Lecture 3. Submanifolds

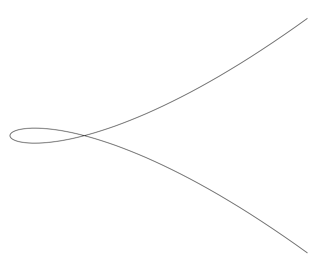
In this lecture we will look at some of the most important examples of manifolds, namely those which arise as subsets of Euclidean space.

2.1 Definition of submanifolds

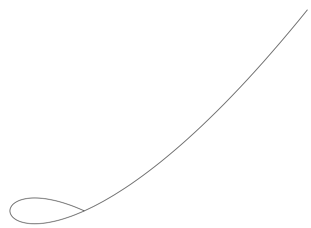
Definition 3.1.1 A subset M of $\mathbb{R}^N$ is a k**-dimensional submanifold** if for every point x in M there exists a neighbourhood V of x in $\mathbb{R}^N$, an open set $U \subseteq \mathbb{R}^k$, and a smooth map $\xi : U \to \mathbb{R}^N$ such that ξ is a homeomorphism onto $M \cap V$, and $D_y\xi$ is injective for every $y \in U$.

Remark. The meaning of 'homeomorphism onto $M \cap V$' in this definition warrants some explanation. This means that the map ξ is continuous and $1 : 1$, maps U onto $M \cap V$, and the inverse is also continuous. The last statement is to be interpreted as continuity with respect to the 'subspace topology' on M induced from the inclusion into $\mathbb{R}^N$. Since open sets in the subspace topology are given by restrictions of open sets in $\mathbb{R}^N$, this is equivalent to the statement that for every open set $A \subseteq U$ there exists an open set $B \subseteq \mathbb{R}^N$ such that $\xi(A) = M \cap B$.

Example 3.1.2 Consider the following examples of curves in the plane, which illustrate the conditions in the definition of submanifold above: The homeomorphism property can be violated if the function ξ is not injective, for example if $\xi(t) = (t^2, t^3 - t)$ for $t \in (-2, 2)$:

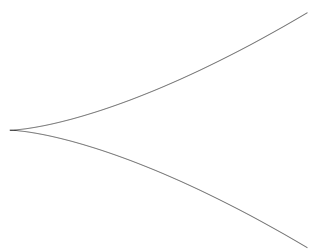


Even if ξ is injective, it might still fail to be a homeomorphism, as in the example $\xi(t) = (t^2, t^3 - t)$ for $t \in (-1, 2)$:



This is not a homeomorphism since $\xi((-0.1, 0.1))$ is not the intersection of the curve with any open set in $\mathbb{R}^2$.

Finally, a smooth map ξ might fail to define a submanifold if the derivative is not injective — in the case of curves, this means the derivative vanishes somewhere. An example is $\xi(t) = (t^2, t^3)$, which describes a cusp:



3.2 Alternative characterisations of submanifolds

Often the definition we have given is not the most convenient way to check whether a given subset of Euclidean space is a submanfold. There are several alternative characterisations of submanifolds, which are given by the following result:

Proposition 3.2.1 *Let M be a subset of Euclidean space $\mathbb{R}^N$. Then the following are equivalent:*

(a). M is a k-dimensional submanifold;

(b). M is a k-dimensional manifold, and can be given a differentiable structure in such a way that the inclusion $i : M \to \mathbb{R}^N$ is an embedding;

(c). *For every* $x \in M$ *there exists an open set* $V \subseteq \mathbb{R}^n$ *containing* x *and an open set* $W \subseteq \mathbb{R}^N$ *and a diffeomorphism* $F : V \to W$ *such that* $F(M \cap V) = (\mathbb{R} \times \{0\}) \cap W$*;*

(d). M *is locally the graph of a smooth function: For every* $x \in M$ *there exists an open set* $V \subseteq \mathbb{R}^N$ *containing* x*, an open set* $U \subseteq \mathbb{R}^k$*, an permutation* $\sigma \in S_N$*, and a smooth map* $f : U \to \mathbb{R}^{N-k}$ *such that*

$$M \cap V = \{(y^1, \dots, y^N) \in \mathbb{R}^N \mid (y^{\sigma(k+1)}, \dots, y^{\sigma(N)}) = f(y^{\sigma(1)}, \dots, y^{\sigma(k)})\}.$$

(e). M *is locally the zero set of a submersion: For every* $x \in M$ *there exists an open set* V *containing* x *and a submersion* $G : V \to Z \subseteq \mathbb{R}^{N-k}$ *such that* $M \cap V = G^{-1}(0)$*.*

Our main tool in the proof will be the Inverse function theorem:

Theorem 3.2.2 (Inverse Function Theorem) *Let* F *be a smooth function from an open neighbourhood of* $x \in \mathbb{R}^N$ *to* $\mathbb{R}^N$*, such that the derivative* D_xF *is an isomorphism. Then there exists an open set* A *containing* x *and an open set* B *containing* $F(x)$ *such that* $F|_A$ *is a diffeomorphism from* A *to* B*.*

For a proof of Theorem 3.2.2 see Appendix A.

Proof of Proposition 3.2.1: (b)$\Longrightarrow$(a) and (c)$\Longrightarrow$(a) are immediate, as are (d)$\Longrightarrow$(c) and (c)$\Longrightarrow$(e).

Suppose (a) holds, and fix $x \in M$. Choose $\xi_x : U_x \to V_x$ with $x \in M \cap V$ as given in the definition of submanifolds. Let $y = \xi_x^{-1}(x)$. Since $D_y\xi_x$ is injective, we can choose a bijection $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ such that rows $\sigma(1), \dots, \sigma(k)$ of $D_y\xi_x$ are linearly independent. Define $\pi : \mathbb{R}^N \to \mathbb{R}^k$ by $\pi(z^1, \dots, z^N) = (z^{\sigma(1)}, \dots, z^{\sigma(k)})$. Then $D_y(\pi \circ \xi_x)$ is an isomorphism, so by the Inverse Function Theorem there are open sets $A \subseteq U$ and $B \subseteq \pi(V) \subset \mathbb{R}^k$ and a smooth map $\eta_x : B \to A$ which is the inverse of $\pi \circ \xi_x|_A$.

Define $\mathcal{A} = \{\varphi_x = \eta_x \circ \pi : \ x \in M\}$. Each of these maps is a homeomorphism, since it has a continuous inverse $\xi|_A$. For $x_2 \neq x_1$ we have $\varphi_{x_2} \circ \varphi_{x_1}^{-1} = \varphi_{x_2} \circ \xi_{x_1}$, which is smooth. Therefore $\mathcal{A}$ is a differentiable atlas making M into a differentiable manifold. Finally, the inclusion $i : M \to \mathbb{R}^N$ is a homeomorphism, and for any chart η_x as above, we have $\mathrm{Id} \circ i \circ \eta_x^{-1} = \xi_x$, which has injective derivative. Therefore i is an embedding, and we have established (b). From the proof we can also establish (d), by taking f to be components $\sigma(k+1), \dots, \sigma(N)$ of $\xi \circ \eta$.

Finally, suppose (e) holds. Let $x \in M$, and let $G : V \to \mathbb{R}^{N-k}$ be a submersion as given in the Proposition. Let $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ be a bijection, such that columns $\sigma(k+1), \dots, \sigma(N)$ of D_xG are independent. Define $F : V \to \mathbb{R}^N$ by $F(z) = (z^{\sigma(1)}, \dots, z^{\sigma(k)}, G(z))$. Then D_xF is an isomorphism, so there exists (locally) an inverse by the Inverse Function Theorem. Define $f(z^1, \dots, z^k)$ to be components $\sigma(k+1), \dots, \sigma(N)$ of $F^{-1}(z^1, \dots, z^k, 0)$. Then (d) holds with this choice of f. $\square$

This proposition gives us a rich supply of manifolds, such as:

(a). The sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$;

(b). The cylinder $(x, y) \in \mathbb{R}^m \oplus \mathbb{R}^n : |x| = 1\}$;

(c). The torus $\mathbb{T}^2 = (x, y) \in \mathbb{R}^2 \oplus \mathbb{R}^2 : |x| = |y| = 1\}$;

(d). The special linear group $SL(n, \mathbb{R})$ consisting of $n \times n$ matrices with determinant equal to 1 (what dimension is this manifold?);

(e). The orthogonal group $O(n)$ consisting of $n \times n$ matrices A satisfying $\mathsf{A}^T\mathsf{A} = \mathsf{I}$, where I is the $n \times n$ identity matrix (what dimension is this?)

Exercise 3.2.1 Consider the subset of $\mathbb{R}^4$ given by the image of the map $\varphi : \mathbb{R} \to \mathbb{R}^4$ defined by

$$\varphi(t) = (\cos t, \sin t, \cos(\sqrt{2}t), \sin(\sqrt{2}t)).$$

Is this a submanifold of $\mathbb{R}^4$?

Definition 3.2.1 A subset M of a manifold N is a k- dimensional **submanifold** of N if for every $x \in M$ and every chart $\varphi : U \to V$ for N with $x \in U$, $\varphi(M \cap U)$ is a k-dimensional submanifold of V.

Exercise 3.2.2 Show that if $M \subset N$ is a submanifold of N then the restriction of every smooth function F on N to M is smooth.

Exercise 3.2.3 Show that the multiplication map $\rho : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ given by $\rho(\mathsf{A}, \mathsf{B}) = \mathsf{AB}$ is smooth.

Exercise 3.2.4 Let M and N be manifolds, and $\chi : M \to N$ a smooth map. Suppose Σ is a submanifold of M, and Γ a submanifold of N.

(i). Show that the restriction of χ to Σ is a smooth map from Σ to N.

(ii). If the $\chi(M) \subset \Gamma$, show that χ is a smooth map from M to Γ.

Example 3.2.1 The groups $SL(n, \mathbb{R})$ and $O(n)$. Each of these groups is contained as a submanifold in $GL(n, \mathbb{R})$. This implies that $SL(n, \mathbb{R}) \times SL(n, \mathbb{R})$ and $O(n) \times O(n)$ are submanifolds of $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$. Therefore the restriction of the multiplication map ρ on $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$ to either of these submanifolds is smooth, and has image contained in $SL(n, \mathbb{R})$ or $O(n)$ respectively. Hence by the result of Exercise 3.2.2, the multiplication on $SL(n, \mathbb{R})$ and $O(n)$ are smooth maps.

3.3 Orientability

Definition 3.3.1 An atlas $\mathcal{A}$ for a differentiable manifold M is **orientable** if whenever φ and η in $\mathcal{A}$ have nontrivial common domain of definition, the Jacobian $\det D(\eta \circ \varphi^{-1})$ is positive. A differentiable manifold is **orientable** if there exists such an atlas. An **orientation** on an orientable manifold is an equivalence class of oriented atlases, where two oriented atlases are equivalent if their union is an oriented atlas.

Exercise 3.3.1 Show that every one-dimensional manifold is orientable.

Exercise 3.3.2 Show that every connected manifold has either zero or two orientations.

Example 3.3.1 Hypersurfaces of Euclidean space A submanifold of dimension n in $\mathbb{R}^{n+1}$ is called a *hypersurface*. An orientation on a hypersurface M is equivalent to the choice of a unit normal vector continuously over the whole of M: Given an orientation on the hypersurface, choose the unit normal $\mathbf{N}$ such that for any chart φ in the oriented atlas for M,

$$\det\left[D\varphi^{-1}(e_1), \ldots, D\varphi^{-1}(e_2), \ldots, D_\varphi(e_n), \mathbf{N}\right] > 0. \qquad (+)$$

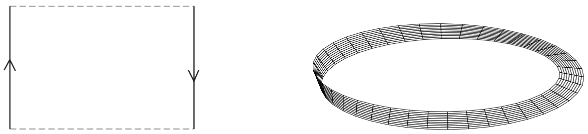
This is continuous on M since it is continuous on overlaps of charts. Conversely, given $\mathbf{N}$ chosen continuously over all of N, we choose an atlas for M consisting of all those charts for which $(+)$ holds.

Exercise 3.3.3 Suppose $F : \mathbb{R}^{n+1} \to \mathbb{R}$ has non-zero derivative everywhere on $M = F^{-1}(0)$. Show that M is orientable.

Example 3.3.2 The Möbius strip and the Klein bottle. The Möbius strip is the topological quotient of $\mathbb{R} \times \mathbb{R}$ by the equivalence relation $\sim$ which identifies (s,t) with $(s{+}1, -t)$ for every s and t in $\mathbb{R}$. M can be given an atlas as follows: We take a chart $\varphi_1 : (0,1) \times \mathbb{R}/\sim \to (0,1) \times \mathbb{R}$ to be the inverse of the map which takes (s,t) to $[(s,t)]$, and $\varphi_2 : (-1/2, 1/2) \times \mathbb{R}/\sim \to (-1/2, 1/2) \times \mathbb{R}$ similarly. Then

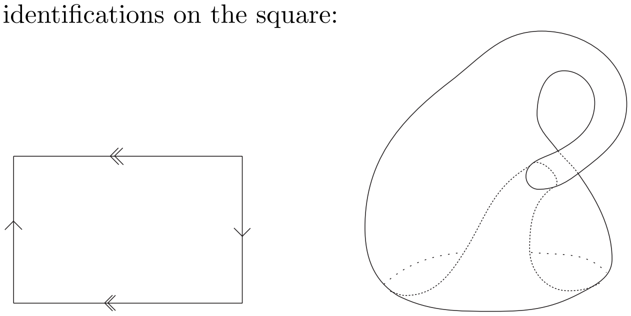
$$\varphi_1 \circ \varphi_2^{-1}(s,t) = \begin{cases} (s,t) & \text{if } s \in (0, 1/2) \\ (s+1, t) & \text{if } s \in (-1/2, 0) \end{cases}$$

which is smooth on $((-1/2, 0) \cup (0, 1/2)) \times \mathbb{R}$. The other transition map is similar.



The Möbius strip is not orientable. I will not prove this rigorously yet. Heuristically, the idea is that if we take an oriented pair of vectors at some point $(s, 0)$, and 'slide' them around the Möbius strip to $(s+1, 0)$, then if there were an oriented atlas it would have to be the case that the vectors obtained in this way were still oriented with respect to the original pair. But this is not the case.

The Klein bottle is another non-orientable surface, given by making the following identifications on the square:



Example 3.3.3 The real projective space $\mathbb{R}P^2$ is another example of a non-orientable surface. One way to visualise $\mathbb{R}P^2$ is as the semisphere with points on the equator identified with their antipodal points.

Some general nonsense: Definition 3.3.1 is one of many similar definitions for special classes of manifolds. More generally, suppose we have a class of maps $\mathcal{M} = \{\varphi : U \to V\}$ between open subsets of a Euclidean space $\mathbb{R}^n$, such that

(*) $\mathcal{M}$ is closed under composition: If $\varphi : U \to V$ and $\eta : W \to Z$ are in $\mathcal{M}$, and $V \cap W$ is non-empty, then $\eta \circ \varphi : \varphi^{-1}(V \cap W) \to Z$ is in $\mathcal{M}$

then one can consider $\mathcal{M}$-manifolds, by requiring the transition maps $\varphi \circ \eta^{-1}$ of an atlas to be in the class $\mathcal{M}$. Some examples are:

(1). The class of continuous maps. This gives rise to **topological manifolds**;
(2). The class of maps which are k times continuously differentiable. The resulting manifolds are C^k **manifolds**;
(3). The class of maps for which each component has a convergent power series (i.e. is a real-analytic function). This gives **real-analytic manifolds**;
(4). The class of maps between open sets of $\mathbb{C}^n$ which are holomorphic – that is, each complex component of the map is given by a convergent power series in the n complex variables. This defines **complex manifolds**;
(5). The class of maps of the form $x \mapsto \mathsf{M}x + v$ where M is in some subgroup G of the general linear group $GL(n, \mathbb{R})$ — such as $SL(n, \mathbb{R})$ (which gives **affine-flat manifolds**), or $O(n)$ (which gives **Euclidean manifolds** or **flat manifolds**);
(6). The class of maps F which have derivative DF in a subgroup G of $GL(n, \mathbb{R})$.

and so on...