Lecture 6. Differential Equations

Our aim is to prove the basic existence, uniqueness and regularity results for ordinary differential equations on a manifold.

6.1 Ordinary differential equations on a manifold.

Any vector field on a differentiable manifold M is naturally associated with a differential equation: If $V \in \mathcal{X}(\mathcal{M})$, and $x \in M$, the basic problem of ODE theory is to find a smooth map $\gamma : I \to M$ for some interval I containing 0, such that

$$\gamma'(t) = V_{\gamma(t)}$$

for all $t \in I$, and

$$\gamma(0) = x.$$

This is the general *initial value problem.* We would like to known several things about this problem: First, that solutions exist; second, that they are unique; and third, that the solutions depend in a smooth way on the initial point $x \in M$. The following proposition incorporates all three aspects:

Proposition 6.1.1 *Let $V \in \mathcal{X}(\mathcal{M})$ and $x \in M$. Then there exists $\delta > 0$, a neighbourhood U of x in M, and a unique smooth map $\Psi : U \times (-\delta, \delta) \to M$ which satisfies* [4]

$$\frac{\partial}{\partial t}\Psi(y,t) = V_{\Psi(y,t)}$$
$$\Psi(y,0) = y$$

for all $y \in U$ and $t \in (-\delta, \delta)$. For each $t \in (\delta, \delta)$, the map $\Psi_t : U \to M$ defined by $\Psi_t(y) = \Psi(y,t)$ is a local diffeomorphism, and

$$\Psi_t \circ \Psi_s = \Psi_{s+t}$$

whenever both sides are defined.

[4] Note that There is a natural vector field ∂_t defined on on $U \times (-\delta, \delta)$ by $(\partial_t f)(x) = \frac{\partial}{\partial s} f(x, t+s)\Big|_{s=0}$. The ODE means that $D_{(y,t)}\Psi(\partial_t) = V_{\Psi(y,t)}$ for all y and t.

The smoothness of Ψ as a function of y amounts to smooth dependence of solutions on their initial conditions, and we get the added bonus that the maps Ψ_t (called the (local) flow of V for time t) are local diffeomorphisms, and they form a "local group": $\Psi_t \circ \Psi_s = \Psi_{s+t}$ as one would expect for a group of diffeomorphisms, but this might hold only on a rather restricted domain.

Example 6.1.1: Problems with V. This example demonstrates why we may only be able to define the flow of a vector field locally, as in the proposition: Take $M = \mathbb{R}$, and take V to be the vector field

$$V_x = x^2 \partial_x.$$

Then for each $x \in \mathbb{R}$ we want to solve the equation

$$\begin{aligned} y'(t) &= y(t)^2; \\ y(0) &= x. \end{aligned}$$

This gives $\Psi(x,t) = y(t) = \frac{x}{1-tx}$, on the time interval $(1/x, \infty)$ if $x < 0$, or $(-\infty, 1/x)$ if $x > 0$ (or $(-\infty, \infty)$ in $x = 0$). Thus the flow of the vector field cannot be defined on $\mathbb{R} \times (\delta, \delta)$ for any $\delta > 0$, and cannot be defined on $U \times \mathbb{R}$ for any open subset U of $\mathbb{R}$. Here the problem seems to arise because the vector field V is not bounded.

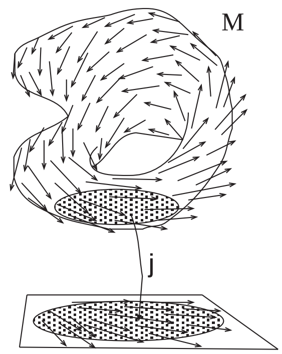
Example 6.1.2: Problems with M. Next consider the example $M = (0,1)$ with the vector field $V_x = \partial_x$: Now we have the flow

$$\Psi(x,t) = x + t$$

which is defined on the region $\{(x,t) : \ t \in (-x, 1-x)\}$. Again, this flow cannot be defined on $M \times (-\delta, \delta)$ for any $\delta > 0$, or on $U \times \mathbb{R}$ for any $U \subset M$. Here the problem seems to arise because the domain manifold has 'edges'.

Remark. In fact the distinction between these types of difficulties is not so clear. The two situations are in some sense the same, since a diffeomorphism can take a bounded interval to an unbounded region, mapping a bounded vector field to an unbounded one: For example, the map $x \mapsto \cot(\pi x)$ maps $(0,1)$ to $\mathbb{R}$ and sends the bounded vector field ∂_x to the unbounded vector field $-\pi/\sin^2(\pi x)\partial_x$. When we come to add some further structure to our manifolds in the form of metrics, we will have some notion of when a vector field is unbounded and when a manifold has a boundary, but without this the notions are not meaningful.

To begin the proof of Proposition 5.1.1, we will first reduce the problem to an ordinary differential equation on $\mathbb{R}^n$, by looking at the flow of the vector field through a chart:



Proposition 6.1.2 *Let $V \in \mathcal{X}(\mathcal{M})$, and suppose $\Psi : U \times (-\delta, \delta) \to M$ satisfies*

$$\begin{aligned} \partial_t \Psi(y,t) &= V_{\Psi(y,t)}; \\ \Psi(y,0) &= y; \end{aligned} \tag{6.1}$$

for each $(y,t) \in U \times (-\delta,\delta)$. Let $\varphi : W \to Z \subset \mathbb{R}^n$ be a chart for M such that $\Psi(U \times (-\delta,\delta)) \subset W$. Then $u(z,t) = \varphi \circ \Psi(\varphi^{-1}(z),t)$ defines a map $u : \varphi(U) \times (-\delta,\delta) \to Z$ which satisfies

$$\begin{aligned} \partial_t u^k(z^1,\ldots,z^n,t) &= \tilde{V}^k(u^1(z^1,\ldots,z^n,t),\ldots,u^n(z^1,\ldots,z^n,t)) \\ u^k(z^1,\ldots,z^n,0) &= z^k, \end{aligned} \tag{6.2}$$

for $k = 1,\ldots,n$ and all $(z^1,\ldots,z^n) \in \varphi(U)$ and $t \in (\delta,\delta)$. Here

$$V\left(\varphi^{-1}(z^1,\ldots,z^n)\right) = \sum_{k=1}^{n} \tilde{V}^k(z^1,\ldots,z^n)\partial_k.$$

Conversely, if u satisfies (6.2) then $\Psi(y,t) = \varphi^{-1}\left(u(\varphi(y),t)\right)$ defines a solution of (6.1).

Proof. We have $D_{(z,t)}u(\partial_t) = D_{\Psi(\varphi^{-1}(z),t)}\varphi \circ D_{(\varphi^{-1}(z),t)}\Psi(\partial_t) = D_\varphi(V)$. By definition $D\varphi(\partial_j) = e_j$ for $j = 1,\ldots,n$, so if $V = \sum \tilde{V}^k\partial_k$, then $D\varphi(V) = \sum \tilde{V}^k e_k$. □

We will prove Proposition 6.1.1 by constructing a unique solution for the ODE (6.2) on a region of $\mathbb{R}^n$.

6.2 Initial value problems.

We begin the proof of Proposition 6.1.1 by constructing solutions of initial value problems:

Proposition 6.2.1 *Let* $F : Z \to \mathbb{R}^n$ *be smooth, where* Z *is an open set of* $\mathbb{R}^n$*. Assume* $\sup_Z \|F\| = M_0 < \infty$*, and* $\sup_Z \|DF\| = M_1 < \infty$*. Let* $z \in Z$*. Then there exists a unique smooth* $\gamma : (-\delta, \delta) \to Z$ *satisfying*

$$\begin{aligned} \frac{d}{dt}\gamma^i(t) &= F(\gamma^1(t), \dots, \gamma^n(t)); \\ \gamma(0) &= z \end{aligned} \tag{6.3}$$

for $i = 1, \dots, n$*. Here* $\delta = \frac{d(z,\partial Z)}{M_0}$.

The curves constructed in this proposition are called the *integral curves* of the vector field $F^i\partial_i$.

Proof. We use the method of successive approximations, or 'Picard iteration'. Begin with some approximation to the solution, say $\gamma^{(0)}(t) = z$ for all $t \in (-\delta, \delta)$. Then we try to improve this approximation by iteration: Suppose we have an approximation $\gamma^{(k)}$. Then produce a new approximation by the formula

$$\gamma^{(k+1)}(t) = z + \int_0^t F\left(\gamma^{(k)}(s)\right) ds.$$

This is based on the following observation: The approximation $\gamma^{(k+1)}$ satisfies the differential equation

$$\frac{d}{dt}\gamma^{(k+1)} = F(\gamma^{(k)}),$$

so we are using the kth approximation to tell us the direction of motion for the $(k+1)$st approximation. If this iteration converges to a limit, the required ODE must be satisfied. Note that this iteration makes sense, because we have

$$\|\gamma^{(k)}(t) - z\| \le |t|M_0 < \delta M_0 = d(z, \partial Z)$$

so $\gamma^{(k)}(t)$ is always an element of Z.

To show that the iteration converges, consider the difference between successive approximations: We will prove that

$$\|\gamma^{(k+1)}(t) - \gamma^{(k)}(t)\| \le \frac{M_0 M_1^k |t|^{k+1}}{(n+1)!} \tag{6.4}$$

for all $k \ge 0$ and $t \in (-\delta, \delta)$. This is true for $k = 0$, since $\|\gamma^{(1)} - \gamma^{(0)}\| = \|\int_0^t F(z)ds\| \le |t|M_0$. We proceed by induction: Suppose the inequality holds for $k - 1$. Then

$$\begin{aligned}\left\|\gamma^{(k+1)}(t)-\gamma^{(k)}(t)\right\| &= \left\|\int_0^t F(\gamma^{(k)}(s))-F(\gamma^{(k-1)}(s))\,ds\right\| \\ &\leq \left|\int_0^t M_1\left\|\gamma^{(k)}(s)-\gamma^{(k-1)}(s)\right\|\,ds\right| \\ &\leq M_1\left|\int_0^t \frac{M_0M_1^{k-1}|s|^k}{k!}\,ds\right| \\ &\leq \frac{M_0M_1^k|t|^{k+1}}{(k+1)!}.\end{aligned}$$

This implies that the sequence $\{\gamma^{(k)}\}$ is a Cauchy sequence in the complete space of continuous maps with respect to uniform convergence, and so converges to a continuous limit γ. The continuity of F and the dominated convergence theorem then imply that

$$\gamma(t) = z + \int_0^t F(\gamma(s))\,ds \tag{6.5}$$

for all $t \in (-\delta, \delta)$, so that γ is differentiable and satisfies the equation

$$\gamma'(t) = F(\gamma(t))$$

for each t, and the initial condition $\gamma(0) = z$. Smoothness follows, since $\gamma \in C^{(k)}$ implies $\gamma \in C^{k+1}$ by the identity (6.5) and the smoothness of F. This establishes the existence of a solution.

To prove uniqueness, suppose γ and σ are two solutions of the initial value problem. Then

$$\begin{aligned}\|\gamma(t)-\sigma(t)\| &= \left\|\int_0^t F(\gamma(s))-F(\sigma(s))\,ds\right\| \\ &\leq M_1\left|\int_0^t \|\gamma(s)-\sigma(s)\|\,ds\right|\end{aligned} \tag{6.6}$$

Let $C = \sup\|\gamma-\sigma\|$. Then an induction similar to that above shows that

$$\|\gamma(t)-\sigma(t)\| \leq \frac{CM_1^k|t|^k}{k!}$$

for any k. Taking $k \to \infty$ gives $\gamma \equiv \sigma$. □

6.3 Smooth dependence on initial conditions.

The result of the previous section produced the unique flow Ψ of the vector field V, but did not address the smoothness of Ψ except in the t direction. Since smoothness is measured by reference to charts, it is enough to show smoothness of the map u from Proposition 6.1.2:

Proposition 6.3.1 *The function u which takes a pair (z,t) to the solution $\gamma(t)$ of the initial value problem (6.3) is smooth on the open set*

$$S = \{(z,t) \in Z \times \mathbb{R} : \ |t| < d(z, \partial Z)/M_0\} .$$

Proof. I will show that u is the C^k limit of a family of smooth functions for any k: Define $u^{(0)}(z,t) = z$, and successively approximate using

$$u^{(k+1)}(z,t) = z + \int_0^t F\left(u^{(k)}(z,s)\right) ds. \tag{6.7}$$

Clearly $u^{(k)}$ maps S to Z for each k, $u^{(k)}$ is smooth for each k, and by (6.4) we have

$$\|u^{(k+1)}(z,t) - u^{(k)}(z,t)\| \leq \frac{M_0 M_1^k |t|^{k+1}}{(k+1)!}.$$

Differentiating (6.7) with t fixed gives

$$\left\|D_z u_t^{(k+1)}\right\| = \left\|\mathrm{I} + \int_0^t DF \circ D_z u_s^{(k)}\, ds\right\| \leq 1 + M_1 \left(\int_0^t \left\|D_z u_s^{(k)}\right\| ds\right)$$

which gives by induction

$$\left\|D_z u_t^{(k)}\right\| \leq \sum_{j=0}^{k} \frac{M_1^j |t|^j}{j!} \leq e^{M_1|t|}$$

independent of k. We also have $\|\partial_t u^{(k)}\| \leq M_0$ for all k, so $\{u^{(k)}\}$ is uniformly bounded in C^1. Similar arguments with higher derivatives give uniform bounds on $\{u^{(k)}\}$ in C^j for every j.

Exercise 6.3.2 Show that a sequence of functions $\{u^{(k)}\}$ which converges in C^0 and is bounded in C^j converges in C^m for $m = 1, \ldots, j-1$ [Hint: The key to this is the following *interpolation inequality*: For any C^j function u, and any $l \in \{1, \ldots, j-1\}$, there is a constant C such that

$$\|D^l u\|_{C^0} \leq C \|u\|_{C^0}^{1-l/j} \|D^j u\|_{C^0}^{l/j}.$$

Prove this by first proving the case $l = 1$, $j = 2$, and then applying this successively to get the other cases. Then apply the estimate to differences $u^{(k)} - u^{(k')}$].

This completes the proof of Proposition 6.3.1, since Exercise 6.3.2 shows that $u = \lim u^{(k)}$ is in C^j for every j. □

Remark. It is possible to show more explicitly that the approximations $u^{(k)}$ converge in C^j for every j, by differentiating the formula $u^{(k+1)} - u^{(k)}$ j times.

6.4 The local group property.

Next we show that the flow Ψ we have constructed satisfies the local group property

$$\Psi_t \circ \Psi_s = \Psi_{t+s}$$

whenever both sides of this equation make sense.

This is very easy: $\Psi_t \circ \Psi_s$ and Ψ_{t+s} both satisfy the same differential equation

$$\partial_t \Psi = V \circ \Psi,$$

and have the same initial condition at $t = 0$. Hence by the uniqueness part of Proposition 6.2.1, they are the same.

6.5 The diffeomorphism property.

We need to show that $D\Psi_t$ is non-singular. Again, this is very easy: It is true for t small (uniformly in space) since Ψ is smooth and $D\Psi_0 = \mathrm{I}$. But by the local group property, $\Psi_t = \left(\Psi_{t/m}\right)^m$, so $D\Psi_t$ is a composition of m non-singular maps, and hence is non-singular.

This completes the proof of Proposition 6.1.1.

6.6 Global flow.

Proposition 6.1.1 gives the existence of the flow of a vector field *locally*, and we have seen that there are examples which show that one cannot in general expect better than this. However there are some very important situations where we can do better:

Proposition 6.6.1 *Let $V \in \mathcal{X}(\mathcal{M})$ be a vector field with compact support – that is, assume that* $\mathrm{supp} V = \overline{\{x \in M : \ V(x) \neq 0\}}$ *is a compact subset of M. Then there exists a unique smooth map $\Psi : M \times \mathbb{R} \to M$ satisfying*

$$\partial_t \Psi = V \circ \Psi; \qquad \Psi(x, 0) = x.$$

The maps $\{\Psi_t\}$ form a one-parameter group of diffeomorphisms of M.

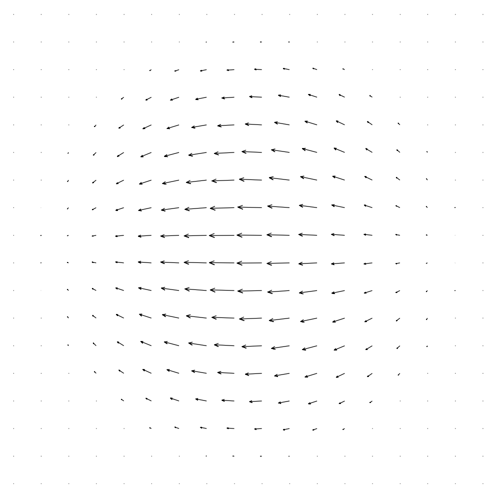


Fig. 6.2: A vector field supported in the unit disk.

Proof. Cover supp$V$ by sets of the form $\varphi_\alpha^{-1}(B_{r_\alpha/2}(0))$ where $\varphi_\alpha : W_\alpha \to Z_\alpha$ is a chart for M and $\overline{B_{r_\alpha}(0)} \subset Z_\alpha$. By compactness there is a finite subcover $\{\varphi_i^{-1}(B_{r_i/2}(0))\}_{i=1}^N$. Let $r = \inf_{i=1,\dots,N} r_i/2 > 0$.

Proposition 6.1.1 gives a local flow Ψ_i on each region $\varphi_i^{-1}(B_{r_i/2}(0)) \times (-\delta, \delta)$, where $\delta = \inf_{i=1,\dots,N} \frac{r}{\sup_{Z_i} \|\tilde{V}_i\|} > 0$. The uniqueness of solutions implies that these local flows agree on the overlaps of these sets, so they combine to give a local flow on the set $M \times (-\delta, \delta)$ (by taking Ψ to be the identity map away from supp$V$). The local group property implies that these maps are diffeomorphisms, since $\Psi_t^{-1} = \Psi_{-t}$ for $|t| < \delta$. Finally, for any $t \in \mathbb{R}$, define $\Psi_t = \left(\Psi_{t/m}\right)^m$, where m is sufficiently large to ensure that $|t|/m < \delta$. Then Ψ satisfies the required differential equation and is defined on $M \times \mathbb{R}$. □

In particular, a smooth vector field on a *compact* manifold M always has a globally defined flow.

Exercise 6.6.1 Show that the exponential map on a Lie group G is a smooth map defined on all of T_eG [Hint: First prove existence in a neighbourhood of the origin. Then use the one-parameter subgroup property to extend to the whole of T_eG].

An important feature of flows of vector fields (i.e. of solving differential equations) is the possibility of substituting new variables in a differential equation. We will formalise this as follows:

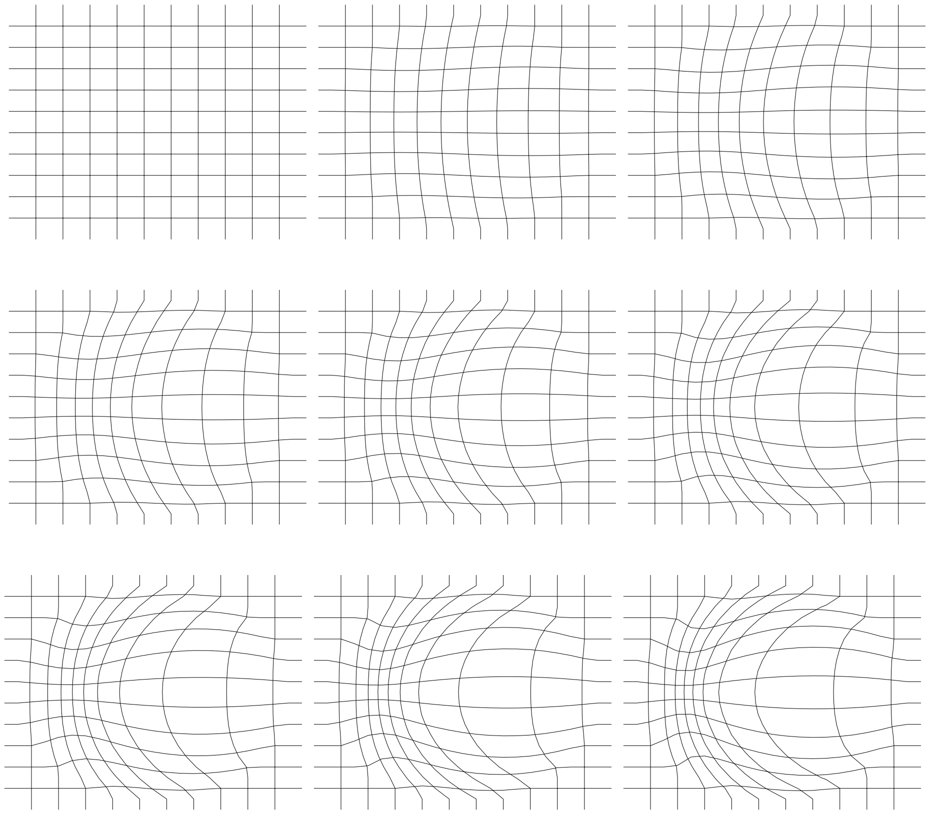


Fig. 6.3: Images of a standard grid at a sequence of times under the flow of the vector field from Fig. 6.2

Definition 6.6.1 Let $F : M \to N$ be smooth. Vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(N)$ are F-related if, for every $x \in X$,

$$D_x F(X_x) = Y_{F(x)}.$$

In particular, if F is a diffeomorphism, every $X \in \mathcal{X}(M)$ has a unique F-related $N \in \mathcal{X}(N)$, which we denote $F_*(X)$, the push forward of X by F.

Proposition 6.6.2 *Let $F : M \to N$ be smooth, and $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ F-related. Let Ψ be the local flow of X, and Φ the local flow of Y. Then*

$$\Phi(F(y), t) = F \circ \Psi(y, t)$$

for all y sufficiently close to x and t sufficiently small.

Proof. The uniqueness of solutions of initial value problems applies, since $\partial_t \left(F \circ \Psi\right)(y, t) = D_{\Psi(y,t)} F(X_{\Psi(y,t)}) = Y_{F \circ \Psi(y,t)}$ and $F \circ \Psi(y, 0) = F(y)$. □