilde{\eta} \circ \eta^{-1}\right) D_{\varphi(x)}\left(\eta \circ F \circ \varphi^{-1}\right) D_{\tilde{\varphi}(x)}\left(\varphi \circ \tilde{\varphi}^{-1}\right).$$

The first and third matrices on the right are non-singular, so the rank of the matrix on the left is the same as the rank of the second matrix on the right. I will not dwell on this, because it is a simple consequence of the construction of tangent spaces for manifolds, which we will work towards in the next few lectures.

Example 2.2.2 We will show that the projection $[.]$ from $\mathbb{R}^{n+1}\backslash\{0\}$ to $\mathbb{R}P^n$ is a submersion. To see this, fix $x = (x_1, \ldots, x_{n+1})$ in $\mathbb{R}^{n+1}\backslash\{0\}$, and assume that $x_{n+1} \neq 0$. On $\mathbb{R}^{n+1}\backslash\{0\}$ we take the trivial chart ι, and on $\mathbb{R}P^n$ we take $\varphi_{n+1} : V_{n+1} \to \mathbb{R}^n$. Then $\varphi_{n+1} \circ [.] \circ \iota^{-1}(x) = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$, and

$$D_x\left(\varphi_{n+1} \circ [.] \circ \iota^{-1}\right) v = \frac{1}{x_{n+1}} \begin{bmatrix} 1 & 0 & \ldots & 0 & -\frac{x_1}{x_{n+1}} \\ 0 & 1 & \ldots & 0 & -\frac{x_2}{x_{n+1}} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -\frac{x_n}{x_{n+1}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n+1} \end{bmatrix}$$

This matrix has rank n, so the map is a submersion.

Exercise 2.2.3 Show that the multiplication operator $\rho : G \times G \to G$ is a submersion if G is a Lie group.

Exercise 2.2.4 Let M and N be differentiable manifolds. Show that for any $x \in M$ the map $i_x : N \to M \times N$ given by $y \mapsto (x, y)$ is an embedding. Show that the projection $pi : M \times N \to M$ given by $\pi(x, y) = x$ is a submersion.

Example 2.2.3 We will show that the inclusion map ι of S^n in $\mathbb{R}^{n+1}$ is an embedding. To show this, we need to show that ι is a homeomorphism (this is immediate since we defined S^n by taking the subspace topology induced by inclusion in $\mathbb{R}^{n+1}$) and that the derivative of $\eta \circ \iota \circ \varphi^{-1}$ is injective.

Here we can take $\eta(x) = x$ to be the trivial chart for $\mathbb{R}^{n+1}$, and φ to be one of the stereographic projections defined in Example 1.1.3, say φ_-. Then $\eta \circ \iota \circ \varphi^{-1}(w) = \frac{(2w, 1-|w|^2)}{1+|w|^2}$. Differentiating, we find

$$\frac{\partial(\eta \circ \iota \circ (\varphi^{-1})^i)}{\partial w^j} = \frac{2}{1+|w|^2}\left(\delta_i^j - \frac{2w^i w^j}{1+|w|^2}\right)$$

and

$$\frac{\partial(\eta \circ \iota \circ (\varphi^{-1})^{n+1})}{\partial w^j} = -\frac{4w^j}{(1+|w|^2)^2}.$$

In particular we find for any non-zero vector $v \in \mathbb{R}^n$

$$\left|D_w(\eta \circ \iota \circ \varphi^{-1})(v)\right|^2 = \frac{4}{(1+|w|^2)^2}|v|^2 > 0.$$

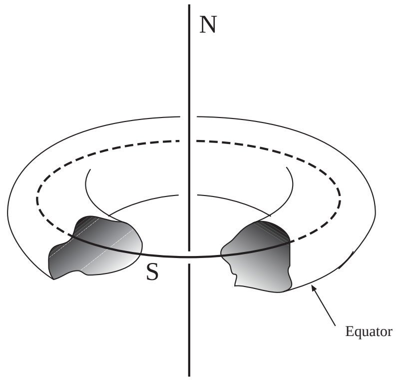
Therefore $D_w(\eta \circ \iota \circ \varphi^{-1})$ has no kernel, hence is injective.

Exercise 2.2.5 Prove that $\mathbb{C}P^1 \simeq S^2$ (Hint: Consider the atlas for S^2 given by the two stereographic projections, and the atlas for $\mathbb{C}P^1$ given by the two projections $[z_1, z_2] \mapsto z_2/z_1 \in \mathbb{C} \simeq \mathbb{R}^2$ and $[z_1, z_2] \mapsto z_1/z_2$. It should be possible to define a map between the two manifolds by defining it between corresponding charts in such a way that it agrees on overlaps).

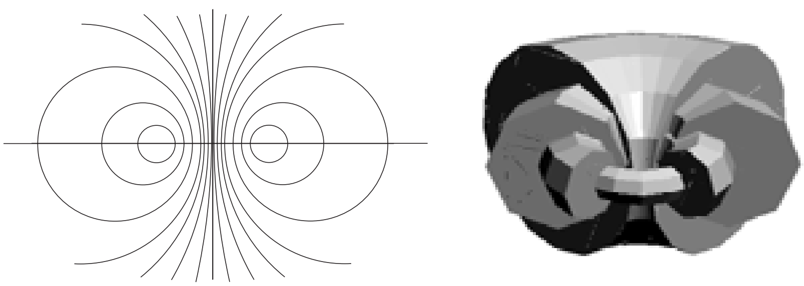
Exercise 2.2.6 (The Hopf map) Consider $S^3 \subseteq \mathbb{R}^4 \simeq \mathbb{C}^2$, and define $\pi : S^3 \to \mathbb{C}P^1 \simeq S^2$ to be the restriction of the canonical projection $(z_1, z_2) \mapsto [z_1, z_2]$. Show that this map is a submersion.

There is a nice way of visualising this map: We can think of S^3 as three-dimensional space (together with infinity). For each point y in $\mathbb{C}P^1$ the set of points in $\mathbb{C}^2$ which project to y is a plane, and the subset of this in S^3 is a circle. Thus we want to decompose space into circles, one corresponding to each point in S^2.

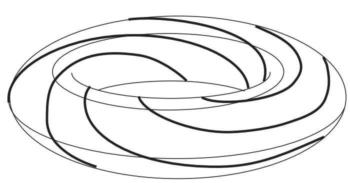
We can start with the north pole, and map this to a circle through infinity (i.e. the z-axis, say). Then we map the south pole to some other circle (say, the unit circle in the x-y plane). Next the equator: Here we have a circle of points, each corresponding to a circle, so this gives us a torus (which is called the Clifford torus):



More generally, consider a circle which moves from south to north pole on S^2. At each 'time' in this sweepout we have a corresponding torus in space, which starts as a thin torus around the unit circle, grows until it agrees with the Clifford torus as our curve passes the equator, and continues growing, becoming larger and larger in size with a smaller and smaller 'tube' down the middle around the z axis as we approach the north pole. This can be obtained by rotating the following picture about the z axis:



Next we should think about how the circles line up on these tori. The crucial thing to keep in mind is that this must be continuous as we move over S^2. In particular, as we approach the south pole the circles have to approach the unit circle, so they have to wind around the torus 'the long way'. On the other hand, as we approach the north pole the circle must start to lie 'along' the z axis. These two things seem to contradict each other, until we realise that we can wind our curves around the tori 'both ways':



Note that each of the circles corresponding to a point in the northern hemisphere intersects the unit disk D_1 in the x-y plane exactly once, and each circle corresponding to a point in the southern hemisphere intersects the right half-plane D_2 in the x-z plane exactly once. We can parametrize each circle through D_1 by an angle θ_1 in such a way that the intersection point with D_1 has $\theta_1 = 0$, and similarly for D_2. Points on the northern hemisphere of S^2 correspond to those circles which pass through a smaller disk $\tilde{D}_1$ in D_1, and those on the southern hemisphere correspond to circles passing through a disk $\tilde{D}_2$ within D_2.

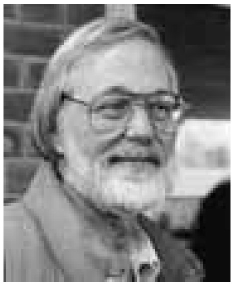
We can parametrize parametrize the boundary of $\tilde{D}_1$ by an angle ϕ_1 (anticlockwise in the x-y plane starting at the right) and that of $\tilde{D}_2$ with an angle ϕ_2 (anticlockwise in the x-z plane starting at the right). Then the angles θ_1 and ϕ_1 give the same point as $\theta_2 = \theta_1 - \phi_1$ and $\phi_2 = \pi - \phi_1$. This describes the 3-sphere as a union of two solid tori $\tilde{D}_1 \times S^1$ and $\tilde{D}_2 \times S^1$, sewn together by identifying points on $\partial\tilde{D}_1 \times S^1$ with points on $\partial\tilde{D}_2 \times S^1$ according to $(\phi, \theta) \in \partial\tilde{D}_1 \times S^1 \sim (\pi - \phi, \theta - \phi) \in \partial\tilde{D}_2 \times S^1$.

Exercise 2.2.7 Suppose we take two solid tori $D_1 \times S^1$ and $D_2 \times S^1$ and glue them together with a different identification, say $(\phi, \theta) \in \partial D_1 \times S^1 \sim (\phi, \theta + k\phi) \in \partial D_2 \times S^1$ for some integer k. How could we decide whether the result is or is not S^3? [This is hard — the answer really belongs to algebraic topology. The idea is to find some quantity which can be associated to a manifold which is invariant under homeomorphism or diffeomorphism (examples are the homotopy groups or homology groups of the manifold). Then if our manifold has a different value of this invariant from S^3, we know it is not diffeomorphic to S^3. Going the other way is much harder, since it is quite possible that non-diffeomorphic manifolds have the same value of any given invariant].

One of the important questions in differential geometry is the study (or classification) of manifolds *up to diffeomorphism*. That is, we introduce an equivalence relation on the space of k-dimensional differentiable manifolds by taking $M \sim N$ if and only if there exists a diffeomorphism from M to N, and study the equivalence classes. Among the results:

- There are only two equivalence classes of connected one-dimensional manifolds, namely those represented by S^1 and by $\mathbb{R}$. In particular the different differentiable structures on $\mathbb{R}$ introduced in Example 1.1.1 are diffeomorphically equivalent (the map $x \mapsto x^{1/3}$ is a diffeomorphism between the two);
- The equivalence class of a compact connected two-dimensional manifold is determined by its genus (an integer representing the number of 'holes' in the surface) and its orientability (which will be defined later).

A more subtle question is the following: Given a topological manifold M, are all differentiable structures on M related by diffeomorphism? If not, how many non-diffeomorphic differentiable structures are there on a given manifold? The answer is that in general there can be more than one differentiable structure on a manifold: It is known that the spheres S^n generally have more than one differentiable structure if $n \geq 7$, and in particular John Milnor proved in 1956 that there are 28 non-diffeomorphic differentiable structures on S^7. Two-dimensional manifolds have unique differentiable structures (this is a consequence of the classification theorem mentioned above), and this is also known for three-dimensional manifolds (although there is no classification of 3-manifolds known). In dimension four it is unknown whether S^4 has more than one differentiable structure, but it is known (through work of Simon Donaldson in the 1980's) that there are four-dimensional manifolds with several (infinitely many) non-diffeomorphic differentiable structures.

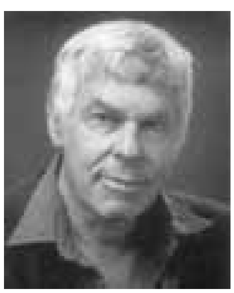

John Milnor


Simon Donaldson


Henri Poincaré

A related question is whether a given topological manifold carries any differentiable structures at all. Again the answer is no, and in particular it is known that there are topological manifolds of dimension 4 which carry no differentiable structure. In dimensions five and higher there are also non-smoothable topological manifolds. In two and three dimensions any manifold carries a differentiable structure.

One further famous problem: Poincaré asked in 1904 whether a 3-manifold which has the same homotopy groups as S^3 must be homeomorphic to S^3 — equivalently, if M is a simply connected compact 3-manifold, is M homeomorphic to S^3? This is the famous Poincaré conjecture, which remains completely open. Remarkably, the analogous problem in higher dimensions — whether an n-manifold with the same homotopy groups as S^n is homeomorphic to S^n — has been solved. Stephen Smale proved this in 1960 for $n \geq 5$ (similar results were also obtained by Stallings and Wallace around the same time). The four-dimensional case was much harder, and was proved by Michael Freedman in 1982 as a result of a truly remarkable theorem classifying closed simply connected four-dimensional topological manifolds. Only the three dimensional case remains unresolved!


Steven Smale


Michael Freedman