Geometry and the Kato Square Root Problem

Menaka Lashitha Bandara
(Lashi Bandara)

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Declaration

I hereby declare that the material in this thesis is my own original work except where stated otherwise.

The material in Section 3.1 is included in the preprint [10] titled "Quadratic estimates for perturbed Dirac type operators on doubling measure metric spaces" and it is to appear in *Proceedings of the AMSI International Conference on Harmonic Analysis and Applications, Feb. 2011, Proc. Centre Math. Appl. Austral. Nat. Univ.*

Chapter 2 is included in the preprint [11] titled "Density problems on manifolds and vector bundles" and it is to appear in *Proc. Amer. Math. Soc.*.

Chapter 4 is based on the material in the preprint [12] titled "Square roots of perturbed sub-Lapalacians on Lie groups" authored by Tom ter Elst, Alan McIntosh and myself and it is to appear in *Studia Math.*

Section 3.2 and Chapter 5 is based on the material in the preprint [13] titled "The Kato square root problem on vector bundles with generalised bounded geometry" authored by Alan McIntosh and myself, and has been submitted for publication.

Lashi Bandara (Menaka Lashitha Bandara)

"Eir Ci!"

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Abstract

The primary focus of this thesis is to consider Kato square root problems for various divergence-form operators on manifolds. This is the study of perturbations of second-order differential operators by bounded, complex, measurable coefficients. In general, such operators are not self-adjoint but uniformly elliptic. The Kato square root problem is then to understand when the square root of such an operator, which exists due to uniform ellipticity, is comparable to its unperturbed counterpart.

A remarkably adaptable operator-theoretic framework due to Axelsson, Keith and McIntosh sits in the background of this work. This framework allows us to take a powerful first-order perspective of the problems which we consider in a geometric setting. Through a well established procedure, we reduce these problems to the study of quadratic estimates.

Under a set of natural conditions, we prove quadratic estimates for a class of operators on vector bundles over complete measure metric spaces. The first kind of estimates we prove are global, and we establish them on trivial vector bundles when the underlying measure grows at most polynomially. The second kind are local, and there, we allow the vector bundle to be non-trivial but bounded in an appropriate sense. Here, the measure is allowed to grow exponentially.

An important consequence of obtaining quadratic estimates on measure metric spaces is that it allows us to consider subelliptic operators on Lie groups. The first-order perspective allows us to reduce the subelliptic problem to a fully elliptic one on a sub-bundle. As a consequence, we are able to solve a homogeneous Kato square root problem for perturbations of subelliptic operators on nilpotent Lie groups. For general Lie groups we solve a similar inhomogeneous problem.

In the situation of complete Riemannian manifolds, we consider uniformly elliptic divergence-form operators arising from connections on vector bundles. Under a set

of assumptions, we show that the Kato square root problem can be solved for such operators. As a consequence, we solve this problem on functions under the condition that the Ricci curvature and injectivity radius are bounded. Assuming an additional lower bound for the curvature endomorphism on forms, we solve a similar problem for perturbations of inhomogeneous Hodge-Dirac operators. A theorem for tensors is obtained by additionally assuming boundedness of a second-order Riesz transform.

Motivated by the study of these Kato problems, where for technical reasons it is useful to know the density of compactly supported functions in the domains of operators, we study connections and their divergence on a vector bundle. Through a first-order formulation, we show that this density property holds for the domains of these operators if the metric and connection are compatible and the underlying manifold is complete. We also show that compactly supported functions are dense in the second-order Sobolev space on complete manifolds under the sole assumption that the Ricci curvature is bounded below, improving a result that previously required an additional lower bound on the injectivity radius.

Contents

Introduction

In the 1960's, Kato studied *abstract evolution equations* of the form

$$\partial_t u(t) + A(t)u(t) = f(t), \quad u(0) = u_0 \in \mathscr{H} \tag{E}$$

where $t \in [0, T]$ on a Hilbert space $\mathscr{H}$ and $A(t)$ are closed, densely-defined operators with domains $\mathcal{D}(A(t))$. They are also assumed to be *maximally accretive*, by which we mean $\operatorname{Re}\langle A(t)u, u\rangle \geq 0$. Under these conditions, fractional powers $A(t)^\alpha$ exist as closed, densely-defined operators, and if for some $\alpha \in (0, 1]$, $\mathcal{D}(A(t)^\alpha)$ is constant, then under certain smoothness conditions on $A(t)$ and $f(t)$, the abstract evolution equation admits a unique, *strict* (continuously differentiable) solution.

The operators $A(t)$ typically arise naturally, associated to a family of sesquilinear forms (or complex energies) $J_t : \mathcal{W} \times \mathcal{W} \to \mathbb{C}$ (where $\mathcal{W} \subset \mathscr{H}$). Namely, the operators $A(t)$ are defined as the operators with largest domains $\mathcal{D}(A(t))$ satisfying: $J_t[u, v] = \langle A(t)u, v\rangle$ for all $u \in \mathcal{D}(A(t))$ and $v \in \mathcal{W}$.

For an $\omega \in [0, \pi/2)$, such a form is said to be ω-sectorial if $\mathcal{W}$ is dense in $\mathscr{H}$, $J_t[u, u] \in \mathrm{S}_{\omega+} = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega\} \cup \{0\}$ (the closed sector of angle ω in the right half complex plane), and $\mathcal{W}$ is complete with respect to the norm $\|u\|_{\mathcal{W}} = \|u\|^2 + \operatorname{Re} J_t[u, u]$. Similarly, we can say that an operator $T : \mathcal{D}(T) \subset \mathscr{H} \to \mathscr{H}$ is ω-accretive if T is densely-defined and closed, $\langle Tu, u\rangle \in \mathrm{S}_{\omega+}$, and $\sigma(T) \subset \mathrm{S}_{\omega+}$. If $\omega = 0$ then T non-negative self-adjoint.

If $A(t)$ is the associated operator to J_t as defined above, and J_t is ω-sectorial, then the Lax-Milgram Theorem guarantees that $A(t)$ is ω-accretive. If $\omega = 0$, this theorem asserts that $A(t)$ is self-adjoint, $\mathcal{D}(\sqrt{A(t)}) = \mathcal{W}$ (so that in particular, it is constant in t) and for all $u, v \in \mathcal{W}$, $J_t[u, v] = \left\langle \sqrt{A(t)}u, \sqrt{A(t)}v \right\rangle$. In the 1961 paper [39], Kato demonstrates that for every $\alpha < 1/2$,

$$\mathcal{D}(A(t)^{\alpha}) = \mathcal{D}(A(t)^{*\alpha}) = \mathcal{D}_{\alpha} \quad \text{and} \quad \|A(t)^{\alpha}u\| \simeq \|A(t)^{*\alpha}u\|. \tag{K_α}$$

Counterexamples were known to say (K_α) is in general false when $\alpha > 1/2$. In the same paper, Kato asks two questions (as Remarks 1 and 2 on page 268) about the critical case $\alpha = 1/2$. We paraphrase these questions in our notation:

(K1) Is $\mathcal{D}(\sqrt{A(t)}) = \mathcal{D}(\sqrt{A(t)^*}) = \mathcal{W}$?

(K2) For the case $\omega = 0$, the Lax-Milgram Theorem tells us that (K1) is true, but is $\|\partial_t \sqrt{A(t)}u\| \lesssim \|u\|_{\mathcal{W}}$ for $u \in \mathcal{W}$?

A year later, Lions demonstrated in [41] that (K1) is in general invalid for maximal accretive operators. In his 1972 paper [42], McIntosh constructed an ω-accretive operator A (with $\omega < \pi/2$) such that $\mathcal{D}(\sqrt{A}) \neq \mathcal{D}(\sqrt{A^*})$, providing a counterexample to (K1) for *regularly accretive* operators. Ten years later, in [43], McIntosh used similar techniques to construct a counterexample for (K2).

While Kato studied fractional powers of accretive operators motivated by abstract evolution equations of the form (E), the study of such equations were themselves motivated by hyperbolic equations on Euclidean space. Thus, attention became focused on the problem of determining:

$$\mathcal{D}(\sqrt{-a \operatorname{div} A\nabla}) = \mathrm{W}^{1,2}(\mathbb{R}^n) \quad \text{and} \quad \|\sqrt{-a \operatorname{div} A\nabla u}\| \simeq \|\nabla u\|, \tag{Kato}$$

where a, A are L^∞ pointwise multiplication operators. The accretivity of the operator follows from assuming an ellipticity condition of the form $\operatorname{Re}\langle A\nabla u, \nabla u\rangle \geq \kappa_1 \|\nabla u\|^2$, for $\kappa_1 > 0$. This was the reincarnation of the abstract problem (K1), which in fact implies (K2) as a consequence of allowing for *complex* coefficients. It is this problem which became popularly known as the *Kato square root problem*.

The Kato square root problem on $\mathbb{R}^n$ was finally settled in 2002 by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [5]. Indeed, much development towards the resolution of this problem happened between the years of 1982 and 2002, particularly in terms of $T(b)$ theorems in harmonic analysis. The book [6] by Auscher and Tchamitchian and the paper [35] by Hofmann are a good source of information on the history of the problem. The survey paper [36] by Hofmann and McIntosh illustrates the development and importance of $T(b)$ theorems.

A very significant development for our purposes occurred in 2005 with the paper [8] by Axelsson (Rosén), Keith and McIntosh. There, the authors developed a first-order perspective of the Kato square root problem and rephrased it in terms of the perturbation of *Dirac type* operators. In this paper, they were able to use this *Axelsson-Keith-McIntosh* (AKM) framework to consolidate many of the previous results in harmonic analysis, and in addition, solve similar problems for systems, differential forms on $\mathbb{R}^n$, and such problems on compact manifolds. This framework is further adapted in [7] to solve inhomogeneous problems on Lipschitz domains $\Omega \subset \mathbb{R}^n$.

We remark that, while [8] proves a Kato square root problem on compact manifolds, the significant advancement in the manifold setting is due to the work of Morris in [46]. There, Morris solves a Kato square root problem on submanifolds of Euclidean space whose second fundamental form is uniformly bounded.

The power of the AKM framework lies in the fact that the operator theory and functional calculi are written out at the level of an abstract Hilbert space, and quadratic estimates, which imply solutions to the Kato square root problem, are obtained under a further set of more concrete hypotheses. More explicitly, the authors consider closed, densely-defined, nilpotent operators Γ on a Hilbert space $\mathscr{H}$, and bounded operators B_1, B_2. Defining $\Pi_B = \Gamma + B_1\Gamma^* B_2$, they show that $\mathcal{D}(\sqrt{{\Pi_B}^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$ and $\|\sqrt{{\Pi_B}^2} u\| \simeq \|\Pi_B u\| \simeq \|\Gamma u\| + \|\Gamma^* B_2 u\|$ under a set of hypotheses (H1)-(H8).

In this thesis, we are primarily concerned with formulating and solving Kato square root problems on manifolds. Similar to the case of boundary value problems, we expect the formulation of such problems to be inhomogeneous (due to geometry). Let $\mathcal{M}$ be a manifold with metric g, $S = (\mathrm{I}, \nabla) : \mathrm{W}^{1,2}(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$ and write $\mathrm{L}_A = aS^*AS$, where $A = (A_{ij})$ is a matrix of L^∞ pointwise multiplication operators and $a \in \mathrm{L}^\infty(\mathcal{M})$. Explicitly, the operator $\mathrm{L}_A u = -a \operatorname{div}(A_{11}\nabla u) - a \operatorname{div}(A_{10} u) + aA_{01}\nabla u + aA_{00} u$, which can be seen as an L^∞ perturbation of the Bochner Laplacian $\Delta_B = -\operatorname{div} \nabla$ with lower-order terms. We require L_A to be uniformly elliptic in order to talk about its square roots. The *Kato square root problem on manifolds* is then to determine conditions on the geometry for which we can obtain:

$$\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{W}^{1,2}(\mathcal{M}) \quad \text{and} \quad \|\sqrt{\mathrm{L}_A} u\| \simeq \|u\|_{\mathrm{W}^{1,2}} = \|u\| + \|\nabla u\|. \tag{Kato$_\mathcal{M}$}$$

In the setting of a Lie group, which has more structure than an abstract manifold, we can ask questions about perturbations of *subelliptic* operators. These questions are the natural sort to generalise the homogeneous problem (Kato). Let $\mathcal{G}$ be a Lie group and $\mathfrak{a}$ an *algebraic basis* for the Lie algebra $\mathfrak{g}$. We can consider the right translation and span of these vectors to yield global vector fields. Associated with these vector fields is a natural sub-Riemannian distance as well as a sub-connection ∇ (which lands in the span of the co-vector fields obtained from $\mathfrak{a}$ rather than the whole co-tangent space). Letting $\mathrm{W}^{1,2}(\mathcal{G})'$ denote the domain of ∇, the homogeneous Kato square root problem for subelliptic operators on Lie groups is then to determine when:

$$\mathcal{D}(\sqrt{-a \operatorname{div} A\nabla}) = \mathrm{W}^{1,2'}(\mathcal{G}) \quad \text{and} \quad \|\sqrt{-a \operatorname{div} A\nabla} u\| \simeq \|\nabla u\|, \tag{Kato$_{\mathcal{G}}$}$$

where $a, A \in \mathrm{L}^\infty$ are multiplication operators and the operator $-\operatorname{div} A\nabla$ is uniformly subelliptic. In fact, this particular problem has been asked by ter Elst since the resolution of the Kato square root problem on $\mathbb{R}^n$ in 2002.

For the purposes of the research we present in this thesis, the profound resolution of the Kato square problem in 2002 is where we begin. We propose the term *geometric Kato square root problem* to describe the geometric problems of the form (Kato$_{\mathcal{M}}$) and (Kato$_{\mathcal{G}}$). The material we present in this thesis advances the understanding of the problem by providing some natural geometric conditions under which (Kato$_{\mathcal{M}}$) and (Kato$_{\mathcal{G}}$) can be solved in the affirmative.

The way in which we attack these problems is to make modifications to the AKM framework and list a set of mild, operator theoretic hypotheses (H1)-(H6) in §1.8. Quadratic estimates, the fundamental objects which allow us to prove Kato square root type estimates, are obtained under two additional hypotheses. We label these hypotheses geometric and outline them in Chapter 3.

A first advancement to the theory we provide is to show that the quadratic estimates can be obtained on measure metric spaces. There are a few difficulties in generalising the proofs in [8] to the metric space setting, but the primary one is the lack of a differentiable structure. We compensate for this by rephrasing the argument in terms of Lipschitz functions and controlling such functions using the *pointwise Lipschitz constant*. Our first theorem is then the following, which allows us to obtain *homogeneous* Kato square root estimates (Kato$_{\mathcal{G}}$), directly generalising (Kato).

Theorem 1 (Theorem 3.1.1)**.** *Let $(\mathcal{X}, d, \mu)$ be a complete measure metric space which is at most polynomial in growth and (Γ, B_1, B_2) satisfy (H1)-(H6) (§1.8) ,* *(H7-H) and (H8-H) (§3.1). Then,* Π_B *satisfies the quadratic estimate*

$$\int_0^\infty \|\Pi_B(\mathrm{I}+t^2{\Pi_B}^2)^{-1}u\|^2\ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$

We are predominantly interested in obtaining solutions to ($\mathrm{Kato}_{\mathcal{M}}$) for manifolds under *intrinsic* geometric conditions. It is worth mentioning here that, at least from our perspective, the quadratic estimates see the geometry because we need to analyse a first-order formulation of the problem on the space $\mathrm{L}^2(\mathcal{M}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M})$. In [46], Morris obtains quadratic estimates on $\mathrm{L}^2(\mathbb{C}^N)$ and uses bounds on extrinsic curvature (second fundamental form) to move from analysis on $\mathrm{L}^2(\mathrm{T}^*\mathcal{M})$ to $\mathrm{L}^2(\mathbb{C}^N)$.

Motivated by the existence of *harmonic coordinates* which provides us with the right kind of uniform intrinsic control on the geometry, we formulate a notion called *generalised bounded geometry* for a vector bundle $\mathcal{V}$ on a measure metric space $\mathcal{X}$ (§1.4). This notion captures a uniform locally Euclidean structure on the bundle. We allow for this very general situation of vector bundles because from our perspective, they capture the notion of *systems* in the presence of geometry. The quadratic estimates used to solve inhomogeneous problems of the form ($\mathrm{Kato}_{\mathcal{M}}$) is then guaranteed by the following theorem.

Theorem 2 (Theorem 3.2.4)**.** *Suppose that* $(\mathcal{X}, d, \mu)$ *is a complete measure metric space that is at most exponential in growth,* $\mathcal{V}$ *satisfies the generalised bounded geometry criterion (§1.4), and that* (Γ, B_1, B_2) *satisfy (H1)-(H6) (§1.8), (H7-I) and (H8-I) (§3.2). Then,* Π_B *satisfies the quadratic estimate*

$$\int_0^\infty \|t\Pi_B(\mathrm{I}+t^2{\Pi_B}^2)^{-1}u\|^2\ \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{V})$.

We acknowledge that the harmonic analytic technology which Morris develops in [46] assists a great deal in proving this theorem. However, this theorem is by no means an immediate consequence. A crucial difficulty is that, in the harmonic analysis, we use techniques such as integral averaging in our proofs. This is not, in general, well defined on a vector bundle. Hence, we are forced to define notions of integration and averaging after fixing a set of coordinates seen through an abstract dyadic decomposition that is valid up to a certain scale. These coordinates may indeed jump at the boundaries of the cubes, but our techniques allow us to absorb such jumps. This construction permits us to import the Euclidean tools to this vector bundle setting.

As aforementioned, the point of these quadratic estimates is that they allow us to solve Kato square root problems. The following theorem is obtained through analysis on Lie groups and due to our first-order perspective of the problem. We also mention that obtaining the quadratic estimates on measure metric spaces is of paramount importance in proving the following theorem as we need to depart from any Riemannian structure of the Lie group to a sub-Riemannian setting.

Theorem 3 (Theorem 4.3.1)**.** *Let $(\mathcal{G}, d, \mu)$ be a connected, nilpotent Lie group with $\mathfrak{a}$ an algebraic basis, d the associated subelliptic distance, and μ the left Haar measure. Suppose that $a, A \in \mathrm{L}^\infty$ and that there exist $\kappa_1, \kappa_2 > 0$ satisfying*

$$\operatorname{Re} \langle au, u \rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \operatorname{Re} \langle A\nabla v, \nabla v \rangle \geq \kappa_2 \|\nabla v\|^2,$$

for every $u \in \mathrm{L}^2(\mathcal{G})$ and $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then, (Kato$_{\mathcal{G}}$) holds.

Through the use of the quadratic estimates in Theorem 2, we are able to solve the following inhomogeneous problem on general Lie groups. In the following theorem, the operator $S = (\mathrm{I}, \nabla)$ where ∇ is the sub-connection associated to the algebraic basis $\mathfrak{a}$.

Theorem 4 (Theorem 4.4.1)**.** *Let $(\mathcal{G}, d, \mu)$ be a connected Lie group, $\mathfrak{a}$ an algebraic basis, d the associated subelliptic distance, and μ the left Haar measure. Let $a, A \in \mathrm{L}^\infty$ such that*

$$\operatorname{Re} \langle au, u \rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \operatorname{Re} \langle ASv, Sv \rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2'}},$$

for all $u \in \mathrm{L}^2(\mathcal{G})$ and $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then, $\mathcal{D}(\sqrt{aS^*AS}) = \mathrm{W}^{1,2}(\mathcal{G})'$ with $\|\sqrt{aS^*AS}u\| \simeq \|u\|_{\mathrm{W}^{1,2'}} = \|u\| + \|\nabla u\|$. That is, we solve (Kato$_{\mathcal{M}}$) with ∇ being a sub-connection, which is the inhomogeneous version of (Kato$_{\mathcal{G}}$).

Since Lie groups have trivial tangent bundles, Theorem 2 is actually a much stronger result than necessary to prove Theorem 4. A relatively easy adaptation of the proof in [46] can be used instead.

As we have previously mentioned, the real strength of our quadratic estimates in Theorem 2 is that they allow us to tackle problems on manifolds with non-trivial tangent bundles. We present the following highlight theorem solving the Kato problem on manifolds for functions. The notation Ric stands for the Ricci curvature of $\mathcal{M}$ and inj its injectivity radius.

Theorem 5 (Theorem 5.3.3)**.** *Let* $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$. *Let* $a, A \in \mathrm{L}^\infty$ *and suppose that the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \quad \textit{and} \quad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$. *Then the problem* ($\mathrm{Kato}_{\mathcal{M}}$) *is solved in the positive.*

This theorem is obtained as a consequence of a more general theorem on tensors, which is obtained through an even more general theorem on vector bundles satisfying generalised bounded geometry. We refrain from listing these more general results here because they require additional technical assumptions which are arduous to define. These results are presented in detail in Chapter 5.

Let us remark that Theorems 3, 4, 5 provide solutions to Kato's question (K1). However, these theorems also have an accompanying stability theorem of the type (K2). More explicitly, these theorems assert that the (non-linear) operator $(a, A) \mapsto \mathrm{L}_A = \sqrt{aS^*AS}$ (or in the case of nilpotent Lie groups the operator $(a, A) \mapsto \sqrt{-a \operatorname{div} A\nabla}$), satisfies Lipschitz bounds. Since we allow the coefficients a and A to be *complex*, we are able to obtain solutions to (K2) immediately from (K1). These results are listed in detail in §4.5 and §5.4.

The fundamental objects of the AKM setup are Dirac type operators, and indeed, the authors of [8] obtain a version of the Kato square root problem for differential forms. We are able to consider a similar, inhomogeneous problem in our case. We emphasise again that the following theorem is possible because we obtain quadratic estimates on non-trivial bundles. In this theorem, $\mathbf{\Omega}(\mathcal{M})$ denotes the space of differential forms on $\mathcal{M}$, R the curvature endomorphism (§6.3) (which can be seen to generalise Ricci curvature to forms), and $\mathrm{D}_A = \mathrm{d} + A^{-1}\mathrm{d}^* A$ for an invertible $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$.

Theorem 6 (Theorem 6.4.3)**.** *Let* $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold and let* $\beta \in \mathbb{C} \setminus \{0\}$. *Suppose there exist* $\eta, \kappa > 0$ *such that* $|\mathrm{Ric}| \leq \eta$ *and* $\mathrm{inj}(\mathcal{M}) \geq \kappa$. *Furthermore, suppose there exists* $\zeta \in \mathbb{R}$ *satisfying* $\mathrm{g}(\mathrm{R}\,\omega, \omega) \geq \zeta\,|\omega|^2$ *for* $\omega \in \mathbf{\Omega}_x(\mathcal{M})$ *and all* $x \in \mathcal{M}$, *and that* $A \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ *such that there exists*

$\kappa_1 > 0$ *satisfying*

$$\mathrm{Re}\,\langle Au, u\rangle \geq \kappa_1 \|u\|^2$$

for all $u \in \mathrm{L}^2(\boldsymbol{\Omega}(\mathcal{M}))$*. Then,* $\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\mathrm{d}^* A)$ *and* $\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \|\,\mathrm{D}_A\, u\| + \|u\|$.

The proofs of the Kato square root theorems we have presented here rely upon the analysis of canonical differential operators such as ∇ and $\mathrm{div} = -\nabla^*$ to obtain estimates such as cancellations with respect to these operators or the control of the Hessian via the Laplacian. To obtain such results, and more generally in PDE, it is useful to know that smooth compactly supported functions are dense in the domains of such operators. While a slight departure from geometric Kato square problems, but very much motivated by it, we study these *density problems*. The first theorem we present is the following very general result on vector bundles, and it is made possible by taking a first-order perspective, very much motivated by the AKM framework. We note that to say ∇ and a metric h on a vector bundle are compatible means that ∇ satisfies a product rule with respect to h.

Theorem 7 (Theorem 2.2.6)**.** *Let* $\mathcal{M}$ *be a smooth, complete Riemannian manifold with smooth metric* g *and let* $\mathcal{V}$ *be a smooth vector bundle over* $\mathcal{M}$ *with smooth metric* h *and connection* ∇*. Suppose that* h *and* ∇ *are compatible. Then,* $\mathrm{C}_c^\infty(\mathcal{V})$ *is dense in* $\mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$ *and* $\mathrm{C}_c^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is dense in* $\mathcal{D}(\mathrm{div})$*. Furthermore,* $\mathrm{C}_c^\infty(\mathcal{V})$ *is dense in* $\mathcal{D}(-\mathrm{div}\,\nabla)$ *(Bochner Laplacian) and* $\mathrm{C}_c^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is dense in* $\mathcal{D}(-\nabla\,\mathrm{div})$ *(co-Laplacian).*

In the book [33], Hebey states that the best known conditions for determining that compactly supported functions are dense in the second-order Sobolev space $\mathrm{W}^{2,2}(\mathcal{M})$ (associated to the Hessian) is under both Ricci curvature bounds and injectivity radius bounds. To the best of our knowledge, no improvement has been made to this result since the writing of this book in 2001. We make the following improvement by dispensing the injectivity radius bounds.

Theorem 8 (Theorem 2.4.1)**.** *Let* $\mathcal{M}$ *be a smooth, complete Riemannian manifold with metric* g *and Levi-Cevita connection* ∇*. If there exists* $\eta \in \mathbb{R}$ *such that* $\mathrm{Ric} \geq \eta\mathrm{g}$*, then* $\mathrm{C}_c^\infty(\mathcal{M})$ *is dense in* $\mathrm{W}^{2,2}(\mathcal{M})$*.*

We conclude this thesis by listing a set of open problems in Chapter 7, whose solutions we expect would advance our understanding of geometric Kato square root problems.

1

Preliminaries

In this chapter, we provide a background for the material we present in this thesis. We give a brief exposition of the ideas that will be of use to us from measure theory, metric spaces, Riemannian manifolds, Lie groups and operator theory. More importantly, the solutions to the Kato problems we present are all based on an important framework first presented by Axelsson (Rosén), Keith, and McIntosh. We present a refinement and adaptation of this framework suited to the class of problems we study.

Before we proceed, we remark that throughout this thesis, unless stated otherwise, we assume *Einstein summation convention*. That is, whenever a raised index appears multiplicatively against a lowered index, we sum over that index. Furthermore, for the convenience of the reader, we provide a list of notation towards the end of this thesis.

1.1 Measure theory

The notion of a measure is fundamental to the material that we present in this thesis. Although we only require a theory of measures for metric spaces in application, we outline this theory on topological spaces since the basic ideas we require are most clearly expressed in this generality. We refer the reader to the book [51] by Simon, [14] by de Barra, and [32] by Halmos as excellent guides to a more detailed exposition of measure theory. For a comprehensive account of the subject, the two volume book

[15] by Bogachev is an excellent resource.

Let $\mathcal{T}$ be a topological space. A *measure* μ on $\mathcal{T}$ is a set function $\mu : \mathscr{P}(\mathcal{T}) \to [0, \infty]$ satisfying

(i) $\mu(\varnothing) = 0$,

(ii) $\mu(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$ whenever $U \subset \cup_i U_i$. (Sub-additivity)

We take a moment here to remark that this notion of measure in the popular literature is called an *outer measure*. For us, the terms outer measure and measure are synonymous since this is a more geometric philosophy of measure theory. For instance, see [51].

We call a set $M \subset \mathcal{T}$ *measurable* if for all sets $S \subset \mathcal{T}$,

$$\mu(S) = \mu(M \cap S) + \mu(M \setminus S).$$

The geometric intuition of this criterion is that a measurable set should "cut" up an arbitrary set in a sensible (additive) way. Let us remark that there exist *non-measurable* sets, with perhaps the most simplest being the *Vitali construction*. See Theorem D in §16 of [32].

Recall that a σ-algebra is a nonempty collection of sets closed under countable intersections and countable unions. The set of all measurable sets denoted by $\mathscr{M}$ forms a σ-algebra. One can check that on $\mathscr{M}$, the following *additivity* property

$$\mu(\cup_i A_i) = \sum_i \mu(A_i)$$

is satisfied whenever $\{A_i\} \subset \mathscr{M}$ is a mutually disjoint collection.

Let us remark here that, traditionally, measure theory has been introduced as an additive set function ν on a σ-algebra $\mathscr{S}$. As a consequence, it is easy to check that $\mathscr{S} \subset \mathscr{M}$ in the notation that we have just introduced.

Another important σ-algebra is called the *Borel* σ-algebra denoted by $\mathscr{B}$. This is defined as the smallest σ-algebra that contains the topology of $\mathcal{T}$. If for a measure μ we have that $\mathscr{B} \subset \mathscr{M}$, then we say that μ is a *Borel measure*. If further for each $S \subset \mathcal{T}$, there exists $S' \in \mathscr{B}$ such that $S \subset S'$ and $\mu(S) = \mu(S')$, then the measure is said to be *Borel-regular*. If μ is Borel-regular and $\mathcal{T} = \cup_{i=1}^{\infty} U_i$ with each U_i open and $\mu(U_i) < \infty$, then we have that

$$\mu(A) = \inf\{\mu(U) : U \text{ open}, \ A \subset U\} \text{ and } \mu(B) = \sup\{\mu(C) : C \text{ closed}, \ C \subset A\}$$

for every $A \subset \mathcal{T}$ and for every measurable $B \subset \mathcal{T}$. See Theorem 1.3 in [51].

A measure is *Radon* if it is Borel-regular and finite on compact sets.

Specifying a measure with a σ-algebra becomes useful when considering product measures. Given measures μ_1 on $\mathcal{T}_1$ over a σ-algebra $\mathcal{S}_1$ and μ_2 on $\mathcal{T}_2$ over a σ-algebra $\mathcal{S}_2$, we can define the measure $\mu_1 \times \mu_2$ on the σ-algebra $\mathcal{S}_1 \otimes \mathcal{S}_2$. Of particular importance is when $\mathcal{S}_i = \mathscr{B}(\mathcal{T}_i)$, the Borel σ-algebra over $\mathcal{T}_i$. It is easy to see that in this case, each μ_i is a Borel measure, but it is not immediate that $\mu_1 \times \mu_2$ is Borel, and in general, it is not. However, Lemma 6.4.2 in [15] shows that if $\mathcal{T}_1$, $\mathcal{T}_2$ are Hausdorff, and $\mathcal{T}_2$ has a countable basis, then $\mathscr{B}(\mathcal{T}_1 \times \mathcal{T}_2) = \mathscr{B}(\mathcal{T}_1) \otimes \mathscr{B}(\mathcal{T}_2)$, which particularly implies that $\mu_1 \times \mu_2$ is a Borel measure. Theorem 7.14.1 in [15] gives conditions under which $\mu_1 \times \mu_2$ can be extended to a Borel measure, but such an extension may not be unique.

Let $f : \mathcal{T} \to \mathbb{R}$ be a function. We say that f is a *μ-measurable function* (or simply measurable function when the measure is clear from context) if $f^{-1}[\alpha, \infty) \in \mathscr{M}$ for all $\alpha < \infty$. A function $f : \mathcal{T} \to \mathbb{R}$ is called *simple* if $f(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$ where A_k are measurable sets. Then, the *integral* with respect to the measure μ is defined as

$$\int_{\mathcal{T}} f(x)\ d\mu(x) = \sum_{i=1}^{n} a_k \mu(A_k).$$

For each non-negative measurable function $f : \mathcal{T} \to \mathbb{R}$, there exists a sequence of simple functions s_n such that $s_n \leq f$ and $s_n \to f$ pointwise. Thus, we extend the definition of integral by

$$\int_{\mathcal{T}} f\ d\mu = \sup_n \int_{\mathcal{T}} s_n\ d\mu.$$

For an arbitrary $f : \mathcal{T} \to \mathbb{R}$, we split $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then, we set

$$\int_{\mathcal{T}} f\ d\mu = \int_{\mathcal{T}} f^+\ d\mu - \int_{\mathcal{T}} f^-\ d\mu$$

provided that $\int_{\mathcal{T}} f^+\ d\mu$ or $\int_{\mathcal{T}} f^-\ d\mu$ is finite. If both are finite, then we say that f is *summable.* In the wider literature, this is sometimes called *integrable.*

When μ is Borel and $f : \mathcal{T} \to \mathcal{T}'$, where $\mathcal{T}'$ is another topological space, we call f μ-measurable if $f^{-1}(U)$ is a measurable set for each open set $U \subset \mathcal{T}'$. If $f^{-1}(U)$ is a Borel set, then f is said to be *Borel*.

1.2 Metric spaces

From this point onwards, our discussion will almost always be limited to the more specific situation of *metric spaces*. Throughout this thesis, we let $(\mathcal{X}, d)$ denote a metric space, where $\mathcal{X}$ is the underlying set and $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is a metric. Recall that the metric d is a function satisfying:

(i) $d(x, y) \geq 0$ for all $x, y \in \mathcal{X}$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$, (Symmetry)

(iii) $d(x, y) = 0$ if and only if $x = y$, and

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

The topology of a metric space is then the topology generated by by the sub-basis $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$ for $x \in \mathcal{X}$ and $r > 0$. The set $B(x, r)$ is called an open ball centred at x and of radius r.

An important property of a metric space is the notion of *completeness*. Recall that a sequence $(x_n) \subset \mathcal{X}$ is called *Cauchy* if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. If for each such sequence, there exists an $x \in \mathcal{X}$ such that $d(x_n, x) \to 0$, then we say that $\mathcal{X}$ is complete.

On a metric space, we can define an important type of function called a *Lipschitz* function. Let $\mathcal{X}_1, \mathcal{X}_2$ be two metric spaces with metrics d_1 and d_2 respectively. Then, a function $f : \mathcal{X}_1 \to \mathcal{X}_2$ is called Lipschitz if there exists a $C > 0$ such that for all $x, y \in \mathcal{X}_1$,

$$d_2(f(x), f(y)) \leq C d_1(x, y).$$

The smallest constant C for which this inequality is satisfied is called the *Lipschitz constant* of f, which we denote by by $\mathbf{Lip}\, f$. The first observation is the following simple lemma, which shows that we can use Lipschitz functions to separate sets of positive distance with control on the Lipschitz constant. Certainly, this result was well known to the authors of [8], and in fact, they cite it in the proof of their Proposition 5.2. Note that in the following lemma, the distance between two sets $E, F \subset \mathcal{X}$ is defined as

$$d(E,F) = \inf_{x\in E, y\in F} d(x,y).$$

Lemma 1.2.1 (Lipschitz separation lemma)**.** *Suppose $E, F \subset \mathcal{X}$ satisfy $d(E,F) > 0$. Then, there exists a Lipschitz function $\eta : \mathcal{X} \to [0,1]$, and a set $\tilde{E} \supset E$ with $d(\tilde{E}, F) > 0$ such that*

$$\eta|_E = 1, \quad \eta|_{\mathcal{X}\setminus\tilde{E}} = 0 \quad \textit{and} \quad \mathbf{Lip}\,\eta \leq 4/d(E,F).$$

Proof. Define $\tilde{E} = \{x \in \mathcal{X} : d(x,E) < 1/4d(E,F)\}$. By construction, $E \subset \tilde{E}$ and from the triangle inequality for d and taking infima,

$$d(\tilde{E},F) + \sup_{x\in\tilde{E}} d(x,E) \geq d(E,F),$$

and since $\sup_{x\in\tilde{E}} d(x,E) \leq 1/4d(E,F)$, it follows that $d(\tilde{E},F) \geq 3/4d(E,F) > 0$.

Now, define:

$$\eta(x) = \begin{cases} 1 - \frac{4d(x,E)}{d(E,F)} & x \in \tilde{E} \\ 0 & x \notin \tilde{E} \end{cases}.$$

We consider the three possible cases.

(i) First, suppose that $x, y \notin \tilde{E}$. Then,

$$|\eta(x) - \eta(y)| = 0 \leq \frac{4d(x,y)}{d(E,F)}.$$

(ii) Now, suppose that $x, y \in \tilde{E}$. By the triangle inequality, we have $d(x,z) \leq d(x,y) + d(y,z)$ and by taking an infima over $z \in E$ and invoking the symmetry of distance, $|d(x,E) - d(y,E)| \leq d(x,y)$. Therefore,

$$\begin{aligned}|\eta(x) - \eta(y)| &= \left|\frac{1 - 4d(x,E)}{d(E,F)} - 1 + \frac{4d(y,E)}{d(E,F)}\right| \\ &= \frac{4}{d(E,F)}\,|d(x,E) - d(y,E)| \leq \frac{4}{d(E,F)} d(x,y).\end{aligned}$$

(iii) Lastly, suppose that $x \in \tilde{E}$ and $y \notin \tilde{E}$. Then $\eta(y) = 0$ and since $d(x,E) \leq$

$\frac{1}{4}d(E,F)$,

$$|\eta(x)-\eta(y)| = |\eta(x)| = \eta(x) = 1 - \frac{4d(x,E)}{d(E,F)} = \frac{d(E,F)-4d(x,E)}{d(E,F)}.$$

But we also have the triangle inequality $d(E,x)+d(x,y) \geq d(y,E)$ and by the choice of y we have that $d(y,E) \geq 1/4d(E,F)$. Therefore, $d(x,y) \geq d(y,E)-d(x,E) \geq \frac{1}{4}d(E,F)-d(x,E)$ which implies that

$$\frac{4d(x,y)}{d(E,F)} \geq \frac{d(E,F)-d(x,E)}{d(E,F)} = |\eta(x)-\eta(y)|.$$

$\square$

As a conclusion to this section, we introduce a tool which provides finer control of Lipschitz maps than the Lipschitz constant. We follow the notation of Cheeger in [16] in the following definition.

Definition 1.2.2 (Lipschitz pointwise constant)**.** *Let $f : \mathcal{X} \to \mathcal{N}$ denote a Lipschitz map into the normed space $\mathcal{N}$ with norm $|\cdot|$. Then, define*

$$\operatorname{Lip} f(x) = \limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)}$$

when x is not an isolated point and $\operatorname{Lip} f(x) = 0$ *otherwise.*

If in the previous Lemma 1.2.1 the set E was chosen to be open, then we can see that $\operatorname{Lip}\eta(x) = 0$ on E. Thus, the Lipschitz pointwise constant has support properties similar to that of *differentiation* and the class of quantities to which $\operatorname{Lip} f$ belongs are called *upper gradients*. We will exploit this feature in the proofs of Chapter 3.

1.3 Measure metric spaces

We say that $\mathcal{X}$ is a *measure metric space* if in addition to it being a metric space, it is equipped with a measure μ. The measures we deal with are at the very least Borel-regular and thus, we shall make this assumption from this point onwards. We sometimes denote the measure metric space by the triple $(\mathcal{X}, d, \mu)$.

An arbitrary measure can behave wildly, so we will impose certain growth assumptions on the measure. The following is one of two growth conditions with which we shall be concerned with in this thesis.

Definition 1.3.1 (Polynomial growth)**.** *Suppose that there exist $C > 0$ and $p > 0$ such that*

$$0 < \mu(B(x, tr)) \le Ct^p \mu(B(x, r)) < \infty \tag{Pol}$$

for every $r > 0$ and $t \ge 1$. Then, we say that μ exhibits polynomial growth *or that it is a* doubling measure*.*

An alternative definition of a doubling measure is to ask that there exists a $C > 0$ so that

$$0 < \mu(B(x, 2r)) \le C\mu(B(x, r)) < \infty.$$

This condition is, in fact, equivalent to condition that the measure exhibits polynomial growth. Also, note that such a measure is Radon.

There are many examples of spaces of polynomial growth. For instance, $\mathbb{R}^n$ with the usual *Lebesgue measure* $\mathscr{L}$ is a space of polynomial growth. A more exotic example is a *nilpotent* Lie group with the Haar measure (see [24]). Also, the canonical volume measure of a complete Riemannian manifold whose Ricci curvature is non-negative is also such a space by the Bishop-Gromov comparison theorem. See, for instance, [17].

Measure metric spaces that exhibit polynomial growth admit the following important *dyadic decomposition* of the underlying space.

Theorem 1.3.2 (The existence of Christ cubes)**.** *There exists a countable collection of open subsets $\left\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\right\}$ with each $z_\alpha^k \in Q_\alpha^k$ (called the centre of Q_α^k) where I_k are index sets (possibly finite), and constants $\delta \in (0,1)$, $a_0 > 0$, $\eta > 0$ and $C_1, C_2 < \infty$ satisfying:*

(i) *For all $k \in \mathbb{Z}$, $\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$,*

(ii) *If $l \ge k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \varnothing$,*

(iii) *For each (k, α) and each $l < k$ there exists a unique β such that $Q_\alpha^k \subset Q_\beta^l$,*

(iv) $\operatorname{diam} Q_\alpha^k \le C_1 \delta^k$*,*

(v) *$B(z_\alpha^k, a_0\delta^k) \subset Q_\alpha^k$,*

(vi) *For all* k, α *and for all* $t > 0$, $\mu \left\{ x \in Q_\alpha^k : d(x, \mathcal{X} \setminus Q_\alpha^k) \leq t\delta^k \right\} \leq C_2 t^\eta \mu(Q_\alpha^k)$.

This important construction was proved by Christ as Theorem 11 in [20]. The sets Q_α^k are sometimes called *Christ cubes*. It is a primary tool that we use to perform harmonic analysis on such spaces. We remark that this theorem holds in greater generality, when the underlying space is equipped with only a *quasi-metric*.

Next, we introduce the following weaker growth condition.

Definition 1.3.3 (Exponential volume growth)**.** *Suppose there exists* $c \geq 1$, $\kappa, \lambda \geq 0$ *such that*

$$0 < \mu(B(x, tr)) \leq ct^\kappa \mathrm{e}^{\lambda tr} \mu(B(x, r)) < \infty \tag{Exp}$$

for all $t \geq 1$, $r > 0$ *and* $x \in \mathcal{X}$. *Then we say that* μ *is* exponentially locally doubling.

Harmonic analytic consequences about such measures are detailed by Morris in [46], and this nomenclature is attributed to him. Note again that such a measure is automatically Radon.

Every measure with polynomial growth is also an exponential one. Moreover, so is the induced volume measure of a Riemannian manifold whose Ricci curvature is uniformly bounded below by a real number. This is again a consequence of the Bishop-Gromov comparison theorem.

As in the polynomial case, these manifolds admit a certain dyadic decomposition as outlined in the following theorem. Note, however, that this decomposition is only valid "below" a certain fixed scale.

Theorem 1.3.4 (Existence of a truncated dyadic structure)**.** *There exists a countable collection of open subsets* $\left\{ Q_\alpha^k \subset \mathcal{M} : \alpha \in I_k,\ k \in \mathbb{Z}_+ \right\}$, $z_\alpha^k \in Q_\alpha^k$ *(called the* centre *of* Q_α^k*), index sets* I_k *(possibly finite), and constants* $\delta \in (0, 1)$, $a_0 > 0$, $\eta > 0$ *and* $C_1, C_2 < \infty$ *satisfying:*

(i) *for all* $k \in \mathbb{Z}_+$, $\mu(\mathcal{M} \setminus \cup_\alpha Q_\alpha^k) = 0$,

(ii) *if* $l \geq k$, *then either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$,

(iii) *for each* (k, α) *and each* $l < k$ *there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$,

(iv) $\operatorname{diam} Q_\alpha^k \leq C_1 \delta^k$,

(v) $B(z_\alpha^k, a_0\delta^k) \subset Q_\alpha^k$,

(vi) *for all* k, α *and for all* $t > 0$, $\mu\left\{x \in Q_\alpha^k : d(x, \mathcal{M} \setminus Q_\alpha^k) \leq t\delta^k\right\} \leq C_2 t^\eta \mu(Q_\alpha^k)$.

This theorem was proved by Morris and can be found as Proposition 4.2 in [46]. Note that in comparison to Theorem 1.3.2, the index here is over $\mathbb{Z}_+$ rather than $\mathbb{Z}$. This is why this structure is said to be *truncated*.

It is easy to see that every measure with polynomial growth exhibits exponentially locally doubling growth. In what is to follow, we will see that under the polynomial growth condition, we are able to obtain certain *homogeneous estimates* and under the exponential growth condition, similar estimates that are *inhomogeneous*. These homogeneous estimates are stronger in the sense that they imply the inhomogeneous kind. This is partly due to the fact that for a space of exponential growth, the best we can do is construct a truncated dyadic decomposition as opposed to the full decomposition as in the polynomial setting.

1.4 Vector bundles on metric spaces

On Euclidean spaces, functions and systems have been the focus of study for the kind of problems we consider here. Our situation requires the generality of *vector bundles*, as they are a natural extension to the language of functions and systems that also encode geometry.

Let us begin by recalling the definition of a vector bundle of *rank* $N \in \mathbb{N}$. Let $\mathcal{V}$ be a topological space and $\pi_{\mathcal{V}} : \mathcal{V} \to \mathcal{X}$ a continuous surjection such that $\pi_{\mathcal{V}}^{-1}\{x\}$ is an N dimensional vector space. Furthermore, suppose that for each $x \in \mathcal{X}$, there exists an open neighbourhood U_x of x and a homeomorphism $\psi_x : U_x \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(U_x)$ satisfying:

(i) $\pi_{\mathcal{V}}\psi_x(y, v) = y$ for all $y \in U_x$ and $v \in \pi_{\mathcal{V}}^{-1}\{y\}$, and

(ii) $v \mapsto \psi_x(y, v)$ is a vector space isomorphism from $\mathbb{C}^N$ to $\pi_{\mathcal{V}}^{-1}\{y\}$, for all $y \in U_x$.

The space $\mathcal{V}$ is called a *vector bundle* of rank N, the set $\mathcal{X}$ is called the *base space* and $\pi_{\mathcal{V}}$ the *bundle projection*. The maps ψ_x are called *local trivialisations* of the bundle. We write $\mathcal{V}_x = \pi_{\mathcal{V}}^{-1}\{x\}$ for $x \in \mathcal{X}$ which is called the *fibre* over x. Note that we can write $\mathcal{V} = \sqcup_{x\in\mathcal{X}}\mathcal{V}_x$.

In our definition, the fibres are complex vector spaces. Real bundles are defined with $\mathbb{R}^N$ in place of $\mathbb{C}^N$ in their definition. When we encounter real vector bundles, we implicitly identify them under their *complexification*. That is, if $\mathcal{V}_{\mathbb{R}}$ is a real vector bundle, then we consider $\mathcal{V}_{\mathbb{R}}$ to be identified with $\mathcal{V} = \mathcal{V}_{\mathbb{R}} \oplus \imath\mathcal{V}_{\mathbb{R}}$.

We also note that we can consider the notion of a *normed bundle*, by which we mean that each $\pi_{\mathcal{V}}^{-1}\{x\}$ is a normed space $\mathscr{N}$, possibly infinite dimensional. Such an object is defined by requiring the trivialisations to be normed space isomorphisms in condition (ii).

The vector bundles we consider in this thesis usually have more structure than just local trivialisability. We say that a continuous map $\mathrm{h} : \mathcal{X} \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is a *metric* if $\mathrm{h}_x = \mathrm{h}(x) : \mathcal{V}_x \times \mathcal{V}_x \to \mathbb{C}$ is an inner product. In particular, we shall always assume that h_x is non-degenerate. We remark that the choice of calling h a metric is unfortunate, since we call d on a metric space also a metric. However, the distinction will be clear from context. Normed bundles typically come with a norm $|\cdot|_x : \mathcal{V}_x \times \mathcal{V}_x \to \mathbb{C}$ on each fibre $x \in \mathcal{X}$. We often use the notation $|\cdot|$ omitting the dependence on x. As with the case of a metric, we typically require the norm to be continuous.

We say that $\xi : \mathcal{X} \to \mathcal{V}$ is a *section* of $\mathcal{V}$ if $\xi(x) \in \mathcal{V}_x$. Note that this is a natural generalisation of a *vector field*. When $\mathcal{X}$ is a measure metric space with measure μ, we write $\mathbf{\Gamma}(\mathcal{V})$ to denote the μ-measurable sections of $\mathcal{V}$. This is departing from the traditional definition of $\mathbf{\Gamma}(\mathcal{V})$ which assumes additional smoothness properties.

Since we allow sections to be rough, we define the *Lebesgue spaces* as follows. First, we let $\mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ denote the space of all $\xi \in \mathbf{\Gamma}(\mathcal{V})$ such that

$$\int_K |\xi| \ d\mu < \infty$$

for all compact sets $K \subset \mathcal{X}$. Then, for $1 \leq p < \infty$, we define $\mathrm{L}^p(\mathcal{V})$ to be $\xi \in \mathbf{\Gamma}(\mathcal{V})$ such that

$$\int_{\mathcal{X}} |\xi|^p \ d\mu < \infty.$$

We equip this space with the norm

$$\|\xi\|_p = \left(\int_{\mathcal{X}} |\xi|^p \ d\mu \right)^p.$$

We will be primarily concerned with the special case of $p = 2$ and for simplicity, we often write $\|\cdot\| = \|\cdot\|_2$. When $p = \infty$, we define $\mathrm{L}^\infty(\mathcal{V})$ to be the $\xi \in \mathbf{\Gamma}(\mathcal{V})$ such that there exists a $C > 0$ so that $|\xi| \leq C$ for μ-almost all $x \in \mathcal{X}$. Then, $\|\xi\|_\infty$ is defined as the smallest such C. We write $\mathrm{L}^p(\mathcal{X})$ in place of $\mathrm{L}^p(\mathcal{X} \times \mathbb{C})$ to keep accord with tradition. When the vector bundle has a metric h, the norm induced by this metric is given by $|\cdot|_x = \sqrt{\mathrm{h}_x(\cdot\,,\cdot\,)}$. In this case, $\mathrm{L}^2(\mathcal{V})$ is a Hilbert space with inner product

$$\langle u, v \rangle = \int_{\mathcal{M}} \mathrm{h}(u, v) \ d\mu.$$

Let us give some examples of vector bundles. First consider $\mathcal{V} = \mathcal{X} \times \mathbb{C}$. This is readily checked to be a vector bundle of rank 1, and the sections of this bundle are simply measurable functions over $\mathcal{X}$. Similarly, let $\mathcal{V} = \mathcal{X} \times \mathbb{C}^N$. Then, this is a vector bundle of rank N and corresponds to the situation of systems. We note that both these examples admit *global trivialisations*. A vector bundle for which there exists a trivialisation $U_x = \mathcal{X}$ is called a *trivial* bundle. Amongst the rank N trivial bundles, it is easy to see that the only such bundle (up to isomorphism) is $\mathcal{X} \times \mathbb{C}^N$. We shall see some natural examples of vector bundles that are non-trivial in the next section.

The metric h is responsible for the *geometry* of $\mathcal{V}$. By choosing a partition of unity subordinate to trivialisations and pasting together the usual inner product in $\mathbb{C}^N$, we can see that vector bundles admit many different metrics or geometries. We define a special class of vector bundles that will be important to us. In the following definition, I denotes the standard Euclidean inner product.

Definition 1.4.1 (Generalised bounded geometry). *Suppose there exists $\rho > 0$, $C \geq 1$, such that for each $x \in \mathcal{X}$, there exists a trivialisation $\psi_x : B(x, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x, \rho))$ satisfying*

$$C^{-1}\mathrm{I} \leq \mathrm{h} \leq C\mathrm{I}$$

in the basis $\{e^i = \psi_x(y, \tilde{e}^i)\}$, where $\{\tilde{e}^i\}$ is the standard orthonormal basis for $\mathbb{C}^N$. Then, we say that $\mathcal{V}$ has generalised bounded geometry *or* GBG. *We call ρ the* GBG radius.

The comparison of h and I here are in the sense of bilinear forms. The generalised bounded geometry criterion captures vector bundles that are *uniformly* locally Euclidean. We choose to call these generalised since their definition is motivated by the existence of coordinates systems that allow us to control the geometry of manifolds uniformly under curvature bounds. We shall discuss the existence of such coordinates in the following section.

1.5 Manifolds and smooth vector bundles

Although we work at the generality of metric spaces to obtain quadratic estimates, our applications will primarily be on differentiable manifolds. We devote this section to recall some facts we need about such objects.

Throughout this thesis, when dealing with manifolds, we will be exclusively concerned with smooth manifolds. For us, an n-dimensional manifold $\mathcal{M}$ is a *second countable Hausdorff space* such that at each point $x \in \mathcal{M}$, there exists an open set U_x and a continuous injection $\psi_x : U_x \to \mathbb{R}^n$ homeomorphic to its image. If $\varphi_y : U_y \to \mathbb{R}^n$ is some other such map with $V = U_y \cap U_x \neq \varnothing$, then we require $\varphi_x \circ \varphi_y^{-1} : \varphi_y(V) \to \varphi_x(V)$ to be a C^∞ diffeomorphism. The collection of charts $\psi_x : U_x \to \mathbb{R}^n$ (keeping track of both the map ψ_x and open set U_x) defines a *differentiable structure* on $\mathcal{M}$.

When we consider vector bundles over manifolds, we ask them to have greater regularity than their barren definition over metric spaces. More specifically, the existence of a differentiable structure on the manifold permits us to demand that the trivialisations be smooth. We call such a bundle a *smooth bundle*. Furthermore, we also ask the bundle metric to be smooth.

At each $x \in \mathcal{M}$, a smooth manifold admits a tangent space which we denote by $\mathrm{T}_x\mathcal{M}$. This is an equivalence class of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $\gamma(0) = x$ with the relation that two curves γ and σ are equivalent if

$$\frac{d}{dt}|_{t=0}(\psi_x \circ \gamma)(t) = \frac{d}{dt}|_{t=0}(\psi_x \circ \sigma)(t).$$

The tangent space at x is denoted by $\mathrm{T}_x\mathcal{M}$, and the cotangent space at x, the dual space of $\mathrm{T}_x\mathcal{M}$, is denoted by $\mathrm{T}^*_x\mathcal{M}$. The tangent bundle and cotangent bundle are then constructed from these fibres and are denoted $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ respectively.

These bundles are in general non-trivial. Sections of $\mathrm{T}\mathcal{M}$ are often called *vector fields* and sections of $\mathrm{T}^*\mathcal{M}$ are called *covector fields*.

The bundle of $(0,0)$-tensors is denoted by $\mathcal{T}^{(0,0)}\mathcal{M} = \mathcal{M} \times \mathbb{R}$ and the (p,q)-tensors (when either $p \geq 1$ or $q \geq 1$) is defined as

$$\mathcal{T}^{(p,q)}\mathcal{M} = \left(\oplus_{j=1}^{p} \mathrm{T}^*\mathcal{M}\right) \bigoplus \left(\oplus_{l=1}^{q} \mathrm{T}\mathcal{M}\right).$$

These are also sometimes referred to as the tensors of *covarient* rank p and *contravarient* rank q. The (p,q)-tensors are a vector bundle of rank n^{p+q}. The p-forms are denoted by $\mathbf{\Omega}^p(\mathcal{M})$ and it is a sub-bundle of $\mathcal{T}^{(p,0)}(\mathcal{M})$. Note that we have $\mathbf{\Omega}^0(\mathcal{M}) = \mathcal{M} \times \mathbb{R}$ and $\mathbf{\Omega}^1(\mathcal{M}) = \mathrm{T}^*\mathcal{M}$. The *exterior bundle* $\mathbf{\Omega}(\mathcal{M}) = \oplus_{p=0}^{n} \mathbf{\Omega}^p\mathcal{M}$ is then a rank 2^n vector bundle, and it is an algebra with respect to the *exterior product* $\wedge$. For a p-form η and q-form ω, $\eta \wedge \omega$ is a $(p+q)$-form, and with respect to this operation, the exterior bundle is a *graded* algebra, often called the *exterior algebra*.

Since we assume that the vector bundle trivialisations are smooth, we are able to define C^k spaces and differential operators on such bundles. By $\mathrm{C}^k(\mathcal{V})$ we mean $\xi : \mathcal{M} \to \mathcal{V}$, $\xi(x) \in \mathcal{V}_x$ such that at each point $x \in \mathcal{M}$, there is a trivialisation $\psi_x : U_x \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(U_x)$ which makes $\psi_x^{-1} \circ \xi : U_x \to U_x \times \mathbb{C}^N$ a C^k map. The spaces $\mathrm{C}^k_{\mathrm{c}}(\mathcal{V})$ are defined to be the set of $\xi \in \mathrm{C}^k(\mathcal{V})$ such that the support of ξ, $\mathrm{spt}\,\xi$, is compact. As we have done previously for the Lebesgue spaces, we write $\mathrm{C}^k(\mathcal{M})$ and $\mathrm{C}^k_{\mathrm{c}}(\mathcal{M})$ in place of $\mathrm{C}^k(\mathcal{M} \times \mathbb{C})$ and $\mathrm{C}^k_{\mathrm{c}}(\mathcal{M} \times \mathbb{C})$.

In each coordinate system, we have the notion of a *coordinate* basis for $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ (and similarly for $\mathcal{T}^{(p,q)}(\mathcal{M})$ and $\mathbf{\Omega}(\mathcal{M})$). For a coordinate system $\{x^i\}$ (giving a name to the coordinates in ψ_x near x), we write the coordinate directions as ∂_{x^i} (or ∂_i when the context is clear). Then $\{\partial_i\}$ forms a local basis for $\mathrm{T}\mathcal{M}$ near x. These are, in fact, differential operators with $\partial_i f$ being the partial derivative of $f \in \mathrm{C}^\infty(\mathcal{M})$ along the coordinate direction e_i in $\mathbb{R}^n$ as seen through ψ_x. The corresponding co-directions for $\mathrm{T}^*\mathcal{M}$ near x are given by $\{dx^i\}$. For the tensors, the basis is either given as a tensor product of these objects or in the case of forms, a wedge product in dx^i.

We also have global or coordinate independent differential operators on these bundles. On the exterior algebra, the fundamental object is the *exterior derivative* $\mathrm{d} : \mathrm{C}^\infty(\mathbf{\Omega}^p(\mathcal{M})) \to \mathrm{C}^\infty(\mathbf{\Omega}^{p+1}(\mathcal{M}))$. Furthermore, we have the notion of the *Lie derivative* of two vector fields $X,\ Y \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$ denoted by $[X,Y]$. On a smooth vector bundle, the fundamental differential operator is called a *connection*. This is an $\mathbb{R}$-linear map $\nabla : \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ satisfying the *Leibniz rule*

$$\nabla(f\eta) = f\nabla\eta + \mathrm{d}\, f \otimes \eta$$

whenever $f \in \mathrm{C}^\infty(\mathcal{M})$ and $\eta \in \mathrm{C}^\infty(\mathcal{V})$. In general, connections are not canonical objects, but rather, they are prescribed. Writing $\{e^i\}$ as the basis induced by a trivialisation for $\mathcal{V}$ near x and $\{\partial_j\}$ for $\mathrm{T}\mathcal{M}$ near x, the *connection coefficients* Γ^i_{jk} are defined as

$$\nabla_{\partial_j} e^i = \Gamma^i_{jk} e^k.$$

We emphasise that the connection coefficients are entirely dependent on the choice of trivialisation and do not behave tensorially.

For a smooth metric h, we have the notion that the metric and connection are *compatible*. This is the requirement that

$$X\mathrm{h}(Y, Z) = \mathrm{h}(\nabla_X Y, Z) + \mathrm{h}(Y, \nabla_X Z)$$

holds for $X \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$ and $Y,\ Z \in \mathrm{C}^\infty(\mathcal{V})$. It expresses that ∇ respects the product rule with respect to h.

If in addition we are given a connection ∇ on $\mathrm{T}\mathcal{M}$, we define the *Hessian* of $X \in \mathrm{C}^\infty(\mathcal{V})$ by

$$\nabla^2 X(Y, Z) = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

for $Y,\ Z \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$. The *curvature* is a property of the connection. It is the lack of symmetry of the Hessian given by the expression

$$\begin{aligned}\mathcal{R}(X, Y)Z &= \nabla^2(Z)(X, Y) - \nabla^2(Z)(Y, X) \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z.\end{aligned}$$

Suppose now that $\mathcal{M}$ is equipped with a metric g, by which we mean that g is a metric on the tangent bundle $\mathrm{T}\mathcal{M}$. Then, for $T \in \mathrm{C}^\infty(\mathcal{T}^{(p,q)}\mathcal{M} \otimes \mathcal{V})$, let

$$\mathrm{tr}_{ij}\, T$$

denote the *trace* of T in the i and j tensor index (for $i, j < p + q$) with respect to g.

Then, the *connection Laplacian* on the vector bundle is given by

$$\Delta X = \operatorname{tr} \nabla^2 X$$

where $X \in \mathrm{C}^\infty(\mathcal{V})$ and where by tr, we mean tr_{12}.

Let us fix a metric g on the tangent bundle, and assume that at the very least it is continuous. The metric actually turns the manifold into a metric space. To see this, take a smooth curve $\gamma : [0,1] \to \mathcal{M}$ and define its length as

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}(t)| \ dt,$$

where $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$. A connected manifold is then a metric space with respect to

$$d_{\mathrm{g}}(x,y) = \inf \left\{ \ell(\gamma) : \gamma(0) = x,\ \gamma(1) = y \right\}.$$

We say that $\mathcal{M}$ is *complete* if $(\mathcal{M}, d_{\mathrm{g}})$ is a complete metric space. In fact, the metric also induces a *volume measure* μ. Locally, writing $\mathrm{g} = \mathrm{g}_{ij}\ dx^i \otimes dx^j$ near x, the volume measure μ takes the expression

$$d\mu = \sqrt{\det(\mathrm{g}_{ij})}\ d\mathscr{L}.$$

This expression is easily checked to be coordinate independent.

Since we now view the manifold as a measure metric space which also has a differentiable structure, we are able to formulate *Sobolev spaces* on it. In fact, we do this at the generality of a vector bundle. Let us fix a metric and connection on a vector bundle and let S^p_k be the set of $u \in \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{V})$ such that $\nabla^i u \in \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{T}^{(i,0)}\mathcal{M} \otimes \mathcal{V})$ for $1 \leq i \leq k$. Then, the Sobolev norm is defined as

$$\|u\|_{\mathrm{W}^{k,p}} = \|u\|_p + \sum_{i=1}^k \|\nabla^i u\|_p$$

for $u \in S^p_k$. We define the Sobolev space $\mathrm{W}^{k,p}(\mathcal{V})$ as the completion of S^p_k under this norm. The Sobolev space $\mathrm{W}^{k,p}_0(\mathcal{V})$ is defined as the completion of $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V})$ under the same norm. As before, when the bundle $\mathcal{V}$ is $\mathcal{M} \times \mathbb{C}$, we write $\mathrm{W}^{k,p}(\mathcal{M})$ and $\mathrm{W}^{k,p}_0(\mathcal{M})$ in place of $\mathrm{W}^{k,p}(\mathcal{M} \times \mathbb{C})$ and $\mathrm{W}^{k,p}_0(\mathcal{M} \times \mathbb{C})$ respectively. Although we define these spaces for arbitrary k and p, we mainly work with $p = 2$ and $k = 1$ or 2.

Equipping a manifold with a metric has an important and far reaching consequence. The metric provides the manifold with geometry as it determines how distances are measured. It also provides us with a unique way of differentiating vector fields as made precise in the following theorem (which can be found as Theorem 3.3.1 in [38]).

Theorem 1.5.1 (The fundamental theorem of Riemannian geometry)**.** *Let* g *be a smooth metric. Then, there exists a unique connection* $\nabla : \mathrm{C}^\infty(\mathrm{T}\mathcal{M}) \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{M})$ *that is metric compatible and torsion-free (by which we mean* $\nabla_X Y - \nabla_Y X = [X, Y]$ *for* $X, Y \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$*).*

The connection guaranteed in this theorem is called the *Levi-Cevita* connection. The connection coefficients in a coordinate chart for the Levi-Cevita connection are called the *Christoffel symbols*. The metric on $\mathrm{T}\mathcal{M}$ induces a canonical metric on $\mathrm{T}^*\mathcal{M}$ (sometimes called the *co-metric*), and we write $\mathrm{g} = \mathrm{g}_{ij}\ dx^i \otimes dx^j$ for the metric and $\mathrm{g} = \mathrm{g}^{kl}\ \partial_k \otimes \partial_l$ for the co-metric in coordinate form. For a Levi-Cevita connection, the Christoffel symbols can be written purely in terms of the metric, co-metric and their derivatives:

$$\Gamma^k_{ij} = \frac{1}{2}\mathrm{g}^{km}(\partial_j \mathrm{g}_{mi} + \partial_i \mathrm{g}_{mj} - \partial_m \mathrm{g}_{ij}).$$

Furthermore, the connection ∇ on $\mathrm{T}\mathcal{M}$ induces a canonical connection on $\mathrm{T}^*\mathcal{M}$ by requiring $\nabla(X(Y)) = \nabla X(Y) + X(\nabla(Y))$, for $X \in \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M})$ and $Y \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$. In coordinates, we have that

$$\nabla_{\partial_j} dx^k = -\Gamma^k_{jl} dx^l$$

where Γ^k_{jl} are the Christoffel symbols. These objects are extended canonically to $\mathcal{T}^{(p,q)}\mathcal{M}$ and $\boldsymbol{\Omega}(\mathcal{M})$. By convention, we take $\nabla = \mathrm{d}$ on $\boldsymbol{\Omega}^0(\mathcal{M}) = \mathcal{T}^{(0,0)}\mathcal{M} = \mathcal{M} \times \mathbb{R}$.

With a Levi-Cevita connection, we can define the following natural curvatures. The most fundamental is the *Riemannian* curvature given by

$$\mathrm{Rm}(X, Y, Z, W) = \mathrm{g}(\mathcal{R}(X, Y)Z, W)$$

where $X,\ Y,\ Z,\ W \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$. It is easy to see that $\mathrm{Rm} \in \mathrm{C}^\infty(\mathcal{T}^{(4,0)}\mathcal{M})$. Also, in coordinates, Rm_{ijkl} can be written purely in terms of Γ^k_{ij} and $\partial_l \Gamma^k_{ij}$, and therefore, the curvature of ∇ is determined by the geometry. The *Ricci* curvature is then defined as

$$\mathrm{Ric} = \mathrm{tr}_{13}\,\mathrm{Rm},$$

and the *scalar* curvature is given by

$$\mathrm{R_s} = \operatorname{tr}\mathrm{Ric}\,.$$

For $X,\ Y \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$ linearly independent vector fields, the *sectional* curvature spanned by X and Y are given by

$$\mathrm{K}(X,Y) = \frac{\mathrm{Rm}(X,Y,Y,X)}{|X|^2\,|Y|^2 - \mathrm{g}(X,Y)}.$$

We say that a smooth curve $\gamma : (-\varepsilon,\varepsilon) \to \mathcal{M}$ is *parallel* if

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0.$$

Locally, parallel curves are *geodesics*, which are curves realising shortest distances. That is, given a point $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$, there exists a unique local geodesic $\gamma_v : I \to \mathcal{M}$, where I is a maximal open interval in $\mathbb{R}$ containing 0 and satisfying $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. The *exponential map* $\exp_x : V_x \subset \mathrm{T}_x\mathcal{M} \to \mathcal{M}$ is then defined as

$$\exp_x(v) = \gamma_v(1),$$

where γ is the geodesic starting at x with velocity v. The Hopf-Rinow theorem (see Theorem 1.7.1 in [38]) asserts that $V_x = \mathrm{T}_x\mathcal{M}$ if and only if $\mathcal{M}$ is complete. However, this map is still only guaranteed to be a *local diffeomorphism*. The largest radius $r > 0$ for which $\exp_x : B(x,r) \subset \mathrm{T}_x\mathcal{M} \to B(x,r) \subset \mathcal{M}$ is a diffeomorphism is called the *injectivity radius* and is denoted by $\mathrm{inj}(\mathcal{M},x)$. The injectivity radius of the entire manifold is defined to be

$$\mathrm{inj}(\mathcal{M}) = \inf_{x\in\mathcal{M}} \mathrm{inj}(\mathcal{M},x).$$

By the identification $\mathrm{T}_x\mathcal{M} \cong \mathbb{R}^n$, the exponential map affords us with coordinates called *normal* or *geodesic coordinates* at x. By what we have already said, these coordinates have largest radius $\mathrm{inj}(\mathcal{M},x)$. In these coordinates, at its centre x, we can arrange $\mathrm{g}_{ij} = \delta_{ij}$ and $\partial_k\mathrm{g}_{ij} = 0$ (see Theorem 1.4.4 in [38]). These coordinates are extremely useful in calculations. Furthermore, we have the existence of the following coordinates called *harmonic coordinates*, which will later allow us to demonstrate a generalised bounded geometry structure of the tensor bundle on certain manifolds.

Theorem 1.5.2 (Existence of global harmonic coordinates)**.** *Suppose there exist* $\kappa,\eta > 0$ *such that* $\mathrm{inj}(M) \geq \kappa$ *and* $|\mathrm{Ric}| \leq \eta$*. Then, for any* $A > 1$ *and* $\alpha \in (0,1)$*,* *there exists* $r_H = r_H(n, A, \alpha, \kappa, \eta) > 0$ *such that* $B(x, r_H)$ *corresponds to a coordinate system satisfying:*

(i) $A^{-1}\mathrm{I} \leq \mathrm{g} \leq A\mathrm{I}$, *and*

(ii) $$\sum_l r_H \sup_{y \in B(x,r_H)} |\partial_l \mathrm{g}_{ij}(y)| + \sum_l r_H^{1+\alpha} \sup_{y \neq z} \frac{|\partial_l \mathrm{g}_{ij}(z) - \partial_l \mathrm{g}_{ij}(y)|}{d(y,z)^\alpha} \leq A - 1.$$

A proof sketch of this theorem can be found in the observations following Theorem 1.2 in [33] and a more detailed version can be found in [3]. We end this section with a historical remark that these coordinates in dimension greater than two were first used in 1916 by Einstein in [25] on *Lorentzian manifolds*. Remarkably, it was not until 1976 that these coordinates were developed in the context of Riemannian geometry by Sabitov and Šefel in [49]. Typically such developments flow from Riemannian to Lorentzian geometry, with the latter being the more difficult situation.

1.6 Lie groups

The study of *subelliptic* problems which we undertake in Chapter 4 will be on a class of manifolds called *Lie groups*. In this study, we will depart from any Riemannian structure of this space and instead work in a *sub-Riemannian* setting. We include this section to recall some facts and ideas that we shall need later. We refer the reader to the book [24] by Dungey, ter Elst and Robinson for a treatment of Lie groups that is better suited to our requirements, the book [55] by Varopoulos, Saloff-Coste, and Coulhon (particularly Chapter 3) and [31] by Hall for a more general treatment of Lie groups focused on their structural properties.

Throughout this thesis, by "Lie group $\mathcal{G}$," we mean a smooth manifold $\mathcal{G}$ of dimension n which is also a *group*, with the extra requirement that multiplication and inversion are smooth operations. Let e denote the identity element of $\mathcal{G}$ and let the associated *Lie algebra* of $\mathcal{G}$, the tangent space $\mathrm{T}_\mathrm{e}\mathcal{G}$, be denoted by $\mathfrak{g}$.

The algebra $\mathfrak{g}$ has an associated Lie bracket, which is a skew-symmetric product satisfying the *Jacobi identity*

$$[[a,b],c] + [[c,a],b] + [[b,c],a] = 0.$$

Let $\mathrm{r}_g : \mathcal{G} \to \mathcal{G}$ be denoted by $\mathrm{r}_g\, h = hg$, which is simply *right multiplication* by g. Given a vector $v \in \mathfrak{g}$, we can extend it to the whole of $\mathcal{G}$ by *right translation.* More precisely, we can define $v(g) = \mathrm{d}\,\mathrm{r}_g(v)$, where $\mathrm{d}\,\mathrm{r}_g$ is the differential of r_g. A vector field X is then said to be *right-invariant* if $\mathrm{d}\,\mathrm{r}_g(X)(h) = X(hg)$. For a function $u : \mathcal{G} \to V$, where V is some other vector space, the *left regular representation* of u is given by $(\mathrm{L}_g\, u)(h) = u(g^{-1}h)$.

For every $v \in \mathfrak{g}$, the existence of a right-invariant vector field $\gamma_v : \mathbb{R} \to \mathcal{G}$ extending it is unique. The *exponential map* $\exp : \mathfrak{g} \to \mathcal{G}$ is then defined as $\exp(v) = \gamma_v(1)$.

With this notation, it can be shown that for $u, v \in \mathfrak{g}$,

$$[u, v] = [\gamma_u, \gamma_v]\,(e),$$

where γ_u and γ_v are the right-invariant extensions to $u, v \in \mathfrak{g}$ respectively.

We have the following first important proposition about Lie groups.

Proposition 1.6.1. *Every Lie group is parallelisable. That is, its tangent bundle is trivial.*

Proof. Let $\{e_i\} \subset \mathfrak{g}$ be a basis for $\mathfrak{g}$. Then, $\exp(-te_i)$ is a curve that generates e_i. The map $t \mapsto \exp(-te_i)g$ then is a curve through g, and we define

$$X_i(g) = \frac{d}{dt}t|_{t=0} \exp(-te_i)g,$$

and for $\tilde{e_i}$ the canonical basis for $\mathbb{R}^n$, define $\Phi(g, \tilde{e_i}) = X_i(g)$. It is straightforward that this is a global trivialisation for $\mathrm{T}\mathcal{G}$. □

Let $\mathfrak{a} = \{a_i\}_{i=1}^m$, with $m \leq n$ $(= \dim \mathcal{G})$. We say that $\mathfrak{a}$ is an *algebraic basis* if $\mathfrak{a}$ is linearly independent and we can obtain a basis for $\mathfrak{g}$ through finite multi-commutation of the elements of $\mathfrak{a}$. We remark that typically, the linear independence is not asserted for an algebraic basis, but we do so here since it has a simplifying effect on applications later. Next, let $\mathcal{A}$ denote the span of the right-translation of $\mathfrak{a}$. Thus, $\mathcal{A} \subset \mathrm{T}\mathcal{G}$ and by replicating the argument in the proof above, it is a trivial bundle. We highlight the following well known theorem of Carathéodory-Chow which we consider to be the fundamental theorem of sub-Riemannian Lie group theory.

Theorem 1.6.2 (Carathéodory-Chow)**.** *Let $\mathcal{G}$ be a connected group, and $\mathfrak{a}$ an algebraic basis. Then, between any two points $g, h \in \mathcal{G}$, there exists an absolutely continuous curve $\gamma : [0,1] \to \mathcal{G}$ such that $\dot{\gamma}(t) \in \mathcal{A}_{\gamma(t)}$.*

Since this theorem is central to all our Lie group applications, from this point onwards (unless stated otherwise), we assume that $\mathcal{G}$ is connected. First, we define a *sub-Riemannian distance* for $\mathcal{G}$. Let $g, h \in \mathcal{G}$ and let γ be a curve connecting g and h guaranteed by Theorem 1.6.2. The absolute continuity of the curve means that it is almost everywhere differentiable in t and therefore, we define the length as

$$\ell(\gamma) = \int_0^1 \left(\sum_{i=1}^m \left| \dot{\gamma}^i(t) \right|^2 \right)^{\frac{1}{2}} dt,$$

where $\gamma = \gamma^i D_i$ with D_i being the right-translated vectorfields of a_i. Then, we define the distance between g and h by

$$d(g, h) = \inf \left\{ \ell(\gamma) : \gamma(0) = g, \gamma(1) = h \right\},$$

and the balls $B(g, r)$ are defined with respect to d.

We show that $(\mathcal{G}, d)$ corresponds to a *sub-Riemannian* structure given by $\mathcal{A}$. To see this, let us write

$$D_i\big|_g = \frac{d}{dt} t\big|_{t=0} \exp(-ta_i) g,$$

which defines a frame for $\mathcal{A}$. First, let us highlight that by Theorem 1.6.2, we can consider $\mathcal{A}$ as a *sub-tangent* space and its sections as *algebraic vector fields*. Now, consider the metric g given by $\mathrm{g}(D_i, D_j) = \delta_{ij}$. Then, the distance d is the canonical distance corresponding to the metric g on $\mathcal{A}$. Furthermore, the following proposition highlights that, in fact, the topology induced by d is equivalent to the natural topology of $\mathcal{G}$. For a proof, see Proposition III.4.1 and the remarks following this result in [55].

Proposition 1.6.3. *The topology of $\mathcal{G}$ is equivalent to the topology of $(\mathcal{G}, d)$ as a metric space. Furthermore, $(\mathcal{G}, d)$ is a complete metric space and for all $g \in \mathcal{G}$ and $r > 0$, the closed balls $\overline{B(g, r)}$ are compact.*

Let us now introduce natural measures on $\mathcal{G}$. First, we say that a Borel measure μ on $\mathcal{G}$ is *left-invariant* if $\mu(gS) = \mu(S)$ for all Borel sets S and $g \in \mathcal{G}$. The theory of Lie groups guarantees the existence of a unique (up to constant) left-invariant measure which is called the *Haar measure*. If the Haar measure is also *right-invariant*, by which we mean $\mu(Sg) = \mu(S)$, then we way that it is bi-invariant and the group $\mathcal{G}$ is *unimodular*. The following proposition gives us growth estimates on the measure metric space $(\mathcal{G}, d, \mu)$. For a proof, see the discussion following II.4.4 in [24].

Proposition 1.6.4. *There exists $C, C' > 0$ and $D \in \mathbb{N}$ such that $C'r^D \leq \mu(B(g,r)) \leq Cr^D$ for all $g \in \mathcal{G}$ and $r \in (0,1]$. Furthermore, $\mu(B(g,r)) \leq Ce^{\lambda r}$ for all $r \geq 1$.*

This proposition illustrates that every $(\mathcal{G}, d, \mu)$ satisfies (Exp). However, by assuming a little extra structure, we can obtain a class of $(\mathcal{G}, d, \mu)$ for which μ grows at most polynomially. First, let us inductively define the spaces $\mathfrak{g}_k$ by setting $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$. These are a decreasing sequence of ideals in k. If there exists some K such that $\mathfrak{g}_K = 0$, then the group $\mathcal{G}$ is said to be *nilpotent*. The smallest such K is called the *rank of nilpotency*. A non-trivial example of a nilpotent Lie group is the *Heisenberg group*. See Example II.4.16 and II.6.1 in [24]. Nilpotent Lie groups are the typical objects that are suited to homogeneous analysis by the following proposition. The proof of this proposition is a consequence of Theorem II.4.6 coupled with the discussion immediately following Corollary II.4.12 in [24].

Proposition 1.6.5. *If $(\mathcal{G}, d, \mu)$ is nilpotent, then it satisfies* (Pol)*.*

Let us now return to the situation of considering an algebraic basis $\mathfrak{a}$ of $\mathfrak{g}$. We consider D_i as a differential operator $D_i : \mathrm{C}^\infty(\mathcal{G}) \to \mathrm{C}^\infty(\mathcal{G})$ by its action on functions. Let us denote the dual of $\mathcal{A}$ by $\mathcal{A}^*$, and write $D^i = D_i^*$. By the theorem of Carathéodory-Chow we can consider $\mathcal{A}^*$ as a *sub-cotangent space*. Furthermore, let us call the sections of $\mathcal{A}^*$ *algebraic covector fields*. Let $\mathcal{V}$ be any vector bundle on $\mathcal{G}$, and write $\nabla f = D_i f\ D^i$ for $f \in \mathrm{C}^\infty(\mathcal{G})$, which is a natural *exterior sub-differential* associated to $\mathcal{A}$. This leads us to define the following on a vector bundle $\mathcal{V}$ over $\mathcal{G}$.

Definition 1.6.6 (Sub-connection)**.** *We say that $\nabla : \mathrm{C}^\infty(\mathcal{V}) \to \mathrm{C}^\infty(\mathcal{A}^* \otimes \mathcal{V})$ is a* sub-connection *if it is an $\mathbb{R}$-linear map satisfying the Leibniz rule*

$$\nabla(fX) = \nabla f \otimes X + f\nabla X$$

for all $X \in \mathrm{C}^\infty(\mathcal{V})$ and $f \in \mathrm{C}^\infty(\mathcal{G})$.

Remark 1.6.7. *Note that ∇* does not *target the entire tangent space, only $\mathcal{A}$. This is how it differs from a connection and why we call it a sub-connection.*

Let us now assume that $\mathcal{V}$ is equipped with a metric h. Then, the $\mathrm{L}^p(\mathcal{V})$ spaces are defined in the natural way with respect to $(\mathcal{G}, d, \mu)$. As in our development of the Sobolev spaces arising from conventional connections on vector bundles, let ${S_p^k}'$ denote the space of all $u \in \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{V})$ such that $\nabla^i u \in \mathrm{C}^\infty \cap \mathrm{L}^p(\otimes_{p=1}^i \mathcal{A}^* \otimes \mathcal{V})$ for $1 \leq i \leq k$. As in the case of connections, define the Sobolev norm as

$$\|u\|_{\mathrm{W}^{k,p'}} = \|u\| + \sum_{i=1}^{k} \|\nabla^i u\|_p.$$

Then, the *sub-Sobolev* space $\mathrm{W}^{k,p}(\mathcal{V})'$ is defined as the closure of ${S_p^k}'$ with respect to this norm. The Sobolev space $\mathrm{W}_0^{k,p}(\mathcal{V})'$ is the closure of $\mathrm{C}_\mathrm{c}^\infty(\mathcal{V})$ under the same norm. The spaces $\mathrm{W}_0^{k,p}(\mathcal{G})'$ and $\mathrm{W}^{k,p}(\mathcal{G})'$ are the sub-Sobolev spaces on functions associated with the canonical exterior sub-differential which we defined earlier. We emphasise that these sub-Sobolev spaces depend on the algebraic basis $\mathfrak{a}$.

1.7 Operator theory and functional calculi

In this section, we provide an exposition of some ideas from operator theory that will of use to us later. We refer the reader to the excellent books [40] by Kato and [56] by Yosida which provides a more complete description of operator theory.

Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces with inner products $\langle \cdot\,, \cdot \rangle_{\mathscr{H}_1}$ and $\langle \cdot\,, \cdot \rangle_{\mathscr{H}_2}$ respectively. We say that a linear map $T : \mathcal{D}(T) \subset \mathscr{H}_1 \to \mathscr{H}_2$ is an *operator* with *domain* $\mathcal{D}(T) \subset \mathscr{H}_1$. The *range* of T is denoted by $\mathcal{R}(T) = \{Tu : u \in \mathcal{D}(T)\} \subset \mathscr{H}_2$ and the *null space* by $\mathcal{N}(T) = \{u \in \mathcal{D}(T) : Tu = 0\} \subset \mathscr{H}_1$. If S is an operator such that $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Tu = Su$ for $u \in \mathcal{D}(S)$, then we write $S \subset T$ and say that T *extends* S. We emphasise that an operator is characterised by both map and domain.

An operator T is said to be *densely-defined* if $\overline{\mathcal{D}(T)} = \mathscr{H}_1$ and it is said to be *closed* if its graph, $\mathscr{G}(T) = \{(u, Tu) : u \in \mathcal{D}(T)\}$, is a closed subset of $\mathscr{H}_1 \times \mathscr{H}_2$. The latter notion is equivalent to requiring that whenever $u_n \in \mathcal{D}(T)$ such that $u_n \to u \in \mathscr{H}_1$ and $Tu_n \to v \in \mathscr{H}_2$, then $u \in \mathcal{D}(T)$ and $Tu = v$. Define the *operator norm* of T by $\|u\|_T = \|u\| + \|Tu\|$ whenever $u \in \mathcal{D}(T)$. Then, an operator T is closed if and only if $(\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. When we say "X is dense in $\mathcal{D}(T)$," we mean that $X \subset \mathcal{D}(T)$ is dense in the operator norm of T. We make a motivational remark that the notions closed and densely-defined are particularly useful when studying differential operators.

By the closed graph theorem (see Theorem 5.20 in [40]), every closed operator T with $\mathcal{D}(T) = \mathscr{H}_1$ is *bounded*, by which we mean there exists $C > 0$ such that $\|Tu\| \leq C\|u\|$ for $u \in \mathscr{H}_1$. Boundedness of an operator is equivalent to it being continuous. The set of bounded operators which map $\mathscr{H}_1$ to $\mathscr{H}_2$ is denoted by $\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$. When $\mathscr{H} = \mathscr{H}_1 = \mathscr{H}_2$, this space is an algebra and we write $\mathcal{L}(\mathscr{H})$.

A notion that is important is that of an operator T being *closable*. By this, we mean that $\overline{\mathscr{G}(T)}$ is equal to the graph of another operator $\overline{T}$ called the *closure* of T. An operator T is closable if and only if $u_n \in \mathcal{D}(T)$ with $u_n \to 0$ and $Tu_n \to v$ implies $v = 0$. It is immediate that $T \subset \overline{T}$.

We say that an operator $S : \mathscr{H}_2 \to \mathscr{H}_1$ *is adjoint* to $T : \mathscr{H}_1 \to \mathscr{H}_2$ if the adjoint formula $\langle Tu, v\rangle_{\mathscr{H}_2} = \langle u, Sv\rangle_{\mathscr{H}_1}$ holds for all $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(S)$. For such operators, if one is densely-defined, then the other is closable (see Theorem 5.28 in [40]). This is an important fact that we often use. There can be many operators S that are adjoint to T. However, if T is densely-defined, then there exists a unique maximal operator T^* adjoint to T called *the adjoint* of T. By maximal, we mean that if S is any operator adjoint to T, then $S \subset T^*$. We construct T^* in the following way. Let $v \in \mathcal{D}(T^*)$ if there exists $f \in \mathscr{H}_2$ such that $\langle Tu, v\rangle_{\mathscr{H}_1} = \langle u, f\rangle_{\mathscr{H}_2}$ for all $u \in \mathcal{D}(T)$ and define $T^*v = f$. The uniqueness of T^* follows since $\mathcal{D}(T)$ is dense in $\mathscr{H}_1$. If further T is closable, then T^* is densely-defined and closed, and furthermore, $T^{**} = \overline{T}$ (see Theorem 5.29 in [40]).

We restrict ourselves to the situation $\mathscr{H} = \mathscr{H}_1 = \mathscr{H}_2$ to talk about the *spectral theory* of operators. The *resolvent* set $\rho(T)$ consists of all $\zeta \in \mathbb{C}$ such that $(\zeta\mathrm{I} - T)$ is one-one, onto and has a bounded inverse. The resolvent $\mathrm{R}_T : \rho(T) \to \mathcal{L}(\mathscr{H})$ is defined as $\mathrm{R}_T(\zeta) = (\zeta\mathrm{I} - T)^{-1}$. The spectrum is then $\sigma(T) = \mathbb{C} \setminus \rho(T)$. Not every operator will have a spectral theory, but for closed operators, $\sigma(T)$ is a closed set and for bounded ones, $\sigma(T)$ is compact.

We say that an operator T is *(hermitian) symmetric* if $\langle Tu, v\rangle = \langle u, Tv\rangle$, for all $u, v \in \mathcal{D}(T)$. The operator T is said to be *skew-symmetric* if $\langle Tu, v\rangle = \langle u, -Tv\rangle$. A symmetric operator is said to be *self-adjoint* if $T^* = T$. Explicitly, this means $Tu = T^*u$ for all $u \in \mathcal{D}(T) = \mathcal{D}(T^*)$. Note that a self-adjoint operator is necessarily densely-defined and closed. A densely-defined, closable operator T is said to be *essentially self-adjoint* if $\overline{T}$ is self-adjoint.

For $0 \leq \omega < \frac{\pi}{2}$, define the bisector

$$\mathrm{S}_\omega = \{\zeta \in \mathbb{C} : |\arg \zeta| \leq \omega \text{ or } |\arg(-\zeta)| \leq \omega \text{ or } \zeta = 0\}.$$

The open bisector is then defined as

$$\mathrm{S}_\omega^{\mathrm{o}} = \{\zeta \in \mathbb{C} : |\arg \zeta| < \omega \text{ or } |\arg(-\zeta)| < \omega,\ \zeta \neq 0\}.$$

An operator T on $\mathscr{H}$ is called *ω-bisectorial* if it is closed, $\sigma(T) \subset \mathrm{S}_\omega$, and the following *resolvent bounds* hold: for each $\mu \in (\omega, \pi/2)$, there exists C_μ such that whenever $\zeta \in \mathbb{C} \setminus \mathrm{S}_\mu$, the inequality $|\zeta| \, \|(\zeta \mathrm{I} - T)^{-1}\| \leq C_\mu$ holds. Depending on context, we refer to such operators simply as bisectorial and ω is then called the *angle of bisectoriality*.

Now fix $\mu \in (\omega, \pi/2)$ and let $\Psi(\mathrm{S}_\mu^{\mathrm{o}})$ denote the space of holomorphic functions $\psi : \mathrm{S}_\mu^{\mathrm{o}} \to \mathbb{C}$ for which there exists $\alpha > 0$ satisfying

$$|\psi(\zeta)| \lesssim \frac{|\zeta|^\alpha}{\mathrm{I} + |\zeta|^{2\alpha}}.$$

For such functions, we define a *functional calculus* similar to the *Riesz-Dunford functional calculus* by

$$\psi(T) = \frac{1}{2\pi\imath} \oint_\gamma \psi(\zeta)(\zeta \mathrm{I} - T)^{-1} \, d\zeta,$$

where γ is a contour in $\mathrm{S}_\mu^{\mathrm{o}}$ enveloping S_ω parametrised anti-clockwise. The integral is defined via Riemann sums and converges absolutely as a consequence of the decay of ψ, coupled with the resolvent bounds of T. The operator $\psi(T)$ is bounded. We say that T has a *bounded holomorphic functional calculus* if there exists $C > 0$ such that

$$\|\psi(T)\| \leq C \|\psi\|_\infty$$

for all $\psi \in \Psi(\mathrm{S}_\mu^{\mathrm{o}})$. We say that a family $\{T(\zeta)\}_{\zeta \in U}$ of ω-bisectorial operators has a *uniformly bounded holomorphic functional calculus* if each operator $T(\zeta)$ has a bounded holomorphic functional calculus on the same sector $S_\mu^o \cup \{0\}$ with a bound which is uniform in $\zeta \in U$.

For an ω-bisectorial operator $T : \mathcal{D}(T) \subset \mathscr{H} \to \mathscr{H}$, we have the splitting $\mathscr{H} = \mathcal{N}(T) \oplus \overline{\mathcal{R}(T)}$ (see Theorem 3.8 and following remarks in [22]). We remark that the direct sum is typically non-orthogonal. An ω-bisectorial operator T has a bounded holomorphic functional calculus if and only if there exists $\mu \in (\omega, \pi/2)$ and $\psi \in$

$\Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$ satisfying the *quadratic estimate*

$$\int_0^\infty \|\psi(tT)u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(T)}$. See [1] for the case when T is one-one and [22] for the general case.

Let $\mathbf{P}_{\mathcal{N}(T)}$ denote the projection of $\mathscr{H}$ onto $\mathcal{N}(T)$ which is zero on $\mathcal{R}(T)$. Let $\mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\mu})$ denote the space of all bounded functions $f : \mathrm{S}^{\mathrm{o}}_{\mu} \cup \{0\} \to \mathbb{C}$ which are holomorphic on $\mathrm{S}^{\mathrm{o}}_{\mu}$. For such a function f, there exists a uniformly bounded sequence (ψ_n) of functions in $\Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$ which converges to $f|_{\mathrm{S}^{\mathrm{o}}_{\mu}}$ in the compact-open topology over $\mathrm{S}^{\mathrm{o}}_{\mu}$. If T has a bounded holomorphic functional calculus, and $u \in \mathscr{H}$, then the limit $\lim_{n\to\infty} \psi_n(T)u$ exists in $\mathscr{H}$, and we define

$$f(T)u = f(0)\, \mathbf{P}_{\mathcal{N}(T)}\, u + \lim_{n\to\infty} \psi_n(T)u \ .$$

This defines a bounded operator independent of the sequence (ψ_n), and we have the bound $\|f(T)\| \leq C\|f\|_\infty$ for some $C > 0$.

In particular the functions χ^+, χ^- and $\mathrm{sgn} = \chi^+ - \chi^-$ belong to $\mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_{\mu})$, where $\chi^+(\zeta) = 1$ when $\mathrm{Re}(\zeta) > 0$ and 0 otherwise, and $\chi^-(\zeta) = 1$ when $\mathrm{Re}(\zeta) < 0$ and 0 otherwise. Therefore, if T has a bounded functional calculus in $\mathscr{H}$, then $\chi^\pm(T)$ are bounded projections, and $\mathscr{H} = \mathcal{N}(T) \oplus \mathcal{R}(\chi^+(T)) \oplus \mathcal{R}(\chi^-(T))$. The bounded operator $\mathrm{sgn}(T)$ equals 0 on $\mathcal{N}(T)$, equals 1 on $\mathcal{R}(\chi^+(T))$, and equals -1 on $\mathcal{R}(\chi^-(T))$.

A fuller treatment of functional calculi for such operators can be found in [1], [22] and a local version of this theory can be found in [45].

1.8 The Axelsson-Keith-McIntosh framework

In [8], the authors engineer a remarkable framework consisting of a set of eight hypotheses which guarantee the solution to the Kato square root problem on $\mathbb{R}^n$, while simultaneously providing a unified approach to other important problems in harmonic analysis. Their efforts culminated in producing a first-order perspective which was absent in previous work. These authors had applications to compact manifolds as a consequence, but the applicability of this framework to a non-compact geometric setting was first demonstrated by Morris in [46]. In this section, we describe an adaptation of this framework to suit our purpose and christen it the *Axelsson-Keith-McIntosh* framework (or *AKM* framework for short).

The adaptability and power of this framework cannot be exaggerated. Here, we present hypotheses (H1)-(H6), with (H4)-(H6) modified to suit our setting. Two sets of additional geometric hypotheses will be stated in Chapter 3, each reflecting a different scenario. The hypotheses we present here provides the core framework.

First, we present the hypotheses (H1)-(H3). These are phrased in terms of an abstract Hilbert space $\mathscr{H}$ with inner product $\langle \cdot\,, \cdot \rangle$. These hypotheses capture perturbations of abstract Dirac type operators and are taken in verbatim from [8].

(H1) The operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathscr{H} \to \mathscr{H}$ is closed, densely-defined and *nilpotent*, by which we mean $\mathcal{R}(\Gamma) \subset \mathcal{D}(\Gamma)$ and $\Gamma^2 = 0$.

(H2) The operators $B_1, B_2 \in \mathcal{L}(\mathscr{H})$ satisfy

$$\begin{aligned} \operatorname{Re}\langle B_1 u, u\rangle &\geq \kappa_1 \|u\|^2 && \text{whenever } u \in \mathcal{R}(\Gamma^*), \\ \operatorname{Re}\langle B_2 u, u\rangle &\geq \kappa_2 \|u\|^2 && \text{whenever } u \in \mathcal{R}(\Gamma), \end{aligned}$$

where $\kappa_1, \kappa_2 > 0$ are constants.

(H3) The operators B_1, B_2 satisfy $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.

These hypotheses are purely operator theoretic and thus have the following operator theoretic consequences. First, define

$$\Gamma_B^* = B_1 \Gamma^* B_2, \quad \Pi_B = \Gamma + \Gamma_B^* \quad \text{and} \quad \Pi = \Gamma + \Gamma^*.$$

The operator Π is self-adjoint, Π_B is bisectorial, and thus we define the following associated bounded operators:

$$\begin{aligned} &R_t^B = (\mathrm{I} + t\Pi_B)^{-1}, \ P_t^B = (\mathrm{I} + t^2\Pi_B^2)^{-1}, \\ &\qquad\qquad Q_t^B = t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}, \ \Theta_t^B = t\Gamma_B^*(\mathrm{I} + t^2\Pi_B^2)^{-1}. \end{aligned}$$

We write R_t, P_t, Q_t, Θ_t upon setting $B_1 = B_2 = \mathrm{I}$. The full implications of these assumptions are listed in §4 of [8].

We will adjourn from presenting further hypotheses to consider a general Kato square root problem at this level of abstraction. For this, we need to know when the operator Π_B has a bounded holomorphic functional calculus. A very useful criterion is to prove the quadratic estimates with respect to the operator Q_t^B. This is possible since $Q_t^B = \psi(t\Pi_B)$ where $\psi(\zeta) = \zeta/(1+\zeta^2)$ and therefore, $\psi \in \Psi(\mathrm{S}_\mu^o)$ for some $\mu < \pi/2$. In this thesis, we use the following criterion to access methods from harmonic analysis.

Proposition 1.8.1. *Suppose that* (Γ, B_1, B_2) *satisfy (H1)-(H3). Then, the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2 \tag{Qest}$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$ *holds if and only if* Π_B *admits a bounded holomorphic functional calculus.*

Now, we present some *Kato square root type* problems for the operator Π_B. This consists of two parts: the first being *Kato square root type estimates*, and the second being *holomorphic dependence* of the underlying functional calculi. The Kato square root results we present later in this thesis will be a consequence of these two parts.

The following theorem describes the first part, and it is an immediate consequence of noting that if the operator Π_B admits a bounded holomorphic functional calculus, then the projections $\chi^\pm(\Pi_B)$ are bounded, as is the operator $\mathrm{sgn}(\Pi_B) = \chi^+(\Pi_B) - \chi^-(\Pi_B)$.

Theorem 1.8.2 (Kato square root type estimate)**.** *Suppose that* (Γ, B_1, B_2) *satisfy (H1)-(H3) and that* Π_B *admits a bounded holomorphic functional calculus. Then,*

(i) there is a spectral decomposition

$$\mathscr{H} = \mathcal{N}(\Pi_B) \oplus E_B^+ \oplus E_B^-$$

where $E_B^\pm = \mathcal{R}(\chi^\pm(\Pi_B))$ *and the direct sum is in general non-orthogonal, and*

(ii) $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{{\Pi_B}^2})$ *with*

$$\|\Gamma u\| + \|\Gamma_B^* u\| \simeq \|\Pi_B u\| \simeq \|\sqrt{{\Pi_B}^2} u\|$$

for all $u \in \mathcal{D}(\Pi_B)$.

The second part is described below. It is proved easily by a minor adaptation of the proof of Theorem 6.4 in [8].

Theorem 1.8.3 (Holomorphic dependence)**.** *Let $U \subset \mathbb{C}$ be an open set and $B_1, B_2 : U \to \mathcal{L}(\mathscr{H})$ be holomorphic and suppose that $(\Gamma, B_1(\zeta), B_2(\zeta))$ satisfy (H1)-(H3) uniformly for all $\zeta \in U$. Suppose further that $\Pi_{B(\zeta)}$ has a uniformly bounded holomorphic functional calculus on $\mathrm{S}^{\mathrm{o}}_{\mu} \cup \{0\}$ for some $\mu \in (\omega, \frac{\pi}{2})$ (where ω is the angle of bisectoriality). Let $f \in \mathrm{Hol}^{\infty}(\mathrm{S}^{\mathrm{o}}_{\mu})$. Then the map $\zeta \mapsto f(\Pi_{B(\zeta)})$ is holomorphic on U.*

Adapting the construction in [8], define the following Hilbert space

$$\mathcal{K} = \mathrm{L}^2((0,\infty), \mathscr{H}; \frac{dt}{t}).$$

Then, for $\psi \in \Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$, $t > 0$ and almost all $x \in \mathcal{M}$, define the operator $\mathcal{S}_{B(\zeta)}(\psi) : \mathscr{H} \to \mathcal{K}$ by $(\mathcal{S}_{B(\zeta)}(\psi)u)(t) = \psi(t\Pi_{B(\zeta)})u$.

Lemma 1.8.4. *Under the hypotheses of Theorem 1.8.3, whenever $\mu \in (\omega, \frac{\pi}{2})$ (where ω is the angle of bisectoriality), the map $\zeta \mapsto \mathcal{S}_{B(\zeta)}(\psi)$ is holomorphic on U for all $\psi \in \Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$.*

Proof. First, we note that our choice of $\mathcal{K}$ is an adequate replacement to $\mathcal{K}$ in the proof of Theorem 6.4 in [8]. Also, for every $\psi \in \Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$ and $t > 0$, it is easy to see that $\psi_t(\zeta) = \psi(t\zeta) \in \Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$ and $\|\psi_t\|_{\infty} = \|\psi\|_{\infty}$. Therefore, the uniformly bounded holomorphic functional calculus assumption holds uniformly in $t > 0$ and is again an adequate substitution where an invocation of their Theorem 4.4 or our Theorem 1.8.3 is required. The only subtlety in the adaptation is where we require uniform boundedness of $\|\mathcal{S}^n_{B(\zeta)}\|$, where the authors of [8] define $\mathcal{S}^n_{B(\zeta)}$ as above but by setting ψ to 0 outside the interval $1/n < t < n$. We are able to guarantee the existence of such bounds by Proposition 1.8.1, when invoked with our uniformly bounded holomorphic functional calculus assumption. With these remarks, we are able to run the rest of the argument of the proof of Theorem 6.4 in [8]. ☐

With the aid of this lemma, we obtain the following Lipschitz estimates.

Theorem 1.8.5 (Lipschitz estimates)**.** *Let $\mathscr{H}, \Gamma, B_1, B_2, \kappa_1, \kappa_2$ satisfy (H1)-(H3) and take $\eta_i < \kappa_i$. Set $\hat{\omega}_i \in (0, \frac{\pi}{2})$ by $\cos\hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_{\infty} + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Let $A_i \in \mathrm{L}^{\infty}(\mathcal{L}(\mathcal{V}))$ satisfy* *(i)* $\|A_i\|_\infty \leq \eta_i$,

(ii) $A_1 A_2 \mathcal{R}(\Gamma), B_1 A_2 \mathcal{R}(\Gamma), A_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$,

(iii) $A_2 A_1 \mathcal{R}(\Gamma^*), B_2 A_1 \mathcal{R}(\Gamma^*), A_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$, *and*

(iv) $\Pi_{(B_i+\zeta A_i)}$ *has a uniformly bounded holomorphic functional calculus for all* $\zeta \in U = \{\xi \in \mathbb{C} : |\xi| \leq \min\{\eta_1 \|A_1\|_\infty^2,\ \eta_2 \|A_2\|_\infty^2\}\}$.

Letting $\mu \in (\hat{\omega}, \frac{\pi}{2})$, *we have:*

(i) *for all* $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_\mu)$,

$$\|f(\Pi_B) - f(\Pi_{B+A})\| \lesssim (\|A_1\|_\infty + \|A_2\|_\infty)\|f\|_\infty, \qquad \textit{and}$$

(ii) *for all* $\psi \in \Psi(\mathrm{S}^{\mathrm{o}}_\mu)$,

$$\int_0^\infty \|\psi(t\Pi_B)u - \psi(t\Pi_{B+A})u\|^2 \frac{dt}{t} \lesssim (\|A_1\|_\infty^2 + \|A_2\|_\infty^2)\|u\|^2,$$

whenever $u \in \mathscr{H}$.

The implicit constants depend on (H1)-(H3) and η_i.

Proof. The argument proceeds in a similar way to the proof of Theorem 6.5 in [8]. First, we note that conditions (ii) and (iii) on A_i guarantee that $\tilde{B}_i(\zeta) = (B_i + \zeta A_i)$ satisfies (H3). This is, in fact, a necessary amendment to the original proof of Theorem 6.5 in [8] for it to be valid. Furthermore, by the choice of ζ for which we assume a uniformly bounded holomorphic functional calculus is the exact set U used in the proof of Theorem 6.5 in [8]. □

The quadratic estimates presented in Proposition 1.8.1 usually take the form of *global* or *local* estimates. The first proposition, which is essentially the content of Proposition 4.8 of [8], provides a set of conditions under which global quadratic estimates exist.

Proposition 1.8.6 (Global quadratic estimates)**.** *Suppose that* (Γ, B_1, B_2) *satisfies the hypotheses (H1)-(H3) and that there exists* $c > 0$ *such that*

$$\int_0^\infty \|\Theta_t^B P_t u\|^2 \, \frac{dt}{t} \leq c\|u\|^2$$

for all $u \in \mathcal{R}(\Gamma)$*, together with three similar estimates obtained by replacing* (Γ, B_1, B_2) *by* (Γ^*, B_2, B_1)*,* $(\Gamma^*, {B_2}^*, {B_1}^*)$ *and* $(\Gamma, {B_1}^*, {B_2}^*)$*. Then,* Π_B *satisfies*

$$\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$*. Thus,* Π_B *has a bounded holomorphic functional calculus.*

In the case of the local theory, we have the following proposition which is an adaptation of Proposition 5.2 in [46]. The reason we call such estimates local is because in the integral appearing in the hypothesis of the proposition, we integrate to some $t_0 < \infty$ rather than to ∞ as in the global case described previously.

Proposition 1.8.7 (Local quadratic estimates)**.** *Suppose that* (Γ, B_1, B_2) *satisfies the hypotheses (H1)-(H3) along with* $\|u\| \lesssim \|\Pi u\|$ *for* $u \in \mathcal{R}(\Pi)$*, and that there exists* $c > 0$ *and* $t_0 > 0$ *such that*

$$\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \, \frac{dt}{t} \leq c\|u\|^2 \tag{Q1}$$

for all $u \in \mathcal{R}(\Gamma)$*, together with three similar estimates obtained by replacing* (Γ, B_1, B_2) *by* (Γ^*, B_2, B_1)*,* $(\Gamma^*, {B_2}^*, {B_1}^*)$ *and* $(\Gamma, {B_1}^*, {B_2}^*)$*. Then,* Π_B *satisfies*

$$\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathscr{H}$*. Thus,* Π_B *has a bounded holomorphic functional calculus.*

Proof. The proof is similar to the proof of Proposition 5.2 in [46]. The assumption that $\|u\| \lesssim \|\Pi u\|$ allows us to handle the integral from t_0 to ∞. Then, by invocation of Proposition 1.8.6, we obtain the conclusion. □

Chapter 3 is dedicated to showing when these propositions are satisfied under a suitable extra set of assumptions.

We continue to introduce the next set of hypotheses (H4)-(H6). They are essentially the same as the assumptions made in [8] and [46], but modified for vector bundles over measure metric spaces.

To formulate (H5), we need the following terminology that allows us to talk about bounded operators over vector bundles. Let $\mathcal{V}$ be a vector bundle over a metric space $\mathcal{X}$. We define the following.

Definition 1.8.8. *The space $\mathcal{L}(\mathcal{V})$ is defined as the vector bundle whose fibres are $\mathcal{L}(\mathcal{V}_x) \cong \mathcal{L}(\mathbb{C}^N)$.*

It is easy to see that $\mathcal{L}(\mathcal{V})$ is actually an algebra and the following proposition illustrates some of its properties.

Proposition 1.8.9. *The space $\mathcal{L}(\mathcal{V})$ is a vector bundle of rank N^2. If $\psi_x : U_x \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(U_x)$ is a trivialisation for $\mathcal{V}$, then $\varphi_x : U_x \times \mathbb{C}^{N^2} \to \pi_{\mathcal{L}(\mathcal{V})}^{-1}(U_x)$ given by*

$$\varphi_x(x, (a_{ij})) = a_{ij}\varphi_x(x, e^i)\varphi_x(x, e^j)^*,$$

where $\varphi_x(x, e^j)^* \in \mathcal{V}_x^*$, and $(a_{ij}) \in \mathbb{C}^{N^2}$. If further $\mathcal{V}$ is equipped with a metric h, *then $\mathcal{L}(\mathcal{V})$ is a normed bundle under the norm*

$$|T_x|_x = \sup_{|v|=1} |T_x v| ,$$

where $T_x \in \pi_{\mathcal{L}(\mathcal{V})}^{-1}\{x\}$.

Proof. Fix $T \in \mathcal{L}(\mathcal{V})$ and $u \in \mathcal{V}_x$. Fix a trivialisation $\psi_x : U_x \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(U_x)$ at x, and write $u = u_i e^i$ in this trivialisation. Then,

$$T_x u = u_i T_x e^i = u_i (T_x e^i)_j e^j,$$

and $((T_x e^i)_j) \in \mathbb{C}^{N^2}$. This shows that $\mathcal{L}(\mathcal{V})$ is a rank N^2 vector bundle and that $\varphi_x : U_x \times \mathbb{C}^{N^2} \to \pi_{\mathcal{L}(\mathcal{V})}^{-1}(U_x)$ as given above is indeed a trivialisation.

Now, fix $x \in \mathcal{X}$ and let $\{e^j\}$ be an orthonormal basis with respect to h at x. Then,

$$|T_x u|^2 = |u_i|^2 \left|T_x e^i\right|^2 \leq C_{T_x} \sum_i |u_i|^2 = C_{T_x} |u|^2 ,$$

with $C_{T_x} = \max_i |T_x e^i|^2$. This expression is independent of coordinates and shows that $|T_x|$ is indeed a norm, which makes $\mathcal{L}(\mathcal{V})$ a normed bundle. $\square$

With this terminology at hand, we list hypotheses (H4)-(H6).

(H4) The Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathcal{V})$, where $\mathcal{V}$ is a vector bundle with continuous metric h over a complete measure metric space $\mathcal{X}$ with a Borel-regular measure μ.

(H5) The operators B_1, B_2 are multiplication operators, i.e. $B_i \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$.

(H6) The operator Γ is a *weak* first-order differential operator. That is, there exists $C_\Gamma > 0$ such that whenever η is a bounded Lipschitz function, we have that $\eta\mathcal{D}(\Gamma) \subset \mathcal{D}(\Gamma)$ and $\mathrm{M}_\eta u(x) = [\Gamma, \eta(x)I]\, u(x)$ is a multiplication operator satisfying

$$|\mathrm{M}_\eta u(x)| \leq C_\Gamma \, |\mathrm{Lip}\, \eta(x)| \, |u(x)|$$

for all $u \in \mathcal{D}(\Gamma)$ and μ-almost all $x \in \mathcal{X}$.

Remark 1.8.10. *We note as in [46] that (H6) implies the same statement as itself but with Γ replaced by either Γ^* or Π.*

The hypothesis (H6) in [8], [7] and [46] had ∇u appearing instead of $\mathrm{Lip}\, u$ since the authors of these papers had the luxury of a differentiable structure which they could exploit. The use of the Lipschitz pointwise constant was motivated by the work of Cheeger in [16] where the author uses Lipschitz maps in measure metric spaces to talk about differentiability properties.

For first-order operators on smooth manifolds or smooth bundles, the commutator $[\Gamma, \eta]$ for $\eta \in \mathrm{C}^\infty_\mathrm{c}$ is actually the *principal symbol* of the operator. See, for instance, [19]. As do the authors of [8], we use (H6) to *characterise* that Γ is a first-order differential operator through this condition.

A most important consequence of the hypotheses (H1)-(H6) is that they allow us to prove *off-diagonal* or *Davies-Gaffney* estimates.

Proposition 1.8.11 (Off-diagonal estimates)**.** *Let U_t be either R^B_t, P^B_t, Q^B_t or Θ^B_t for $t \in \mathbb{R}^+$. Then, there exists a $C_\Theta > 0$, which only depends on (H1)-(H6) and for every $M > 0$, there exists $C_M > 0$ such that*

$$\|\chi_E U_t u\| \leq C_M \left\langle \frac{|t|}{d(E,F)} \right\rangle^M \exp\left(-C_\Theta \frac{d(E,F)}{t}\right) \|\chi_F u\|$$

whenever E, F are Borel, spt $u \subset F$.

This proposition appeared in this form in [46], and it is essentially contained in [8]. The former proof is easily adapted to yield a proof for this proposition by employing the Lipschitz separation lemma (Lemma 1.2.1) to obtain control on cutoffs due to the fact that our (H6) is formulated in terms of the Lipschitz pointwise constant.

We conclude this section by spelling out the philosophy of how we use these estimates. As we have stated before, the goal is to obtain quadratic estimates (Qest). While there could be many different methods to prove such estimates, we will only focus on techniques from harmonic analysis in this thesis. The off-diagonal estimates are a crucial tool in this endeavour as they provide us exponential decay which we then *pit* against growth of the underlying measure metric space in the analysis.

2

Density problems on manifolds and vector bundles

In the analysis of differential operators, it is often useful in calculations to know that the set of smooth, compactly supported functions (or sections) is dense in the domain of the operator in question. We call this the *density problem*.

In this chapter, we study the density problem for some canonical differential operators on vector bundles over smooth, complete Riemannian manifolds. Under very general assumptions, we show that smooth, compactly supported sections are dense in the domains of these operators. Furthermore, we show that smooth, compactly supported functions are dense in the second-order Sobolev space on such manifolds under the sole additional assumption that the Ricci curvature is uniformly bounded from below. This improves the current known result which also requires a lower bound on the injectivity radius.

2.1 Essential self-adjointness and hyperbolic equations

The perspective that we develop here hinges on the maximal nature of self-adjointness. Running the risk of labouring the point, we present some lemmata that are useful to the proofs we present in this chapter. Recall from §1.7 the notion of self-adjointness and symmetric extension.

Proposition 2.1.1. *Let T be self-adjoint, and S a symmetric extension of T. Then, $S = T$.*

The following proposition yields the same conclusion as above, but it is more useful since we only need to verify that the operators are equal on a dense subset.

Proposition 2.1.2. *Let T and S be self-adjoint. If $\mathcal{D} \subset \mathcal{D}(T) \cap \mathcal{D}(S)$, $\mathcal{D}$ is dense in $\mathcal{D}(T)$, and $Tu = Su$ for all $u \in \mathcal{D}$, then $T = S$.*

Proof. Since S is self-adjoint, in particular it is symmetric, and thus, by Proposition 2.1.1, it suffices to show that $T \subset S$.

Let $u \in \mathcal{D}(T)$. By the hypotheses, there exists a sequence $u_j \in \mathcal{D}$ such that $u_j \to u$ and $Tu_j \to Tu$, and further $Tu_j = Su_j$. Thus, we have that $u_j \to u$ and $Su_j \to Tu$. But the self-adjointness of S in particular means that S is closed and hence $u \in \mathcal{D}(S)$ and $Su = Tu$. This shows that $T \subset S$ as required. $\square$

The work of Chernoff in [19] is central to proving the results we present in this paper. Thus, for the convenience of the reader, we recall some ideas from [19].

Our setting is the following. Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with smooth metric g and volume measure μ. Further, let $\mathcal{V}$ denote a smooth, complex vector bundle of finite rank over $\mathcal{M}$ with smooth metric h. Since $\mathcal{V}$ is complex, we assume that h is hermitian symmetric. We denote the fibre of $\mathcal{V}$ over $x \in \mathcal{M}$ by $\mathcal{V}_x$.

Next, let L be a first-order differential operator on $\mathrm{C}^\infty(\mathcal{V})$, and let $L_c = L$ with domain $\mathrm{C}^\infty_c(\mathcal{V})$. We recall the *symbol* of L from [19]. Let $x \in \mathcal{M}$, $v \in \mathrm{T}^*_x\mathcal{M}$, $e \in \mathcal{V}_x$ and fix $g \in \mathrm{C}^\infty(\mathcal{M})$ and $f \in \mathrm{C}^\infty(\mathcal{V})$ such that $\mathrm{d}\, g(x) = v$ and $f(x) = e$. Then, the symbol of L at x in the direction v acting on e is defined by

$$\sigma(x, v)e = \sigma_L(x, v)e = [L, gI]\, f(x).$$

We note that this is really the *principal symbol* of the operator L and emphasise as in [19] that this definition only applies to first order operators.

Retaining the nomenclature of [19], we write the *speed of propagation* at x as

$$\mathrm{c}(x) = \mathrm{c}_L(x) = \sup_{|v|=1} |\sigma(x, v)|$$

where $|\sigma(x, v)| = \sup_{|e|=1} |\sigma(x, v)e|$. Fix $x_0 \in \mathcal{M}$ and $r > 0$. Then, the speed of propagation of L inside the ball $B(x_0, r)$ is given by

$$\mathrm{c}(r) = \mathrm{c}_L(r) = \sup_{x \in B(x_0, r)} \mathrm{c}(x).$$

With this notation at hand, we present the following theorem which is an immediate consequence of Theorem 2.2 in [19]. This result is the central tool that we use in this chapter.

Theorem 2.1.3. *Let $\mathcal{V}$ be a smooth vector bundle with smooth metric h over a smooth, complete Riemannian manifold $\mathcal{M}$. Let Π be a first-order differential operator on $\mathrm{C}^\infty(\mathcal{V})$ and Π_c the same operator as Π but with domain $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$. If Π_c is symmetric and there exists $C > 0$ such that $\mathrm{c}_\Pi(x) \leq C$ for each $x \in \mathcal{M}$, then every power of Π_c is essentially self-adjoint.*

2.2 Connections, divergence and Laplacians on vector bundles

In this section, we further assume that $\mathcal{V}$ is equipped with a connection ∇ and that it is compatible with h. Let ∇_c be the restriction of ∇ to $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$. Then, we have the following proposition.

Proposition 2.2.1. *Suppose that the connection ∇ and the metric h are compatible. If $T \in \mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ and $P \in \mathrm{L}^2 \cap \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ or if $T \in \mathrm{L}^2 \cap \mathrm{C}^\infty(\mathcal{V})$ and $P \in \mathrm{C}^\infty_\mathrm{c}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$, then*

$$\int_{\mathcal{M}} \langle \nabla T, P \rangle \ d\mu = \int_{\mathcal{M}} \langle T, -\operatorname{tr} \nabla P \rangle \ d\mu. \tag{$\star$}$$

Proof. Fix $x \in \mathcal{M}$ so that we can locally write $T = T_i e^i$ and $P = P_{kl} dx^k \otimes e^l$. Next, define the $\mathcal{V}$ "inner product" yielding a 1-form by $\langle T, P \rangle_{\mathcal{V}} = T_i P_{kl} \mathrm{h}(e^i, e^l) dx^k$.

Now, to make calculations easier, further assume that $\{x^i\}$ are normal coordinates at x. Then, for any $X = X_k dx^k$, we have that the divergence at x is $\operatorname{div} X = \partial_k X_k$.

Thus, at x,

$$\operatorname{div} \langle T, P \rangle_{\mathcal{V}} = \sum_k (\partial_k T_i) P_{kl} \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(\nabla_{\partial_k} e^i, e^l) + \sum_k T_i (\partial_k P_{kl}) \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(e^i, \nabla_{\partial_k} e^l)$$

by the compatibility of ∇ and h.

A calculation at x then shows that $\nabla T = \nabla(T_i e^i) = \partial_k T_i dx^k \otimes e^i + T_i dx^k \otimes \nabla_{\partial_k} e^i$. Also, $\operatorname{tr} \nabla P = \sum_k \partial_k P_{kl} e^l + \sum_k P_{kl} \nabla e^l$, since we assumed normal coordinates at x, making $\nabla dx^k = 0$. Then, a direct calculation shows that

$$\langle \nabla T, P \rangle = \sum_k (\partial_k T_i) P_{kl} \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(\nabla_{\partial_k} e^i, e^l), \quad \text{and}$$

$$\langle T, \operatorname{tr} \nabla P \rangle = \sum_k T_i (\partial_k P_{kl}) \mathrm{h}(e^i, e^l) + \sum_k T_i P_{kl} \mathrm{h}(e^i, \nabla_{\partial_k} e^l).$$

Thus, at x, $\operatorname{div} \langle T, P \rangle_{\mathcal{V}} = \langle \nabla T, P \rangle + \langle T, \operatorname{tr} \nabla P \rangle$.

By the compactness of $\operatorname{spt} T$ or of $\operatorname{spt} P$, it is easy to see that $\operatorname{spt} \langle T, P \rangle_{\mathcal{V}}$, $\operatorname{spt} \langle T, \operatorname{tr} \nabla P \rangle$ and $\operatorname{spt} \langle \nabla T, P \rangle$ are all compact. Thus, we integrate this equation over $\mathcal{M}$ and apply the divergence theorem to obtain the conclusion. □

This means the operators ∇_c and $-\operatorname{tr} \nabla_c$ are adjoint to each other and by the density of $\mathrm{C}^\infty_\mathrm{c}$ in L^2, both these operators are densely-defined and closable.

Define $-\operatorname{div} = {\nabla_c}^*$. Clearly, $\operatorname{tr} \nabla_c \subset \operatorname{div}$, and moreover, $\overline{\operatorname{tr} \nabla_c} \subset \operatorname{div}$. Furthermore, let $\tilde{\nabla} = (-\operatorname{tr} \nabla_c)^*$. It is clear that $\overline{\nabla_c} \subset \tilde{\nabla}$. In this section we show that under appropriate conditions, $\overline{\nabla_c} = \tilde{\nabla}$. The following proposition illustrates the connection of this statement to density problems.

Proposition 2.2.2. *Let $\mathcal{M}$ be a smooth Riemannian manifold and $\mathcal{V}$ a vector bundle with metric* h *and connection ∇ that are compatible. Then, the following are equivalent:*

(i) $\overline{\operatorname{tr} \nabla_c} = \operatorname{div}$,

(ii) $\overline{\nabla_c} = \tilde{\nabla}$,

(iii) $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ *is dense in* $\mathcal{D}(\tilde{\nabla})$,

(iv) $\mathrm{C}^\infty_\mathrm{c}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is dense in* $\mathcal{D}(\mathrm{div})$.

Proof. Suppose that $\overline{\mathrm{tr}\,\nabla_c} = \mathrm{div}$. Since $\overline{\nabla_c}$ is closed and densely-defined, $(-\mathrm{div})^* = {\nabla_c}^{**} = \overline{\nabla_c}$. Also, $(-\mathrm{div})^* = \overline{\mathrm{tr}\,\nabla_c}^* = (\mathrm{tr}\,\nabla_c)^* = \tilde{\nabla}$. Thus (i) implies (ii). By similar reasoning, (ii) implies (i). Now, we note that $\overline{\nabla_c} = \tilde{\nabla}$ if and only if $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ is dense in $\mathcal{D}(\tilde{\nabla})$ since $\nabla_c = \tilde{\nabla}$ on $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$. Thus, (ii) and (iii) are equivalent. By similar argument, (iv) and (i) are equivalent to each other. This concludes the proof. □

Remark 2.2.3. *Note that we do not assume completeness in this proposition.*

We also consider the following self-adjoint operators $\Delta_\mathrm{B} = -\mathrm{div}\,\overline{\nabla_c}$ with domain

$$\mathcal{D}(\Delta_\mathrm{B}) = \left\{u \in \mathrm{L}^2(\mathcal{V}) : u \in \mathcal{D}(\overline{\nabla_c}),\ \overline{\nabla_c} u \in \mathcal{D}(\mathrm{div})\right\} \subset \mathcal{D}(\overline{\nabla_c})$$

and $\Delta_\mathrm{A} = -\overline{\mathrm{tr}\,\nabla_c}\tilde{\nabla}$ with domain

$$\mathcal{D}(\Delta_\mathrm{A}) = \left\{u \in \mathrm{L}^2(\mathcal{V}) : u \in \mathcal{D}(\tilde{\nabla}),\ \tilde{\nabla} u \in \mathcal{D}(\overline{\mathrm{tr}\,\nabla_c})\right\} \subset \mathcal{D}(\tilde{\nabla}).$$

These are two *Laplacians* on the vector bundle. Furthermore, by ($\star$), we find that the *connection Laplacian* $\Delta_c = -\mathrm{tr}\,{\nabla_c}^2$ with domain $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ is densely-defined and closable. Let Δ_0 denote the closure of Δ_c and denote its domain by $\mathcal{D}(\Delta_0)$. Certainly, it is easy to see that $\Delta_0 \subset \Delta_\mathrm{B}$ and $\Delta_0 \subset \Delta_\mathrm{A}$ and we have the following proposition which lists some equivalences.

Proposition 2.2.4. *Under the same hypotheses as Proposition 2.2.2, the following are equivalent:*

(i) Δ_0 *is self-adjoint,*

(ii) $\Delta_0 = \Delta_\mathrm{B} = \Delta_\mathrm{A}$,

(iii) $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ *is dense in* $\mathcal{D}(\Delta_\mathrm{B})$,

(iv) $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$ *is dense in* $\mathcal{D}(\Delta_\mathrm{A})$.

Proof. If Δ_0 is self-adjoint, then since $\Delta_0 \subset \Delta_\mathrm{B}$ and $\Delta_0 \subset \Delta_\mathrm{A}$ where Δ_B and Δ_A are in particular symmetric, Proposition 2.1.1 allows us to conclude that $\Delta_0 = \Delta_\mathrm{B}$ and $\Delta_0 = \Delta_\mathrm{A}$. Thus, (i) implies (ii). It is easy to see that (ii) implies (iii) since $\Delta_0 = \Delta_\mathrm{B}$ on $\mathrm{C}^\infty_\mathrm{c}(\mathcal{V})$. Similarly, (ii) implies (iv). If (iv) holds, then $\Delta_0 = \Delta_\mathrm{A}$ and Δ_A

is self adjoint and so (iv) implies (i). Similarly, (iii) implies (i) and this concludes the proof. $\square$

We also define the following objects that we call *co-Laplacians*. First, define $\rhd = -\overline{\nabla_c}\,\mathrm{div}$ with domain

$$\mathcal{D}(\rhd) = \left\{u \in \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) : u \in \mathcal{D}(\mathrm{div}),\ \mathrm{div}\, u \in \mathcal{D}(\overline{\nabla_c})\right\} \subset \mathcal{D}(\mathrm{div}).$$

Also, let $\tilde{\rhd} = -\tilde{\nabla}\overline{\mathrm{tr}\,\nabla_c}$ with domain

$$\mathcal{D}(\tilde{\rhd}) = \left\{u \in \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) : u \in \mathcal{D}(\overline{\mathrm{tr}\,\nabla_c}),\ \overline{\mathrm{tr}\,\nabla_c} u \in \mathcal{D}(\tilde{\nabla})\right\} \subset \mathcal{D}(\overline{\mathrm{tr}\,\nabla_c}).$$

It is easy to see that both these operators are self-adjoint. The corresponding *connection co-Laplacian* is then given by $\rhd_c$ with domain $\mathrm{C}^\infty_\mathrm{c}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. As for the connection Laplacian, the condition ($\star$) implies that $\rhd_c$ is a densely-defined, closable operator. Thus, let $\rhd_0$ denote the closure of $\rhd_c$ with domain $\mathcal{D}(\rhd_0)$. It is easy to see that $\rhd_0 \subset \rhd$ and $\rhd_0 \subset \tilde{\rhd}$. We have the following list of equivalences whose proof is similar to that of the analogous proposition we proved for the Laplacians.

Proposition 2.2.5. *Under the same hypotheses as Proposition 2.2.2, the following are equivalent:*

(i) $\rhd_0$ *is self-adjoint,*

(ii) $\rhd_0 = \rhd = \tilde{\rhd}$,

(iii) $\mathrm{C}^\infty_\mathrm{c}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is dense in* $\mathcal{D}(\rhd)$,

(iv) $\mathrm{C}^\infty_\mathrm{c}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ *is dense in* $\mathcal{D}(\tilde{\rhd})$.

We now prove the main result of this section. We remark that as a consequence of this theorem, each of the statements of Propositions 2.2.2, 2.2.4, and 2.2.5 hold.

Theorem 2.2.6. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with smooth metric g and let $\mathcal{V}$ be a smooth vector bundle over $\mathcal{M}$ with smooth metric h and connection ∇. Suppose that h and ∇ are compatible. Then, $\overline{\nabla_c} = \tilde{\nabla}$ and Δ_0 and $\rhd_0$ are self-adjoint.*

Proof. Let $\mathscr{H} = \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ and define

$$\Pi = \begin{pmatrix} 0 & -\mathrm{tr}\,\nabla \\ \nabla & 0 \end{pmatrix}$$

with domain $\mathcal{D}(\Pi) = \mathrm{C}^\infty(\mathcal{V}) \oplus \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Let Π_c denote Π with domain $\mathcal{D}(\Pi_c) = \mathrm{C}_\mathrm{c}^\infty(\mathcal{V}) \oplus \mathrm{C}_\mathrm{c}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. First, we show that $\langle \Pi_c u, v\rangle = \langle u, \Pi_c v\rangle$, for $u, v \in \mathcal{D}(\Pi_c)$. Let $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{D}(\Pi_c)$. Then, $\Pi_c u = (-\operatorname{tr} \nabla_c u_2, \nabla_c u_1)$ and $\Pi_c v = (-\operatorname{tr} \nabla_c v_2, \nabla_c v_1)$. Therefore,

$$\begin{aligned}\langle \Pi_c u, v\rangle &= \langle -\operatorname{tr} \nabla_c u_2, v_1\rangle + \langle \nabla_c u_1, v_2\rangle, \text{ and}\\ \langle u, \Pi_c v\rangle &= \langle u_1, -\operatorname{tr} \nabla_c v_2\rangle + \langle u_2, \nabla_c v_1\rangle.\end{aligned}$$

But, by ($\star$),

$$\langle -\operatorname{tr} \nabla_c u_2, v_1\rangle = \langle u_2, \nabla_c v_1\rangle \quad \text{and} \quad \langle \nabla_c u_1, v_2\rangle = \langle u_1, -\operatorname{tr} \nabla_c v_2\rangle.$$

Therefore, $\langle \Pi_c u, v\rangle = \langle u, \Pi_c v\rangle$.

Next, we compute the symbol of Π. Fix $x \in \mathcal{M}$ and $v \in \mathrm{T}^*_x\mathcal{M}$ and $e = (e_1, e_2) \in \mathcal{V}_x \oplus \mathrm{T}^*_x\mathcal{M} \otimes \mathcal{V}_x$. Let $f \in \mathrm{C}^\infty(\mathcal{V}) \oplus \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ such that $f(x) = e$, and $g \in \mathrm{C}^\infty(\mathcal{M})$ such that $\nabla g = v$. Recall the following product rules for ∇ and $\operatorname{tr} \nabla$:

$$\nabla(gf) = \nabla g \otimes f + g\nabla f, \quad \text{and} \quad \operatorname{tr} \nabla(gf) = g \operatorname{tr} \nabla v + \operatorname{tr}(\overline{\nabla g} \otimes f)$$

where by $\overline{\nabla g}$ we mean complex conjugation. Write $f = (f_1, f_2)$. Then,

$$g\Pi f = g(-\operatorname{tr} \nabla f_2, \nabla f_1)$$

and

$$\Pi(gf) = (-g \operatorname{tr} \nabla f_2 - \operatorname{tr}(\overline{\nabla g} \otimes f_2), \nabla g \otimes f_1 + g\nabla f_1).$$

Thus,

$$\begin{aligned}\sigma(x, v)e &= \Pi(gf)(x) - (g\Pi f)(x)\\ &= (-\operatorname{tr}(\overline{\nabla g(x)} \otimes f_2(x), \nabla g(x) \otimes f_1(x)) = (-\operatorname{tr}(\overline{v} \otimes e_2), v \otimes e_1)\end{aligned}$$

since $f(x) = (e_1, e_2)$. Therefore,

$$\begin{aligned}|\sigma(x, v)e|^2 = |-\operatorname{tr}(\overline{v} \otimes e_2)|^2 + |v \otimes e_1|^2 &\leq |v|^2 |e_2|^2 + |v|^2 |e_1|^2\\ &= |v|^2 (|e_2|^2 + |e_1|^2) = |v|^2 |e|^2.\end{aligned}$$

From this, it is easy to see that $|\sigma(x, v)| = |v|$ making $\mathrm{c}(x) = 1$.

These facts demonstrate that the hypotheses of Theorem 2.1.3 are satisfied for Π

and so we conclude that Π_c is essentially self-adjoint. Letting Π_0 denote the closure of Π_c, note that

$$\Pi_0 = \begin{pmatrix} 0 & -\overline{\operatorname{tr}\nabla_c} \\ \overline{\nabla_c} & 0 \end{pmatrix} \quad \text{and} \quad \Pi_c^* = \begin{pmatrix} 0 & -\operatorname{div} \\ \tilde{\nabla} & 0 \end{pmatrix}.$$

By the self-adjointness of Π_0, we find that $\Pi_c^* = \overline{\Pi_0}^* = \Pi_0{}^* = \Pi_0$. Thus, $\overline{\nabla_c} = \tilde{\nabla}$ and $\overline{\operatorname{tr}\nabla_c} = \operatorname{div}$.

Also, we have that

$$\Pi_c^2 = \begin{pmatrix} -\operatorname{tr}\nabla_c{}^2 & 0 \\ 0 & -\nabla_c \operatorname{tr}\nabla_c \end{pmatrix} = \begin{pmatrix} \Delta_c & 0 \\ 0 & \rhd_c \end{pmatrix}.$$

By Theorem 2.3, Π_c^2 is essentially self-adjoint, which implies that $\overline{\Delta_c} = \Delta_0$ and $\overline{\rhd_c} = \rhd_0$ are also self-adjoint. $\square$

Remark 2.2.7. *When $\mathcal{V}$ is $\mathcal{M} \times \mathbb{C}$, i.e. for functions, the self-adjointness of Δ_0 on $\mathrm{L}^2(\mathcal{M})$ under the assumption that $\mathcal{M}$ is complete is a well known fact. It was stated in [19] as an application, and the author of this paper points to [30] by Gaffney and [47] by Roelcke as historical references. Similar results for forms and tensors are obtained by Strichartz in [54].*

We point out the following immediate consequence of this theorem for the tensor Laplacian and co-Laplacian.

Corollary 2.2.8. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g and connection ∇ that are compatible. Then, $\mathrm{C}_\mathrm{c}^\infty(\mathcal{T}^{(p,q)}\mathcal{M})$ is dense in $\mathcal{D}(\Delta_\mathrm{B})$ and $\mathrm{C}_\mathrm{c}^\infty(\mathcal{T}^{(p,q+1)}\mathcal{M})$ is dense in $\mathcal{D}(\rhd)$.*

Remark 2.2.9. *We emphasise that in the corollary, we do not rule out the possibility that ∇ has torsion. That is, the relationship $\nabla_X Y - \nabla_Y X = [X, Y]$ may fail for some $X, Y \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$.*

Next, we show some consequences of Theorem 2.2.6 to Sobolev spaces on vector bundles. Recall the spaces S_k^p from §1.5. Let $\nabla_2 : S_1^2 \to \mathrm{C}^\infty \cap \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ be defined by $\nabla_2 = \nabla$. Then, we have the following operator theoretic characterisations of $\mathrm{W}^{1,2}(\mathcal{V})$ and $\mathrm{W}_0^{1,2}(\mathcal{V})$.

Corollary 2.2.10. *Under the same hypotheses as Theorem 2.2.6, ∇_2 is a densely-defined, closeable operator and $\mathrm{W}^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_2})$. Furthermore, $\mathrm{W}_0^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_c})$ and $\mathrm{W}_0^{1,2}(\mathcal{V}) = \mathrm{W}^{1,2}(\mathcal{V})$.*

Proof. Fix $u \in S_1^2$ and $v \in \mathrm{C}_c^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Then, by Proposition 2.2.1, ∇_2 is densely-defined and closeable. Furthermore, it shows that $\overline{\nabla_2} \subset \tilde{\nabla}$, and it is easy to see that $\overline{\nabla_c} \subset \overline{\nabla_2}$. Then by Theorem 2.2.6, we have that $\overline{\nabla_c} = \tilde{\nabla} = \overline{\nabla_2}$.

Since $\nabla_2 = \nabla$ on S_1^2, it is easy to see that $\mathrm{W}^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_2})$. We find that $\mathrm{W}_0^{1,2}(\mathcal{V}) = \mathcal{D}(\overline{\nabla_c})$ for a similar reason, and this concludes the proof. □

Remark 2.2.11. *In this generality, do not know whether* $\mathrm{W}_0^{k,2}(\mathcal{V}) = \mathrm{W}^{k,2}(\mathcal{V})$ *for* $k \geq 2$*. In fact, even for the case* $\mathcal{V} = \mathcal{M} \times \mathbb{C}$ *with* ∇ *being the Levi-Cevita connection, we use assumptions on curvature to show that* $\mathrm{W}_0^{2,2}(\mathcal{M}) = \mathrm{W}^{2,2}(\mathcal{M})$*. We deal with this situation in §2.4. The case for* $k > 2$ *is considered in Proposition 3.2 in [33].*

In the following result, we provide an alternative proof of the previous proposition which does not rely on Theorem 2.2.6. It shows that *every* connection $\nabla_2 : S_1^2 \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$, not just those on complete manifolds, are densely-defined and closeable. The proof also highlights more explicitly the way in which completeness creeps into the picture when proving that $\mathrm{C}_c^\infty(\mathcal{V})$ is dense in $\mathrm{W}^{1,2}(\mathcal{V})$.

Proposition 2.2.12. *For every vector bundle* $\mathcal{V}$*, metric* h *and connection* ∇ *over an arbitrary manifold* $\mathcal{M}$*,*

(i) the operator ∇_2 *is densely-defined and closed,*

(ii) if the manifold $\mathcal{M}$ *is complete, then* $\mathrm{C}_c^\infty(\mathcal{V})$ *is dense in* $\mathrm{W}^{1,2}(\mathcal{V})$*.*

Proof. We prove (i). That $\mathrm{C}_c^\infty(\mathcal{V})$ is dense in $\mathrm{L}^2(\mathcal{V})$ is shown by a mollification argument after covering the manifold by countably many compact sets corresponding to coordinate charts. Since $\mathrm{C}_c^\infty(\mathcal{V}) \subset \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$, we have that ∇_2 is a densely defined operator.

We show that ∇_2 is a closable operator by reduction to the well known fact that $\nabla_2 = \mathrm{d}$ is closable on scalar-valued functions. Fix $u_n \in S_1^2$ such that $u_n \to 0$ and $\nabla_2 u_n \to v$. For each $x \in \mathcal{M}$, we can find an open set K_x such that $\overline{K_x}$ is compact and with a bundle chart $\psi_x : \overline{K_x} \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(\overline{K_x})$. Let $\{K_x\}$ be a collection of such sets and let $\{K_j\}$ denote a countable subcover. Now, fix K_j and let $\{e^i\}$ denote an orthonormal frame (by a Gram-Schmidt procedure) in $\overline{K_j}$. Write $u_n = (u_n)_i e^i$ and note that

$$\nabla_2 u_n = \nabla_2 (u_n)_i \otimes e^i + (u_n)_i \nabla_2 e^i$$

and therefore,

$$\sum_i |\nabla_2(u_n)_i - v_i|^2 = \left|\nabla_2(u_n)_i \otimes e^i - v\right|^2 \leq |\nabla_2 u_n - v|^2 + \sup_l \left|\nabla_2 e^l\right|^2 |u_n|^2 .$$

Since by assumption $\overline{K_i}$ is compact and the basis e^i is smooth, we have that $\sup_l \left|\nabla_2 e^l\right|^2 \leq C_j$ for some C_j dependent on K_j. Thus,

$$\|\nabla_2(u_n)_i - v_i\|^2_{\mathrm{L}^2(K_j)} \to 0.$$

Thus, we have that $(u_n)_i \to 0$ and $\nabla_2(u_n)_i \to v_i$ in $\mathrm{L}^2(K_j)$, and by the closability of ∇_2 on functions, we have that $v_i = 0$ almost everywhere in K_j. Thus, $v = 0$ almost everywhere in K_j and consequently $v = 0$ almost everywhere in $\mathcal{M}$. This proves that ∇_2 is a closable operator.

Now we prove (ii). Since we assume that $\mathcal{M}$ is complete, the Hopf-Rinow theorem guarantees that closed balls are compact. Fix some centre $x \in \mathcal{M}$ and consider closed balls $\overline{B(x,r)}$. For such balls, we can find $\eta_r \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V})$ such that spt η_r is compact, $0 \leq \eta \leq 1$, $\eta_r \equiv 1$ on $B(x,r)$ and $\sup |\nabla \eta_r| \to 0$ as $r \to \infty$. Now, for any $u \in \mathrm{H}^1(\mathcal{V})$, $\eta_r u$ is compactly supported. Also,

$$\nabla(\eta_r u) = \nabla \eta_r \otimes u + \eta_r \nabla u$$

and

$$\|\nabla(\eta_r u) - \nabla u\| \leq \|\nabla \eta_r \otimes u\| + \|\eta_r \nabla u - \nabla u\| \leq \sup_{x \in \mathcal{M}} |\nabla \eta_r| \, \|u\| + \|\eta_r \nabla u - \nabla u\| \to 0$$

as $r \to \infty$. Thus, it suffices to consider $u \in \mathrm{W}^{1,2}(\mathcal{V})$ with spt u compact. But for each such u, an easy mollification argument in each bundle chart will yield a sequence $u_n \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{V})$ such that $\|u_n - u\|_{\mathrm{H}^1} \to 0$. □

Remark 2.2.13. *This proof of (ii) is somewhat instructive as to why a simple argument of this kind cannot be used to prove a similar result for higher order Sobolev spaces. Let us just consider the case of $k = 2$. In this situation, we would need to find η_r such that $\left|\nabla^2 \eta_r\right|$ is small, and we should expect curvature to serve as an obstruction.*

Let us return back to realising the Sobolev spaces $\mathrm{W}^{k,2}(\mathcal{V})$ as the domains of an appropriate operator. Define $\Xi_k : S^2_k \to \oplus_{i=1}^k \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{T}^{(0,i)}\mathcal{M} \otimes \mathcal{V})$ by $\Xi_k u = (\nabla^i u)_{i=1}^k$. The following proposition characterises Sobolev spaces in terms of this operator.

Proposition 2.2.14. *Under the same hypothesis as Theorem 2.2.6, the operator Ξ_k is closeable and $\mathrm{W}^{k,2}(\mathcal{V}) = \mathcal{D}(\overline{\Xi_k})$.*

Proof. We note the case $k = 1$ is handled in Corollary 2.2.10.

Let $u_n \in S^2_k$ such that $u_n \to 0$ and $\Xi_k u_n \to (v_i)_{i=1}^k$. Then, we have that $v_1 = 0$ by Corollary 2.2.10. Set $w_n = \nabla u_n$ so that $w_n \to v_1 = 0$ and $\nabla w_n \to v_2$. Again, by invoking Corollary 2.2.10, we conclude that $v_2 = 0$. Proceeding this way, we find that $v_i = 0$ for all $i = 1, \ldots, k$. Thus, Ξ_k is closeable.

That $\mathrm{W}^{k,2}(\mathcal{V}) = \mathcal{D}(\overline{\Xi_k})$ follows from the fact that $\|u\|_{\mathrm{W}^{k,2}} = \|u\|_{\Xi_k} = \|u\| + \|\Xi_k u\|$ for $u \in S^2_k$. $\square$

We conclude this section with the following *regularity result.* We only use this result in §2.4 by specialising $\mathcal{V}$ to $\mathcal{M} \times \mathbb{C}$. However, we include it in this section since it is of interest in its own right and holds true at this level of generality.

Proposition 2.2.15. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and $\mathcal{V}$ a smooth vector bundle with metric h and connection ∇ that are compatible. Then, $\mathrm{W}^{2,2}_0 \subset \mathcal{D}(\Delta_{\mathrm{B}})$ and $\mathrm{W}^{2,2}(\mathcal{V}) \subset \mathcal{D}(\Delta_{\mathrm{B}})$ continuously.*

Proof. First, we show that whenever $u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$ and $\operatorname{tr} \nabla^2 u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$, then $u \in \mathcal{D}(\Delta_{\mathrm{B}})$ and $\Delta_{\mathrm{B}} u = -\operatorname{tr} \nabla^2 u$. For that, fix $v \in \mathrm{C}^\infty_c(\mathcal{V})$ and let $w \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ with $\operatorname{tr} \nabla w \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$. Then, since $x \mapsto \mathrm{h}(v(x), -\operatorname{tr} \nabla w(x))$ is compactly supported, we have $\langle -\operatorname{tr} \nabla w, v\rangle = \langle w, \nabla_c v\rangle$. Furthermore, by an application of the Cauchy-Schwarz inequality, we find that $\nabla u \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Thus, upon putting $w = \nabla u$, we have that $\left\langle -\operatorname{tr} \nabla^2 u, v\right\rangle = \langle \nabla u, \nabla_c v\rangle$. Again, since $x \mapsto \mathrm{g}(\nabla u(x), \nabla_c v(x))_x$ is compactly supported, we have that $\langle \nabla u, \nabla_c v\rangle = \left\langle u, -\operatorname{tr} {\nabla_c}^2 v\right\rangle$. In particular, for our choice of v, we have that $\Delta_{\mathrm{B}} v = -\operatorname{tr} {\nabla_c}^2 v$ and since Δ_{B} is self-adjoint, $u \in \mathcal{D}(\Delta_{\mathrm{B}})$ and $\Delta_{\mathrm{B}} u = -\operatorname{tr} \nabla^2 u$.

Now, we show that $\mathrm{W}^{2,2}(\mathcal{V}) \subset \mathcal{D}(\Delta_B)$ continuously. So, fix $u \in \mathrm{W}^{2,2}(\mathcal{V})$ and note that as a consequence of Proposition 2.2.14, there exists $u_n \in \mathrm{C}^\infty \cap \mathrm{L}^2(\mathcal{V})$ such that $u_n \to u$, and $\Xi_2 u_n \to \overline{\Xi_2} u$. In particular, $(\Xi_2 u_n)$ is Cauchy and therefore, $\|\nabla^2 u_n - \nabla^2 u_m\| \to 0$. Coupling this observation with the fact that $|\operatorname{tr} \nabla^2 u_n| \leq |\nabla^2 u_n|$, and since $\Delta_{\mathrm{B}} u_n = -\operatorname{tr} \nabla^2 u_n$, we have that $\Delta_{\mathrm{B}} u_n \to v$. But Δ_{B} is closed and hence, $u \in \mathcal{D}(\Delta_{\mathrm{B}})$ and $v = \Delta_{\mathrm{B}} u$. The continuity of the embedding follows easily.

Since $\mathrm{W}^{2,2}_0(\mathcal{V}) \subset \mathrm{W}^{2,2}(\mathcal{V})$ is a continuous embedding, we conclude that $\mathrm{W}^{2,2}_0(\mathcal{V}) \subset \mathcal{D}(\Delta_{\mathrm{B}})$ continuously. $\square$

2.3 Exterior, interior derivatives and the Dirac operator

For technical clarity, let us denote the exterior and interior derivatives by d_∞ and δ_∞ with domains $\mathcal{D}(\mathrm{d}_\infty) = \mathcal{D}(\delta_\infty) = \mathrm{C}^\infty(\mathbf{\Omega}(\mathcal{M}))$. Recall that they satisfy the local expressions

$$\mathrm{d}_\infty \, \omega = dx^i \wedge \nabla_{\partial_i}\omega \quad \text{and} \quad \delta_\infty \omega = -dx^i \llcorner \nabla_{\partial_i}\omega.$$

On p-forms, we have that $\mathrm{d}_\infty : \mathrm{C}^\infty(\mathbf{\Omega}^p(\mathcal{M})) \to \mathrm{C}^\infty(\mathbf{\Omega}^{p+1}(\mathcal{M}))$ and that $\delta_\infty : \mathrm{C}^\infty(\mathbf{\Omega}^p(\mathcal{M})) \to \mathrm{C}^\infty(\mathbf{\Omega}^{p-1}(\mathcal{M}))$. By d_c and δ_c, denote the operators d_∞ and δ_∞ but with $\mathcal{D}(\mathrm{d}_c) = \mathcal{D}(\delta_c) = \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$. Recall also that

$$\langle \mathrm{d}_c \, \omega, \eta \rangle = \langle \omega, \delta_c \eta \rangle$$

for $\omega, \eta \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$. Thus, d_c and δ_c are densely-defined, closable operators. We denote the closures by d_0 and δ_0 respectively. Also, let $\mathrm{d} = \delta_c^*$ and $\delta = \mathrm{d}_c^*$. Then, we have the following theorem.

Theorem 2.3.1. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with smooth metric g and Levi-Cevita connection ∇. Then, $\mathrm{d}_0 = \mathrm{d}$ and $\delta_0 = \delta$. In other words, $\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$ is dense in $\mathcal{D}(\mathrm{d})$ and in $\mathcal{D}(\delta)$. Furthermore, $\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$ is dense in $\mathcal{D}(\mathrm{d}\,\delta)$ and in $\mathcal{D}(\delta\,\mathrm{d})$.*

Proof. Define

$$\Pi_\infty = \begin{pmatrix} 0 & \delta_\infty \\ \mathrm{d}_\infty & 0 \end{pmatrix}$$

with $\mathcal{D}(\Pi_\infty) = \mathrm{C}^\infty(\mathbf{\Omega}(\mathcal{M})) \oplus \mathrm{C}^\infty(\mathbf{\Omega}(\mathcal{M}))$. Then, let Π_c to be the operator Π_∞ with $\mathcal{D}(\Pi) = \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M})) \oplus \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$.

First, we show that Π_c is symmetric. Fix $u = (u_1, u_2), v = (v_1, v_2) \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}))$, and note that $\Pi_c u = (\delta_c u_2, \mathrm{d}_c \, u_1)$, $\Pi_c v = (\delta_c v_2, \mathrm{d}_c \, v_1)$. Then,

$$\langle \Pi_c u, v \rangle = \langle \delta_c u_2, v_1 \rangle + \langle \mathrm{d}_c \, u_1, v_2 \rangle = \langle u_2, \mathrm{d}_c \, v_1 \rangle + \langle u_1, \delta_c v_1 \rangle = \langle u, \Pi_c v \rangle .$$

Now, fix $v \in \mathrm{T}^*\mathcal{M}$ and $e \in \boldsymbol{\Omega}_x(\mathcal{M})$. Let $g \in \mathrm{C}^\infty(\mathcal{M})$ and $f \in \mathrm{C}^\infty(\boldsymbol{\Omega}(\mathcal{M})) \oplus \mathrm{C}^\infty(\boldsymbol{\Omega}(\mathcal{M}))$ such that $\mathrm{d}\, g(x) = v$ and $f = (f_1, f_2)$ and $f(x) = e = (e_1, e_2)$. Then,

$$\sigma(x,v)e = [\Pi_\infty, gI]\, f(x) = \Pi_\infty(gf) - g\Pi_\infty f = (\delta_\infty(gf_2) - g\delta_\infty f_2, \mathrm{d}_\infty(gf_1) - g\,\mathrm{d}_\infty f_1).$$

Note then that,

$$\mathrm{d}_\infty(gf_1) - g\,\mathrm{d}_\infty f_1 = (\mathrm{d}_\infty g \wedge f_1 + g\,\mathrm{d}_\infty f_1) - g\,\mathrm{d}_\infty f_1 = \mathrm{d}_\infty g \wedge f_1.$$

Thus, at x, $\mathrm{d}_\infty g \wedge f_1 = v \wedge e_1$. By a similar calculation,

$$\delta_\infty(gf_2) - g\delta_\infty f_2 = (g\delta_\infty f_2 - \mathrm{d}_\infty g \llcorner f_2) - g\delta_c f_2 = -\,\mathrm{d}_\infty g \llcorner f_2$$

so that at x, $\mathrm{d}_\infty g \llcorner f_2 = v \llcorner e_2$. We combine these two calculations so that

$$|\sigma(x,v)e|^2 = |v \wedge e_1|^2 + |v \llcorner e_2|^2 \le |v|^2 \left(|e_1|^2 + |e_2|^2\right) \le |v|^2 |e|^2 .$$

Therefore, $|\sigma(x,v)| = |v|$ and thus, $\mathrm{c}(r) = 1$. Therefore, by Theorem 2.1.3, we conclude that Π_c and powers of Π_c are essentially self-adjoint. Thus, $\Pi_c^* = \overline{\Pi}_c^* = \overline{\Pi_c}$ and since

$$\Pi_c^* = \begin{pmatrix} 0 & \delta \\ \mathrm{d} & 0 \end{pmatrix},$$

we conclude that $\overline{\mathrm{d}}_c = \mathrm{d}$ and $\overline{\delta}_c = \delta$.

Next, note that

$$\Pi_c^2 = \begin{pmatrix} \delta_c\,\mathrm{d}_c & 0 \\ 0 & \mathrm{d}_c\,\delta_c \end{pmatrix}$$

and by similar reasoning as above, we have that $\overline{\delta_c\,\mathrm{d}_c}$ and $\overline{\mathrm{d}_c\,\delta_c}$ are self-adjoint. It is easy to see that $\overline{\delta_c\,\mathrm{d}_c} \subset \delta\,\mathrm{d}$ and $\overline{\mathrm{d}_c\,\delta_c} \subset \mathrm{d}\,\delta$. Furthermore, $\delta\,\mathrm{d}$ and $\mathrm{d}\,\delta$ are self-adjoint and so by Proposition 2.1.1, we conclude that $\overline{\delta_c\,\mathrm{d}_c} = \delta\,\mathrm{d}$ and $\overline{\mathrm{d}_c\,\delta_c} = \mathrm{d}\,\delta$. □

Also, recall that the *(smooth) Hodge-Dirac* operator $\mathrm{D}_\infty : \mathrm{C}^\infty(\boldsymbol{\Omega}(\mathcal{M})) \to \mathrm{C}^\infty(\boldsymbol{\Omega}(\mathcal{M}))$ is given by $\mathrm{D}_\infty = \mathrm{d}_\infty + \delta_\infty$. Let $\mathrm{D}_c = \mathrm{d}_c + \delta_c$ with domain $\mathcal{D}(\mathrm{D}_c) = \mathrm{C}^\infty_\mathrm{c}(\boldsymbol{\Omega}(\mathcal{M}))$. An easy calculation shows that

$$\langle \mathrm{D}_c\,\omega, \eta\rangle = \langle \omega, \mathrm{D}_c\,\eta\rangle$$

when $\omega, \eta \in \mathrm{C}^\infty_\mathrm{c}(\boldsymbol{\Omega}(\mathcal{M}))$. Thus, D_c is a densely-defined, closable operator. Denote its closure by D_0 and let $\mathrm{D} = \mathrm{d} + \delta$. The latter operator is the *Hodge-Dirac* operator with domain $\mathcal{D}(\mathrm{D}) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\delta)$. As a consequence of Theorem 2.3.1, D is self-adjoint. The *Hodge-Laplacian* is then denoted by D^2. The following theorem is well known and it is presented in [19] as an application. However, we include it here for completeness.

Theorem 2.3.2. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with smooth metric g endowed with the Levi-Cevita connection ∇. Then, we have that $\mathrm{D}_0 = \mathrm{D}$ and $\overline{\mathrm{D}_c^2} = \mathrm{D}^2$. In other words, $\mathrm{C}_\mathrm{c}^\infty(\boldsymbol{\Omega}(\mathcal{M}))$ is dense in $\mathcal{D}(\mathrm{D})$ and in $\mathcal{D}(\mathrm{D}^2)$.*

Proof. It is straightforward to check that D_c is symmetric. The computation of the symbol was done in the proof of Theorem 2.3.1. Explicitly

$$\sigma(x, v)e = v \wedge e - v \llcorner e,$$

and thus $c(r) = 1$. Thus, by Theorem 2.1.3, we have that D_0 is self-adjoint. But $\mathrm{D}_0 \subset \mathrm{D}$ and D is self-adjoint so by Proposition 2.1.1, we have that $\mathrm{D}_0 = \mathrm{D}$. Similarly, $\overline{\mathrm{D}_c^2} = \mathrm{D}^2$. □

2.4 Second-order Sobolev spaces on manifolds

In this short section, for smooth complete manifolds endowed with the Levi-Cevita connection, we show that $\mathrm{W}_0^{2,2}(\mathcal{M}) = \mathrm{W}^{2,2}(\mathcal{M})$ whenever $\mathrm{Ric} \geq \eta \mathrm{g}$ for some $\eta \in \mathbb{R}$. This improves the current result recorded by Hebey in [33] which requires the additional assumption that $\mathrm{inj}(\mathcal{M}) \geq \kappa > 0$. This is the content of Proposition 3.3 in [33]. This theorem is a consequence of the results which we have obtained in §2.2.

Theorem 2.4.1. *Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric g and Levi-Cevita connection ∇. If there exists $\eta \in \mathbb{R}$ such that $\mathrm{Ric} \geq \eta \mathrm{g}$, then $\mathrm{W}_0^{2,2}(\mathcal{M}) = \mathrm{W}^{2,2}(\mathcal{M})$.*

Proof. We note that the hypotheses of this theorem also satisfy those of Theorem 2.2.6 and Proposition 2.2.15 with $\mathcal{V}$ as $\mathcal{M} \times \mathbb{C}$. By the latter result, we have that $\mathrm{W}_0^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_\mathrm{B})$ and $\mathrm{W}^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_\mathrm{B})$ continuously.

Next, recall the following Bochner-Lichnerowicz-Weitzenböck identity

$$\langle \nabla \Delta u, \nabla u \rangle = \frac{1}{2} \Delta |\nabla u|^2 + \left|\nabla^2 u\right|^2 + \mathrm{Ric}(\nabla u, \nabla u)$$

for $u \in \mathrm{C}^\infty(\mathcal{M})$. By using the bound $\mathrm{Ric} \geq \eta \mathrm{g}$, we obtain that $\|\nabla^2 u\| \lesssim \|(I+\Delta)u\|$ whenever $u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$. Thus, coupled with the fact that $\Delta_0 = \Delta_{\mathrm{B}}$ by Theorem 2.2.6, we find the the reverse inclusion $\mathcal{D}(\Delta_B) \subset \mathrm{W}^{2,2}_0(\mathcal{M})$ holds and this allows us to conclude that $\mathcal{D}(\Delta_B) = \mathrm{W}^{2,2}_0(\mathcal{M})$.

Combining these set inclusions, we have that $\mathrm{W}^{2,2}(\mathcal{M}) \subset \mathcal{D}(\Delta_B) = \mathrm{W}^{2,2}_0(\mathcal{M})$. But $\mathrm{W}^{2,2}_0(\mathcal{M}) \subset \mathrm{W}^{2,2}(\mathcal{M})$ continuously and so we conclude that $\mathrm{W}^{2,2}_0(\mathcal{M}) = \mathrm{W}^{2,2}(\mathcal{M})$. $\square$

3

The quadratic estimates

In this chapter, we prove quadratic estimates (Qest) in two settings - on measure metric spaces of polynomial growth and for such spaces with exponential growth. These estimates depend on the hypotheses (H1)-(H6) which we have already described in §1.8 and two additional geometric hypotheses which we present here.

In the case of polynomial growth on trivial bundles, these hypotheses are (H7-H) and (H8-H). Under these assumptions, we obtain *global quadratic estimates*. For the more general exponential case, we are able to permit the bundle to be non-trivial but bounded. In this setting, we are able to at best prove *local quadratic estimates* under two additional hypotheses (H7-I) and (H8-I).

We acknowledge here the authors of [8] and of [46] whose work provided vital motivation and guidance in obtaining the homogeneous estimates. The material in [46] is of paramount importance in the inhomogeneous case. In fact, we use much of the technology from [46] to obtain these latter estimates. We will cite the exact results from [8] and [46] as we present the proofs of these results throughout this chapter.

3.1 Polynomial growth and trivial bundles

In this section, we obtain global quadratic estimates generalising the quadratic estimates in the paper [8]. It is these estimates that are employed in Chapter 4 to obtain *homogeneous* estimates for the Kato square root problem for subelliptic operators on Lie groups, thus extending the Kato square root problem on $\mathbb{R}^n$.

While the structure of our proof is based on the proof in [8], our proof here does not follow immediately from previous work. As we have previously mentioned, we have divorced the assumptions on the operator Γ from requiring the underlying space to have a differentiable structure. This is a significantly different scenario from the proof in $\mathbb{R}^n$ and requires a more subtle formulation of the framework in which we work.

The metric and measure in the proof of [8] were strongly related, with the measure arising as the canonical measure from the Riemannian metric (the Euclidean inner product) on $\mathbb{R}^n$. We have allowed for more general measures and metrics in our proof. This is of vital importance in the study of subelliptic problems on Lie groups in Chapter 4 as the sub-Riemannian structure requires this weaker interaction between the measure and metric.

The proof in [8] relies upon significant ideas from modern harmonic analysis, including covering theorems, maximal functions, tent spaces, and Carleson measures. In order to prove a generalisation of the quadratic estimates in [8], we have had to formulate and verify such results in our measure metric space setting. While it is possible that these technical results are not new, we present them in detail in a later sub-section.

We obtain the global quadratic estimates under the hypothesis that $\mathcal{V}$ is trivial, in addition to assuming the measure grows at most polynomially.

As a consequence of the triviality assumption, throughout this section, we assume that $\mathcal{V} = \mathbb{C}^N$ for some $N \in \mathbb{N}$. As we have previously stated, we assume that the measure μ is at most polynomial in growth.

Let us now introduce the two geometric hypotheses. The following is the first, and it is called the *cancellation hypothesis.*

(H7-H) For each open ball B, we have

$$\int_B \Gamma u \, d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \, d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with spt $u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with spt $v \subset B$.

Next, we have the following *Poincaré hypothesis*, which contains as a part of it a *coercivity hypothesis*.

(H8-H) -1 (Poincaré hypothesis)

There exists $C' > 0$, $c > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2(\mathcal{N})$ where $\mathcal{N}$ is a normed bundle with norm $|\cdot|_{\mathcal{N}}$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B |\Xi u|_{\mathcal{N}}^2 \, d\mu$$

for all balls $B = B(x, r)$ and $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

-2 (Coercivity hypothesis)

There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,

$$\|\Xi u\|_{\mathrm{L}^2(\mathcal{N})} \leq \tilde{C} \|\Pi u\|.$$

Under these hypotheses, the main theorem we prove in this section is the following.

Theorem 3.1.1. *Let $(\mathcal{X}, d, \mu)$ be a complete measure metric space which is at most polynomial in growth, $\mathcal{V} = \mathbb{C}^N$, and (Γ, B_1, B_2) satisfy (H1)-(H6), (H7-H) and (H8-H). Then, Π_B satisfies the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{X}, \mathbb{C}^N)$

3.1.1 Consequences of the dyadic decomposition

Recall that as a consequence of the polynomial growth assumption (Pol), the space $\mathcal{X}$ admits a dyadic decomposition as described in Theorem 1.3.2. Here, we prove some important estimates which are a consequence of the doubling property using the dyadic decomposition. We recall $C_1, C_2, a_0, \delta, \eta, \mathscr{Q}^k, I_k$ and Q_α^k from Theorem 1.3.2.

Define $\mathscr{Q}^k = \{Q_\alpha^k : \alpha \in I_k\}$ to be the *level k dyadic cubes* and $\mathscr{Q} = \cup_k \mathscr{Q}^k$ to be the collection of dyadic cubes. For $Q_\alpha^k \in \mathscr{Q}^k$, define the *length* as $\ell(Q_\alpha^k) = \delta^k$ and the *centre* as z_α^k.

It is easy to see that each $\mathscr{Q}^k$ is a mutually disjoint collection. Furthermore, we have $\partial(\cup\mathscr{Q}^k) = \cup_{Q\in\mathscr{Q}^k}\partial Q$. These facts coupled with the assumption $\mu(B(x,r)) > 0$ implies that $\mathcal{X} = \overline{\cup\mathscr{Q}^k}$.

Fix a cube $Q \in \mathscr{Q}^j$ and denote the centre of this cube by z. We are interested in counting the number of cubes inside "shells" centred from this cube. We begin with the following definition.

Definition 3.1.2. *Whenever $k \geq 1$, define*

$$\mathscr{C}_k = \left\{Q^j_\alpha \in \mathscr{Q}^j : (k-1)C_1\delta^j \leq d(z, z^j_\alpha) \leq kC_1\delta^j\right\}.$$

Also, let $\tilde{\mathscr{C}_k} = \{Q^j_\alpha \in \mathscr{Q}^j : d(z, z^j_\alpha) \leq kC_1\delta^j\}$.

It is easy to see that $\mathscr{Q}^j = \cup_{k\geq 1}\mathscr{C}_k$. We compute a bound for $\operatorname{card}\mathscr{C}_k$ (where $\operatorname{card} S$ denotes the *cardinality* of a set S). First, we have the following proposition describing the distance of points in $\cup\mathscr{C}_k$ to z.

Proposition 3.1.3. *Let $Q^j_\alpha \in \mathscr{C}_k$. Then,*

(i) $0 \leq d(z,x) \leq (k+1)C_1\delta^j$ for all $x \in Q^j_\alpha$ when $k \leq 2$, and

(ii) $\frac{1}{3}kC_1\delta^j \leq d(z,x) \leq (k+1)C_1\delta^j$ for all $x \in Q^j_\alpha$ when $k \geq 3$.

Proof. Fix $Q^j_\alpha \in \mathscr{C}_k$ and fix $x \in Q^j_\alpha$. Then,

$$d(x,z) \leq d(x, z^j_\alpha) + d(z^j_\alpha, z) \leq \operatorname{diam} Q^j_\alpha + kC_1\delta^j \leq (k+1)C_1\delta^j.$$

Also,

$$(k-1)C_1\delta^j \leq d(z, z^j_\alpha) \leq d(x,z) + d(x, z^j_\alpha) \leq d(x,z) + C_1\delta^j.$$

Combining these two estimates we have

$$(k-2)C_1\delta^j \leq d(z,x) \leq (k+1)C_1\delta^j.$$

This gives us (i). To obtain (ii), note that whenever $k \geq 3$ we have $\frac{1}{3}k \leq k-2$. $\square$

Next, we compare two balls which are separated by an arbitrary distance. In the following proposition (and indeed the rest of the section), let us fix p to be the constant arising in the polynomial growth condition (Pol).

Proposition 3.1.4. *Fix balls $B(x,r), B(y,r) \subset \mathcal{X}$. Then, for all $\varepsilon > 0$,*

$$2^{-p}\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^{-p}\mu(B(y,r)) \leq \mu(B(x,r)) \\ \leq 2^{p}\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^{p}\mu(B(y,r)).$$

Proof. Fix $\varepsilon > 0$ and note that

$$B(x,r), B(y,r) \subset B(x, d(x,y)+r+\varepsilon), B(y, d(x,y)+r+\varepsilon).$$

Therefore,

$$\mu(B(y,r)) \leq \mu\left(B\left(x, \frac{d(x,y)+r+\varepsilon}{r}r\right)\right) \\ \leq 2^{p}\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^{p}\mu(B(x,r)).$$

Similarly, we have

$$\mu(B(x,r)) \leq \mu\left(B\left(y, \frac{d(x,y)+r+\varepsilon}{r}r\right)\right) \\ \leq 2^{p}\left(\frac{d(x,y)+r+\varepsilon}{r}\right)^{p}\mu(B(y,r))$$

which establishes the claim. □

Let us make a parenthetical remark regarding our assumption $0 < \mu(B(x,r)) < \infty$ for all $x \in \mathcal{X}$ and $r > 0$. By coupling the previous proposition with the doubling property, we are able to recover this assumption if we only required $0 < \mu(B(x_0,r_0)) < \infty$ to hold for some $x_0 \in \mathcal{X}$ and $r_0 > 0$. Thus, this assumption is weaker than it originally appears.

We now return back to the problem of estimating $\operatorname{card}\mathscr{C}_k$. The reader will observe that we have been generous in our calculations.

Proposition 3.1.5. *We have* $\operatorname{card}\tilde{\mathscr{C}_k} \leq Ck^{2p}$ *where*

$$C = 4^p\left(\frac{C_1+2a_0}{a_0}\right)^p\left(\frac{2C_1}{a_0}\right)^p.$$

In particular, $\operatorname{card}\mathscr{C}_k \leq Ck^{2p}$.

Proof. Fix $k \geq 1$. Set $\varepsilon = r = a_0\delta^j$ and then

$$d(z, z_\alpha^j) + r + \varepsilon \leq kC_1\delta^j + 2a_0\delta^j \leq (C_1 + 2a_0)\delta^j k$$

when $Q_\alpha^j \in \tilde{\mathscr{C}_k}$. By Proposition 3.1.4,

$$2^{-p}\left(\frac{C_1 + 2a_0}{a_0}\right)^{-p} k^{-p}\mu(B(z, a_0\delta^j)) \leq \mu(B(z_\alpha^k, a_0\delta^j)).$$

Now, note that by Proposition 3.1.3, we have $\sup_{x\in Q_\alpha^j} d(x,z) \leq (k+1)C_1\delta^j$ and so $\cup\tilde{\mathscr{C}_k} \subset B(z, (k+1)C_1\delta^j)$. Then,

$$\begin{aligned}\mu(B(z,(k+1)C_1\delta^j)) &\leq 2^p\left(\frac{(k+1)C_1}{a_0}\right)^p \mu(B(z, a_0\delta^j))\\ &\leq 2^p\left(\frac{2C_1}{a_0}\right)^p k^p\mu(B(z, a_0\delta^j)).\end{aligned}$$

Since $\mu(B(z, a_0\delta^j)) < \infty$ and by combining the two estimates, and the fact that $B(z_\alpha^k, a_0\delta^j) \subset Q_\alpha^j$ for each $Q_\alpha^j \in \tilde{\mathscr{C}_k}$, we compute

$$\begin{aligned}\operatorname{card}\mathscr{C}_k &\leq 2^p\left(\frac{2C_1}{a_0}\right)^p k^p\, 2^p\left(\frac{C_1+2a_0}{a_0}\right)^p k^p\\ &= 4^p\left(\frac{C_1+2a_0}{a_0}\right)^p\left(\frac{2C_1}{a_0}\right)^p k^{2p}.\end{aligned}$$

The observation that $\mathscr{C}_k \subset \tilde{\mathscr{C}_k}$ completes the proof. □

We have the following important consequences. They are useful in many of the calculations we perform in §3.1.3. Following the notation in [8], we write $\langle x\rangle = 1 + |x|$.

Corollary 3.1.6. *Fix $\delta^{j+1} < t \leq \delta^j$ and a cube $Q \in \mathscr{Q}^j$. Then,*

$$\sum_{R\in\mathscr{Q}^j}\left\langle\frac{\operatorname{dist}(R,Q)}{t}\right\rangle^{-M} \leq C\left(1 + 4^p + \left(\frac{3}{C_1}\right)^M\sum_{k=3}^{\infty}k^{2p-M}\right)$$

with C being the constant in the previous proposition.

Proof. First, we note that

$$1 \leq 1 + \frac{\operatorname{dist}(R,Q)}{t} \quad \text{and} \quad \frac{\operatorname{dist}(R,Q)}{\delta^j} \leq 1 + \frac{\operatorname{dist}(R,Q)}{t}.$$

Then,

$$\begin{aligned}\sum_{R \in \mathscr{Q}^j} \left\langle \frac{\operatorname{dist}(R,Q)}{t} \right\rangle^{-M} &\leq \operatorname{card} \mathscr{C}_1 + \operatorname{card} \mathscr{C}_2 + \sum_{k=3}^{\infty} \sum_{R \in \mathscr{C}_k} \left(\frac{\delta^j}{d(R,Q)} \right)^M \\ &\leq C + C2^{2p} + \sum_{k=3}^{\infty} \operatorname{card} \mathscr{C}_k \left(\frac{\delta^j}{\frac{1}{3} k C_1 \delta^j} \right)^M \\ &\leq C \left(1 + 4^p + \left(\frac{3}{C_1} \right)^M \sum_{k=3}^{\infty} k^{2p-M} \right).\end{aligned}$$

□

Corollary 3.1.7. *For each $M > 2p+1$, there exists a constant $A_M > 0$ such that*

$$\sup_Q \sum_{R \in \mathscr{Q}^j} \left\langle \frac{\operatorname{dist}(R,Q)}{t} \right\rangle^{-M} \leq A_M.$$

3.1.2 Maximal functions and Carleson Measures

In this section, we present some results regarding maximal functions, Carleson measures and related topics in our measure metric space setting that is crucial to the proof of the quadratic estimates. A full treatment of the classical theory of maximal functions and Carleson measures can be found in §4 of [52] by Stein. We do not claim that the results we present here are new - they are stated and obtained by straightforward changes to their classical counterparts from this source. We include this material here to make the proof of the quadratic estimates essentially self contained. We further refer to [34] by Heinonen and [21] by Coifman and Weiss as two excellent expositions that touch on some of the issues and ideas presented here.

For a measurable subset S with $0 < \mu(S) < \infty$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}, \mathbb{C}^N)$, we define the *average* of f on S by $\fint_S f = \mu(S)^{-1} \int_S f$. Then, we make the following definition.

Definition 3.1.8 (Maximal function)**.** *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}, \mathbb{C}^N)$. Define the uncentred maximal function of f by:*

$$\mathcal{M} f(x) = \sup_{B \ni x} \fint_B |f| \, d\mu$$

where the supremum is taken over all balls B containing x.

We want to deduce that this $\mathcal{M}$ exhibits a weak type $(1,1)$ estimate and is bounded in $\mathrm{L}^p(\mathcal{X},\mathbb{C}^N)$ for $p>1$. The proof of the following theorem is standard via the Vitali type covering Theorem 1.2 in [21].

Theorem 3.1.9 (Maximal theorem)**.** *There exists a constant $C_1>0$ such that whenever $f\in\mathrm{L}^1(\mathcal{X},\mathbb{C}^N)$, we have*

$$\mu(\{x\in\mathcal{X}:\mathcal{M}f(x)>\alpha\})\leq\frac{C_1}{\alpha}\int_{\mathcal{X}}|f|\ d\mu.$$

Whenever $f\in\mathrm{L}^q(\mathcal{X},\mathbb{C}^N)$ with $q>1$,

$$\|\mathcal{M}f\|_q\leq C_q\|f\|_q$$

where $C_q>0$ is a constant.

In order to set up a theory of Carleson measures, we require an *upper half space.* We define this to be $\mathcal{X}_+=\mathcal{X}\times\mathbb{R}^+$ where $\mathbb{R}^+=(0,\infty)$. The *cone* over a point $x\in\mathcal{X}$ is then defined as $\Gamma(x)=\{(y,t)\in\mathcal{X}_+:d(x,y)<t\}$ which leads to the following.

Definition 3.1.10 (Nontangential maximal function)**.** *Let $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathcal{X}_+,\mathbb{C}^N)$. Define*

$$\mathcal{M}^*f(x)=\sup_{(y,t)\in\Gamma(x)}|f(y,t)|\,.$$

Like its classical counterpart, this maximal function is measurable, as the following proposition illustrates.

Proposition 3.1.11. *The set $\{x\in\mathcal{X}:\mathcal{M}^*f(x)>\alpha\}$ is open and hence $\mathcal{M}^*f$ is measurable.*

Proof. Fix $x\in\mathcal{X}$ with $\mathcal{M}^*f(x)>\alpha$. Then, there exists a $(y,t)\in\Gamma(x)$ such that $|f(y,t)|>\alpha$. Consider the ball $B(y,t)$ and take any $z\in B(y,t)$. Note that since $d(z,y)<t$ we have $(y,t)\in\Gamma(z)$ and so $\mathcal{M}^*f(z)>\alpha$. Therefore, $x\in B(y,t)\subset\{x\in\mathcal{X}:\mathcal{M}^*f(x)>\alpha\}$. $\square$

As a consequence of this proposition, we define the following function space in an analogous way to the classical theory.

Definition 3.1.12 (Nontangential function space)**.** *Let $\mathcal{N}$ denote the space of Borel measurable functions $f : \mathcal{X}_+ \to \mathbb{C}$ such that $\mathcal{M}^* f \in \mathrm{L}^1(\mathcal{X})$. We equip this space with the norm $\|f\|_{\mathcal{N}} = \|\mathcal{M}^* f\|_1$.*

Now, let $B = B(x, r)$ and define the *tent over B* as

$$\mathrm{T}(B) = \{(y, t) \in \mathcal{X}_+ : d(x, y) \leq r - t\}.$$

For an arbitrary open set $O \subset \mathcal{X}$, we define the *tent over O* by $\mathrm{T}(O) = \mathcal{X}_+ \setminus \cup_{x \in \mathcal{X} \setminus O} \Gamma(x)$. The following is an equivalent characterisation of $\mathrm{T}(O)$.

Proposition 3.1.13. *Whenever $(x, t) \in \mathrm{T}(O)$ we have that*

$$(x, t) \in \mathrm{T}(B(x, d(x, \mathcal{X} \setminus O)))$$

and in particular, $\mathrm{T}(O) = \cup_{x \in O} \mathrm{T}(B(x, d(x, \mathcal{X} \setminus O)))$.

Proof. First, note that by de Morgen's law, we can conclude that $\mathrm{T}(O) = \cap_{y \in \mathcal{X} \setminus O} \mathcal{X}_+ \setminus \Gamma(y)$. Fix $(x, t) \in \mathrm{T}(O)$. So, $(x, t) \in \mathcal{X}_+ \setminus \Gamma(y)$ for all $y \in \mathcal{X} \setminus O$. That is, for all $y \notin O$, we have $(x, t) \notin \Gamma(y)$ which implies $d(x, y) \geq t$. Therefore, $d(x, \mathcal{X} \setminus O) \geq t$. Then, by the definition of $\mathrm{T}(B(x, r))$ and setting $r = d(x, \mathcal{X} \setminus O)$, we conclude $(x, t) \in \mathrm{T}(B(x, d(x, \mathcal{X} \setminus O)))$. The converse inclusion is easy since $B(x, d(x, \mathcal{X} \setminus O)) \subset O$. □

Definition 3.1.14 (Carleson function)**.** *Let ν be any Borel measure on $\mathcal{X}_+$. Define*

$$\mathrm{C}(\nu)(x) = \sup_{B \ni x} \frac{\nu(\mathrm{T}(B))}{\mu(B)}.$$

Definition 3.1.15 (Space of Carleson measures)**.** *We define $\mathcal{C}$ to be the space of measures ν that are Borel on $\mathcal{X}_+$ and such that $\mathrm{C}(\nu)$ is bounded. Such a measure is called a* Carleson measure *and we define*

$$\|\nu\|_{\mathcal{C}} = \sup_{x \in \mathcal{X}} \mathrm{C}(\nu)(x)$$

to be the Carleson norm.

Since we have a dyadic structure, we define the *Carleson box* over $Q \in \mathscr{Q}$ by $\mathrm{R}_Q = \overline{Q} \times (0, \ell(Q)]$. Unlike the classical definition, we are forced to take $\overline{Q}$ since $\mathscr{Q}$ is only guaranteed to cover $\mathcal{X}$ almost everywhere. The importance of this subtlety will become apparent in the proof of the following proposition that provides an alternative characterisation of a Carleson measure.

Proposition 3.1.16. *Let ν be a Borel measure on $\mathcal{X}_+$. Then the statement*

$$\sup_B \frac{\nu(\mathrm{T}(B))}{\mu(B)} < \infty \qquad \text{for every ball } B$$

is equivalent to the statement

$$\sup_Q \frac{\nu(\mathrm{R}_Q)}{\mu(Q)} < \infty \qquad \text{for every } Q \in \mathscr{Q}.$$

Proof. First, fix $Q \in \mathscr{Q}^j$ and let x_Q be its centre. Then, we have that $Q \subset B(x_Q, C_1\delta^j)$. Then, certainly, $\mathrm{R}_Q \subset \mathrm{T}(B(x_Q, (C_1+2)\delta^j))$. So,

$$\begin{aligned}\nu(\mathrm{R}_Q) &\leq \nu(\mathrm{T}(B(x_Q,(C_1+2)\delta^j))) \leq \|\nu\|_{\mathcal{C}}\mu(B(x_Q,(C_1+2)\delta^j)\\ &\leq 2^p\left(\frac{C_1+2}{a_0}\right)^p \|\nu\|_{\mathcal{C}}\mu(B(x_Q,a_0\delta^j)) \leq 2^p\left(\frac{C_1+2}{a_0}\right)^p \|\nu\|_{\mathcal{C}}\mu(Q).\end{aligned}$$

The converse is harder. Fix $B = B(x,r)$ and let $j \in \mathbb{Z}$ such that $\delta^{j+1} < r \leq \delta^j$. Let $N(B) = \{Q \in \mathscr{Q}^j : Q \cap B \neq \varnothing\}$. It is an easy fact that $N(B) \neq \varnothing$.

(i) First, we claim that $B \subset \cup_{Q \in N(B)} \overline{Q}$. Suppose $y \in B$ but $y \notin \cup N(B)$. That is, $y \notin Q$ for all $Q \in \mathscr{Q}^j$. Thus, there exists a $Q \in \mathscr{Q}^j$ such that $y \in \partial Q$. That is, for every $\varepsilon > 0$, $B(y,\varepsilon) \cap Q \neq \varnothing$. But there exists an $\varepsilon > 0$ such that $B(y,\varepsilon) \subset B$, and so $Q \cap B \neq \varnothing$. This means that $Q \in N(B)$ and establishes the claim.

(ii) Fix $Q \in N(B)$ as a reference cube and let $Q' \in N(B)$ be any other cube. Since $r < \delta^j$, we note that $d(x, x_Q), d(x, x_{Q'}) \leq \delta^j + C_1\delta^j$. Therefore, $d(x_Q, x_{Q'}) \leq 2(C_1+1)\delta^j$. That is, all the centres of cubes $Q' \in N(B)$ are inside the ball $B(x_Q, 2(C_1+1)\delta^j)$ and hence $\tilde{\mathscr{C}}_{2(C_1+1)}$. Thus, by Proposition 3.1.5,

$$\operatorname{card} N(B) \leq \operatorname{card} \tilde{\mathscr{C}}_{2(C_1+1)} \leq C2^p(C_1+1)^{2p}.$$

(iii) Now, suppose that $(y,t) \in \mathrm{T}(B)$. That is, $y \in B$ and We have $d(y,t) \leq r - t \leq \delta^j$. By (i), there exists a cube $Q \in N(B)$ such that $y \in \overline{Q}$. Therefore, $(y,t) \in \mathrm{R}_Q = \overline{Q}$ and shows that $\mathrm{T}(B) \subset \cup_{Q \in N(B)} \mathrm{R}_Q$.

(iv) Fix $Q \in N(B)$ and so $d(x, x_Q) \le (C_1 + 1)\delta^j$. Set $\varepsilon = r = \delta^{j+1}$ in Proposition 3.1.4 so that

$$\begin{aligned}\mu(B(x_Q, \delta^{j+1})) &\le 2^p \left(\frac{(C_1+1)\delta^j + 2\delta^{j+1}}{\delta^{j+1}} \right) \mu(B(x, \delta^{j+1})) \\ &\le 2^p ((C_1+1)\delta^{-1} + 2)^p \mu(B(x, r)).\end{aligned}$$

Now, by combining (i) - (iv),

$$\begin{aligned}\nu(\mathrm{T}(B)) \le \sum_{Q \in N(B)} \nu(\mathrm{R}_Q) &\lesssim \sum_{Q \in N(B)} \mu(B(x_Q, C_1\delta^j)) \\ &\lesssim \sum_{Q \in N(B)} \mu(B(x_Q, \delta^{j+1})) \lesssim \operatorname{card} N(B) \mu(B(x,r)) \lesssim \mu(B(x,r))\end{aligned}$$

which completes the proof. □

We quote the following covering theorem of Whitney given as Theorem 1.3 in [21].

Theorem 3.1.17 (Whitney Covering Theorem)**.** *Let $O \subsetneq \mathcal{X}$ be open. Then, there exists a set of balls $\mathscr{E} = \{B_j\}_{j \in \mathbb{N}}$ and a constant $c_1 < \infty$ independent of O such that*

(i) The balls in $\mathscr{E}$ are mutually disjoint,

(ii) $O = \bigcup_{j \in \mathbb{N}} c_1 B_j$,

(iii) $4c_1 B_j \not\subset O$.

This allows us to prove the following theorem which is a generalisation of Carleson's Theorem.

Theorem 3.1.18. *Let $f \in \mathcal{N}$ and $\nu \in \mathcal{C}$. Then,*

$$\iint_{\mathcal{X}_+} |f(x,t)| \ d\nu(x,t) \lesssim \|f\|_{\mathcal{N}} \|\nu\|_{\mathcal{C}}$$

where the constant depends only on p and the Whitney constant c_1.

Proof. (i) We prove $\{(x,t) \in \mathcal{X}_+ : |f(x,t)| > \alpha\} \subset \mathrm{T}(E_\alpha)$ where the set $E_\alpha = \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$. Fix $(x,t) \in \mathcal{X}_+$ such that $|f(x,t)| > \alpha$. Then, whenever $y \in B(x,t)$, we also have $x \in B(y,t)$ and

$$\mathcal{M}^* f(y) = \sup_{t>0} \sup_{z \in B(y,t)} |f(z,t)| > |f(x,t)| > \alpha.$$

Therefore, $B(x,t) \subset E_\alpha$ and $(x,t) \in \mathrm{T}(B(x,t)) \subset \mathrm{T}(E_\alpha)$.

(ii) Let $O \subsetneq \mathcal{X}$ be an open set, and let $\mathscr{E} = \{B_j\}_{j\in\mathbb{N}}$ be the Whitney covering guaranteed by Theorem 3.1.17. We prove that

$$\mathrm{T}(O) \subset \cup_j \mathrm{T}(9c_1 B_j).$$

Fix $x \in O$ and let $(x,t) \in T(B(x, d(x, \mathcal{X} \setminus O)))$. Then, there exists a ball $B_j = B_j(x_j, r_j) \in \mathscr{E}$ such that $x \in c_1 B_j$. Let $y \in B(x, d(x, \mathcal{X} \setminus O))$. Since $4c_1 B_j \cap \mathcal{X} \setminus O$, for any $z \in \mathcal{X} \setminus O$ $d(y, \mathcal{X} \setminus O) \le d(x,z) \le 8c_1 r_j$ Then,

$$d(y, x_j) \le d(y,x) + d(x, x_k) \le d(x, \mathcal{X} \setminus O) + d(x, x_k) < 8c_1 r_j + c_1 r_j = 9c_1 r_j.$$

This proves that $B(x, d(x, \mathcal{X} \setminus O)) \subset 9c_1 B_j$ and so $\mathrm{T}(B(x, d(x, \mathcal{X} \setminus O))) \subset \mathrm{T}(9c_1 B_j)$. We apply Proposition 3.1.13 to conclude that $\mathrm{T}(O) \subset \cup_j \mathrm{T}(9c_1 B_j)$.

(iii) Now, we prove that there exists a constant $C > 0$ such that for all open sets $O \subset \mathcal{X}$,

$$\nu(\mathrm{T}(O)) \le C \|\nu\|_{\mathcal{C}} \mu(O).$$

First assume that $O = \mathcal{X}$. If $\mu(\mathcal{X}) = \infty$, then there is nothing to prove. So suppose otherwise. Now, for any $x \in \mathcal{X}$ and any ball $B_r = B(x,r)$,

$$\frac{1}{\mu(B_r)} \nu(\mathrm{T}(B_r)) \le \mathrm{C}(\nu)(x) \le \|\nu\|_{\mathcal{C}}$$

and therefore, $\nu(\mathrm{T}(B_r)) \le \|\nu\|_{\mathcal{C}} \mu(\mathcal{X})$ for every ball B_r of radius r. Now, $\chi_{\mathrm{T}(B_n)} \le 1$ for each $n \in \mathbb{N}$ and $\chi_{\mathrm{T}(B_n)} \to \chi_{\mathrm{T}(\mathcal{X})}$ and $n \to \infty$ pointwise. Then, by application of Dominated Convergence Theorem,

$$\nu(\mathrm{T}(\mathcal{X})) = \int_{\mathcal{X}_+} \lim_{n\to\infty} \chi_{\mathrm{T}(B_n)}\, d\nu = \lim_{n\to\infty} \int_{\mathcal{X}_+} \chi_{\mathrm{T}(B_n)}\, d\nu \le \|\nu\|_{\mathcal{C}} \mu(\mathcal{X}).$$

Now, consider the case when $O \subsetneq \mathcal{X}$. Then, by (ii) and the subadditivity of the measure,

$$\begin{aligned} \nu(\mathrm{T}(O)) \le \sum_j \nu(\mathrm{T}(9c_1 B_j)) &\le \|\nu\|_{\mathcal{C}} \sum_j \mu(9c_1 B_j) \\ &\le 2^p (9c_1)^p \|\nu\|_{\mathcal{C}} \sum_j \mu(B_j) \le (18c_1)^p \|\nu\|_{\mathcal{C}} \mu(O). \end{aligned}$$

(iv) By (i) and (iii),

$$\nu \{(x,t) \in \mathcal{X}_+ : |f(x,t)| > \alpha\} \lesssim \|\nu\|_{\mathcal{C}}\mu \{x \in \mathcal{X} : \mathcal{M}^* f(x) > \alpha\}$$

and integrating both sides with respect to α completes the proof.

$\square$

3.1.3 Harmonic Analysis of Π_B

Here, we combine the estimates of the previous sections to prove Proposition 1.8.6. As a consequence we obtain Theorem 3.1.1.

Let us first describe some machinery required to perform the harmonic analysis in proving Proposition 1.8.6. Let $\mathscr{Q}_t = \mathscr{Q}^j$ for $\delta^{j+1} < t \leq \delta^j$. Following the structure of the proof in [8], for $t \in \mathbb{R}^+$, we define the *dyadic averaging operator* $\mathcal{A}_t : \mathscr{H} \to \mathscr{H}$ as

$$\mathcal{A}_t(x) = \sum_{Q \in \mathscr{Q}_t} \chi_Q(x) \fint_Q u \, d\mu$$

when $x \in \cup \mathscr{Q}_t$ and 0 elsewhere. A straightforward calculation shows that $\mathcal{A}_t \in \mathcal{L}(\mathscr{H})$ and $\|\mathcal{A}_t\| \leq 1$ uniformly in t. Then, the *principal part* is defined as $\gamma_t(x)w = (\Theta_t^B \omega)(x)$ for $w \in \mathbb{C}^N$ and where $\omega(x) = w$ for all $x \in \mathcal{X}$.

Following [8], to prove Proposition 1.8.6, we need to show that

$$\int_0^\infty \|\Theta_t^B P_t u\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for $u \in \mathcal{R}(\Pi)$. Thus, we follow the paradigm in [8], [7] and [44] and decompose this estimate in the following way:

$$\begin{aligned}\int_0^\infty \|\Theta_t^B P_t u\|^2 \frac{dt}{t} \leq \int_0^\infty \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t u\|^2 \frac{dt}{t} \\ + \int_0^\infty \|\gamma_t \mathcal{A}_t (P_t - I) u\|^2 \frac{dt}{t} + \iint_{\mathcal{X}_+} |\mathcal{A}_t u(x)|^2 |\gamma_t(x)|^2 \frac{d\mu(x) dt}{t}.\end{aligned}$$

The purpose of the first two terms is to reduce the estimate down to the third term which is dealt with a Carleson measure estimate.

We first note the following consequences of the off-diagonal estimates.

Corollary 3.1.19. *Let $Q \in \mathscr{Q}_t$ and $0 < s \leq t$ with U_s as specified in Proposition 1.8.11. Then,*

$$\|U_s u\|_{\mathrm{L}^2(Q)} \leq C_M \sum_{R \in \mathscr{Q}_t} \left\langle \frac{\mathrm{dist}(R, Q)}{s} \right\rangle^{-M} \|u\|_{\mathrm{L}^2(R)}$$

whenever $u \in \mathscr{H}$.

In our setting, it is more convenient to deal with the following function space rather than $\mathrm{L}^2_{\mathrm{loc}}$ as used in [8].

Definition 3.1.20. *We define $\mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathbb{C}^N)$ to be the space of measurable functions $f : \mathcal{X} \to \mathbb{C}^N$ such that on each $Q \in \mathscr{Q}_t$,*

$$\int_Q |f|^2 \ d\mu < \infty.$$

We equip this space with the seminorms $\|\cdot\|_{\mathrm{L}^2(Q)}$ indexed by $\mathscr{Q}_t$.

We have the following observations analogous to those on page 478 in [8]. It follows from Propositions 3.1.3, 3.1.4, 3.1.5 coupled with the off-diagonal estimates and by choosing $M > \frac{5p}{2} + 1$. We recall that p is the constant arising in (Pol).

Corollary 3.1.21. *There exists a $C' > 0$ such that for all $t > 0$, U_t extends to a continuous map $U_t : \mathrm{L}^\infty(\mathcal{X}, \mathbb{C}^N) \to \mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathbb{C}^N)$ with*

$$\|U_t u\|_{\mathrm{L}^2(Q)} \leq C' \mu(Q)^{\frac{1}{2}} \|u\|_{\mathrm{L}^\infty}.$$

Corollary 3.1.22. *We have $\gamma_t \in \mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X}, \mathcal{L}(\mathbb{C}^N))$ and for all $Q \in \mathscr{Q}_t$ satisfy*

$$\fint_Q |\gamma_t(x)|^2_{\mathcal{L}(\mathbb{C}^N)} \ d\mu(x) \leq C'^2$$

In particular, $\|\gamma_t \mathcal{A}_t\|_{\mathcal{L}(\mathscr{H})} \leq C'$ uniformly for all $t > 0$. The constant C' is the same as that of the previous corollary.

Controlling the first term in [8] relies primarily on the weighted Poincaré inequality as given in Lemma 5.4 in [8]. We pursue a similar strategy and begin by noting the following simple consequence of (H8-H).

Lemma 3.1.23 (Dyadic Poincaré inequality)**.** *Whenever $Q \in \mathscr{Q}_t$ and $r \geq C_1 \delta^{-1}$*

we have

$$\int_{B(x_Q,rt)} |u(x) - u_Q|^2 \ d\mu(x) \lesssim r^{p+2} \int_{B(x_Q,crt)} |t\Xi u(x)|^2_{\mathcal{N}} \ d\mu(x)$$

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Xi)$.

This yields the following proposition analogous to Lemma 5.4 in [8].

Proposition 3.1.24 (Weighted Poincaré inequality). *Whenever* $Q \in \mathscr{Q}_t$ *and* $M > p+1$, *we have*

$$\int_{\mathcal{X}} |u(x) - u_Q|^2 \left\langle \frac{d(x,Q)}{t} \right\rangle^{-M} \ d\mu(x) \lesssim \int_{\mathcal{X}} |t\Xi u(x)|^2 \left\langle \frac{d(x,Q)}{t} \right\rangle^{p-M} \ d\mu(x)$$

for all $u \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$, *where the constant depends on* M.

Proof. Observe that for $M > 1$, we have

$$\left\langle \frac{d(x,Q)}{t} \right\rangle^{-M} \leq \frac{2C_1}{\delta} \left\langle \frac{d(x,x_Q)}{t} \right\rangle^{-M}.$$

By evaluating the integral

$$\int_{\mathcal{X}} \int_{\theta(x)}^{\infty} |u(x) - u_Q| \ d\nu(r) \ d\mu(x),$$

where $d\nu(r) = Mr^{-M-1}\, dr$, and invoking Lemma 3.1.23 along with Fubini's Theorem establishes the claim. □

This leads to the following proposition which bounds the first term.

Proposition 3.1.25 (First term inequality). *Whenever* $u \in \mathcal{R}(\Pi)$, *we have*

$$\int_0^{\infty} \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 \ \frac{dt}{t} \lesssim \|u\|^2.$$

We omit the proof since it is very similar to the proof of Proposition 5.5 in [8]. It is a simple matter of verification using Corollary 3.1.7 and invoking the weighted Poincaré inequality along with (H8-H)-2.

We now consider bounding the second term. This relies on a suitable substitution for Lemma 5.6 in [8]. The crux of the argument is to be able to perform a cutoff "close" to the boundary of the dyadic cube in question. First, we define the following sets.

Definition 3.1.26 ($\mathcal{E}_\tau, \tilde{\mathcal{E}}_\tau$). *Let $Q \in \mathscr{Q}_t$ and $\tau \leq t$ Define*

$$\mathcal{E}_\tau = \left\{ x \in Q : d(x, \mathcal{X} \setminus Q) > \frac{a_0 \tau}{2} \right\}, \ \tilde{\mathcal{E}}_\tau = \left\{ x \in Q : d(x, \mathcal{X} \setminus Q) \leq \frac{a_0 \tau}{2} \right\}.$$

The following proposition renders a suitable Lipschitz substitution to the smooth cutoff used in Lemma 5.6 in [8] and Lemma 5.7 in [44].

Proposition 3.1.27. *There exists a Lipschitz function $\xi : Q \to [0,1]$ such that $\xi = 1$ on $\mathcal{E}_\tau$, $\operatorname{spt}\, (\operatorname{Lip} \xi) \subset \tilde{\mathcal{E}}_\tau$, and*

$$\mathbf{Lip}\, \xi \leq \frac{16}{a_0 \tau}.$$

Proof. Set

$$F = \left\{ x \in Q : d(x, \mathcal{X} \setminus Q) \leq \frac{a_0 \tau}{4} \right\}$$

and note that $F \subset \tilde{\mathcal{E}}_\tau$. Then,

$$\frac{a_0 \tau}{2} \leq \operatorname{dist}(\mathcal{X} \setminus Q, \mathcal{E}_\tau) \leq \operatorname{dist}(\mathcal{E}_\tau, F) + \operatorname{dist}(\mathcal{X} \setminus Q, F) \leq \operatorname{dist}(\mathcal{E}_\tau, F) + \frac{a_0 \tau}{4}$$

and so $\operatorname{dist}(\mathcal{E}_\tau, F) > \frac{a_0 \tau}{4}$. By application of Lemma 1.2.1, we find $\xi = 1$ on $\mathcal{E}_\tau$, $\xi = 0$ on $Q \setminus F$ and

$$\mathbf{Lip}\, \xi \leq \frac{4}{\frac{a_0 \tau}{4}} = \frac{16}{a_0 \tau}.$$

Now, fix $x \in \mathcal{E}_\tau$. It is a simple matter to verify that $\mathcal{E}_\tau$ is open and nonempty. So there exists an $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subset \mathcal{E}_\tau$. Therefore,

$$\begin{aligned} \operatorname{Lip} \xi(x) &= \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{d(x,y)} \\ &= \limsup_{\varepsilon \to 0} \left\{ \frac{|\xi(x) - \xi(y)|}{d(x,y)} : y \in \mathcal{E}_\tau \cap B(x, \varepsilon) \setminus \{a\} \right\} = 0. \end{aligned}$$

Thus, $\operatorname{spt}\, \xi \subset \tilde{\mathcal{E}}_\tau$. $\square$

This enables us to prove the following lemma. It is of key importance in bounding the second term, as well as in the Carleson measure estimate which allows us to bound the last term.

Lemma 3.1.28. *Let Υ be Γ, Γ^* or Π. Then, whenever $Q \in \mathscr{Q}_t$,*

$$\left| \fint_Q \Upsilon u \; d\mu \right|^2 \lesssim \frac{1}{t^\eta} \left(\fint_Q |u|^2 \; d\mu \right)^{\frac{\eta}{2}} \left(\fint_Q |\Upsilon u|^2 \; d\mu \right)^{1-\frac{\eta}{2}}$$

where the constant depends only on C_1, C_2, a_0, η and p.

Proof. Let $\tau = \left(\fint_Q |u|^2 \; d\mu \right)^{\frac{1}{2}} \left(\fint_Q |\Upsilon u|^2 \; d\mu \right)^{-\frac{1}{2}}$. The case of $t \leq \tau$ is easy. So, suppose that $\tau \leq t \leq \delta^j$ and let ξ be the Lipschitz function guaranteed in Proposition 3.1.27 extended to 0 outside of Q. and so write

$$\left| \int_Q \Upsilon u \; d\mu \right| \leq \left| \int_Q (1-\xi) \Upsilon u \; d\mu \right| + \left| \int_Q [\xi, \Upsilon] u \; d\mu \right| + \left| \int_Q \Upsilon(\xi u) \; d\mu \right|.$$

The last term is 0 by (H7) and so we are left with estimating the two remaining terms. First, noting that spt $(1-\xi) \subset \tilde{\mathcal{E}}_\tau$ we compute

$$\begin{aligned} \left| \int_Q (1-\xi) \Upsilon u \; d\mu \right| &\leq \left| \int_{\tilde{\mathcal{E}}_\tau} (1-\xi) \Upsilon u \; d\mu \right| \leq \left(\int_{\tilde{\mathcal{E}}_\tau} |\Upsilon u|^2 \; d\mu \right)^{\frac{1}{2}} \mu(\tilde{\mathcal{E}}_\tau) \\ &\leq C_2^{\frac{1}{2}} \left(\frac{a_0 \tau}{2\delta^j} \right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2 \; d\mu \right)^{\frac{1}{2}} \\ &\leq C_2^{\frac{1}{2}} \left(\frac{a_0 \tau}{2t} \right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2 \; d\mu \right)^{\frac{1}{2}}. \end{aligned}$$

Now, for the second term. We note that spt $M_\xi \subset$ spt $\operatorname{Lip} \xi \subset \tilde{\mathcal{E}}_\tau$ and compute

$$\begin{aligned} \left| \int_Q [\xi, \Upsilon] u \right| &= \left| \int_{\tilde{\mathcal{E}}_\tau} M_\xi(x) u(x) \; d\mu(x) \right| \leq \left(\int_{\tilde{\mathcal{E}}_\tau} |M_\xi|^2 \; d\mu \right)^{\frac{1}{2}} \left(\int_{\tilde{\mathcal{E}}_\tau} |u|^2 \; d\mu \right)^{\frac{1}{2}} \\ &\leq \mathbf{Lip}\, \xi \; \mu(\tilde{\mathcal{E}}_\tau)^{\frac{1}{2}} \left(\int_Q |u|^2 \right)^{\frac{1}{2}} \leq \frac{16}{a_0} C_2^{\frac{1}{2}} \left(\frac{a_0 \tau}{2t} \right)^{\frac{\eta}{2}} \frac{1}{\tau} \mu(Q)^{\frac{1}{2}} \left(\int_Q |u|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{16}{a_0} C_2^{\frac{1}{2}} \left(\frac{a_0 \tau}{2t} \right)^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2 \right)^{\frac{1}{2}} \end{aligned}$$

where we have used Cauchy-Schwarz inequality to obtain the first inequality, (H6) in the second, the condition (vi) of Theorem 1.3.2 in the third, and substitution for $\frac{1}{\tau}$ in the last. Combining these estimates, we have

$$\left| \int_Q \Upsilon u \; d\mu \right| \leq D \frac{1}{t^{\frac{\eta}{2}}} \tau^{\frac{\eta}{2}} \mu(Q)^{\frac{1}{2}} \left(\int_Q |\Upsilon u|^2 \; d\mu \right)^{\frac{1}{2}}$$

where

$$D = C_2^{\frac{1}{2}} \left(\frac{a_0}{2}\right)^{\frac{\eta}{2}} + \frac{16}{a_0} C_2^{\frac{1}{2}} \left(\frac{a_0}{2}\right)^{\frac{\eta}{2}} \quad \text{and} \quad \tilde{D} = C(2^p C_1^p a_0^{-p})^{\frac{1}{2}}.$$

By Cauchy-Schwartz and multiplying both sides by $\mu(Q)^{-2}$, we find

$$\left| \fint_Q \Upsilon u \, d\mu \right|^2 \leq 2D^2 \frac{1}{t^\eta} \tau^\eta \fint_Q |\Upsilon u|^2 \, d\mu.$$

The proof is complete by making a substitution for τ^η. □

Proposition 3.1.29 (Second term estimate)**.** *For all $u \in \mathscr{H}$, we have*

$$\int_0^\infty \|\gamma_t \mathcal{A}_t (P_t - I)u\| \, \frac{dt}{t} \lesssim \|u\|^2.$$

Again, the proof of this proposition is omitted since it resembles the proof of Proposition 5.7 in [8] with minor differences.

3.1.4 Carleson measure estimate

We begin this section with the following proposition which illustrates that the final term can be dealt with a Carleson measure estimate.

Proposition 3.1.30. *For all $u \in \mathscr{H}$, and $\nu \in \mathcal{C}$,*

$$\iint_{\mathcal{X}_+} |\mathcal{A}_t u(x)|^2 \, d\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2.$$

Proof. First, we show that for almost every $x \in \mathcal{X}$,

$$\mathcal{M}^* |\mathcal{A}_. u|^2 (x) \lesssim \mathcal{M}u(x)^2$$

where the constant depends only on p, C_1, δ and a_0. Let $f \in \mathrm{L}^1_{\text{loc}}(\mathcal{X}_+, \mathbb{C}^N)$. Then, we note that

$$\mathcal{M}^* f(x) = \sup_{t>0} \sup_{y \in B(x,t)} |f(y,t)|.$$

Fix t such that $\delta^{j+1} < t \leq \delta^j$ and fix $x \in \cup \mathscr{Q}_t$. Since $\mathcal{A}_t u(z) = 0$ when $z \notin \cup \mathscr{Q}_t$, take $y \in \cup \mathscr{Q}_t$ such that $d(x,y) < t$. Let $Q \in \mathscr{Q}_t$ be the unique cube with $y \in Q$ and let $y_Q \in Q$ such that $B(y_Q, a_0 \delta^j) \subset Q \subset B(y_Q, C_1 \delta^j)$. Then, $d(y_Q, x) \leq d(y_Q, y) + d(y,x) \leq Ct$, where $C = (C_1 \delta^{-1} + 1)$.

Also

$$\mu(B(y_Q, Ct)) \leq \mu(B(y_Q, C\delta^j)) \leq 2^p C^p {a_0^{-p}} \mu(B(y_Q, a_0\delta^j)) \leq 2^p C^p {a_0^{-p}} \mu(Q)$$

and therefore,

$$|\mathcal{A}_t u(y)| \leq \fint_Q |u| \ d\mu \leq 2^p C^p a_0^{-p} \fint_{B(y_Q, Ct)} |u| \ d\mu.$$

Moreover,

$$|\mathcal{A}_t u(y)|^2 \leq C' \left(\fint_{B(y_Q, Ct)} |u| \ d\mu \right)^2$$

where $C' = 2^{2p} C^{2p} a_0^{-2p}$.

Now, since we have established that $x \in B(y_Q, Ct)$,

$$\sup_{y \in B(x,t)} |\mathcal{A}_t u(y)|^2 \leq C' \sup_{y \in B(x,t)} \left(\fint_{B(y_Q(y), Ct)} |u| \ d\mu \right)^2 \leq C' (\mathcal{M} u(x))^2.$$

Let $\tilde{\mathcal{X}} = \cap_j \cup \mathcal{Q}^j$ and so $\mu(\mathcal{X} \setminus \tilde{\mathcal{X}}) = \mu(\cup_j \mathcal{X} \setminus \cup \mathcal{Q}^j) \leq \sum_j \mu(\mathcal{X} \setminus \cup \mathcal{Q}^j) = 0$. Therefore, $x \in \tilde{\mathcal{X}}$, then $x \in \cup \mathcal{Q}_t$ for all $t > 0$. So, fix $x \in \tilde{\mathcal{X}}$. Then,

$$\mathcal{M}^* \left| \mathcal{A}_. u \right|^2 (x) = \sup_{t > 0} \sup_{y \in B(x,t)} |\mathcal{A}_t u(y)|^2 \leq C' \mathcal{M} u(x)^2$$

which completes the proof.

Next, let $f(x,t) = |\mathcal{A}_t u(x)|^2$. Then, $\|f\|_{\mathcal{N}} = \|\mathcal{M}^* f\|_1 \lesssim \|\mathcal{M} u\|^2 < \infty$ by the Maximal Theorem 3.1.9. Invoking Carleson's Theorem 3.1.18 completes the proof.

□

Thus, to bound the final term, it suffices to prove

$$A \mapsto \iint_A |\gamma_t(x)|^2 \ d\mu(x) \frac{dt}{t}$$

is a Carleson measure. We follow [8] and fix $\sigma > 0$ to be chosen later. Let

$$K_\nu = \left\{ \nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left| \frac{\nu'}{|\nu'|} - \nu \right| \leq \sigma \right\}$$

and let $\mathcal{F}$ be a finite set of $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$ such that $\cup_{\nu \in \mathcal{F}} K_\nu = \mathcal{L}(\mathbb{C}^N) \setminus \{0\}$.

We note as do the authors of [8] that it is enough to show

$$\iint_{(x,t)\in \mathrm{R}_Q, \gamma_t \in K_\nu} |\gamma_t(x)|^2 \ d\mu(x)\frac{dt}{t} \lesssim \mu(Q)$$

for each $\nu \in \mathcal{F}$. A stopping time argument allows us to reduce this to the following.

Proposition 3.1.31. *There exists a $\beta \in (0,1)$ such that for every dyadic cube $Q \in \mathscr{Q}$ and $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$, there exists a collection $\{Q_k\} \subset \mathscr{Q}$ of disjoint subcubes of Q satisfying $\mu(E_{Q,\nu}) > \beta\mu(Q)$ and such that*

$$\iint_{(x,t)\in E^*_{Q,\nu},\ \gamma_t(x)\in K_\nu} |\gamma_t(x)|^2 \ d\mu(x)\frac{dt}{t} \lesssim \mu(Q)$$

where $E_{Q,\nu} = Q \setminus \cup_k Q_k$ and $E^*_{Q,\nu} = \mathrm{R}_Q \setminus \cup_k \mathrm{R}Q_k$.

A proof of this hinges on defining a test function similar to the one found on page 484 in [8]. Here, the authors use a smooth cutoff function in their construction. Again, we rephrase this in terms of a Lipschitz cutoff function whose existence is guaranteed by the following lemma.

Lemma 3.1.32. *Let $Q \in \mathscr{Q}$. Then, there exists a Lipschitz function $\eta : \mathcal{X} \to [0,1]$ such that $\eta = 1$ on $B(x_Q, \tau C_1\, \ell(Q))$ and $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, 2\tau C_1\, \ell(Q))$ with*

$$\mathbf{Lip}\,\eta \leq \frac{4}{\tau C_1}\frac{1}{\ell(Q)}$$

whenever $\tau > 1$.

Proof. Fix $Q \in \mathscr{Q}^j$, and we have $Q \subset B(x_Q, \tau C_1\delta^j) \subset B(x_Q, 2\tau C_1\delta^j)$. Also,

$$d(B(x_Q, \tau C_1\delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1\delta^j)) \geq (2\tau C_1 - \tau C_1)\delta^j = \tau C_1\delta^j.$$

Now, we invoke Lemma 1.2.1 with $E = B(x_Q, \tau C_1\delta^j)$ and $F = \mathcal{X} \setminus B(x_Q, 2\tau C_1\delta^j)$ to find a Lipschitz $\eta : \mathcal{X} \to [0,1]$ with $\eta = 1$ on $B(x_Q, \tau C_1\delta^j)$, $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, \tau C_1\delta^j)$ and

$$\mathbf{Lip}\,\eta \leq \frac{4}{d(B(x_Q, \tau C_1\delta^j), \mathcal{X} \setminus B(x_Q, 2\tau C_1\delta^j))} \leq \frac{4}{\tau C_1}\frac{1}{\delta^j} = \frac{4}{\tau C_1}\frac{1}{\ell(Q)}$$

which completes the proof. $\square$

The *test function* is now defined as follows. Let $Q \in \mathscr{Q}$ and fix $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$. Let η_Q be the Lipschitz map guaranteed by Lemma 3.1.32 and let $w, \hat{w} \in \mathbb{C}^N$ such that $\nu^*(\hat{w}) = w$ with $|w| = |\hat{w}| = 1$. Furthermore, let $w_Q = \eta_Q w$ and define

$$\begin{aligned} f^w_{Q,\varepsilon} &= w_Q - \varepsilon\,\ell(Q)\imath\Gamma(I + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q \\ &= (1 + \varepsilon\,\ell(Q)\imath\Gamma^*_B)(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q. \end{aligned}$$

It is then an easy fact that $\|w_Q\|^2 \leq (4\tau C_1 a_0^{-1})^p \mu(Q)$ and we obtain the following lemma analogous to Lemma 5.10 in [8].

Lemma 3.1.33. *There exists $c > 0$ such that for all $\varepsilon > 0$, $\|f^w_{Q,\varepsilon}\| \leq c\mu(Q)^{\frac{1}{2}}$, $\iint_{\mathrm{R}_Q} \left|\Theta^B_t f^w_{Q,\varepsilon}\right|^2 \, d\mu(x)\dfrac{dt}{t} \leq c\dfrac{1}{\varepsilon^2}\mu(Q)$, and $\left|\fint_Q f^w_{Q,\varepsilon} - w\right| \leq c\varepsilon^{\frac{\eta}{2}}$.*

Proof. The proof of the first two estimates are essentially the same as that of Lemma 5.10 in [8]. To prove the last estimate, note that since $\eta_Q = 1$ on Q, we have on Q that

$$\begin{aligned} f^w_{Q,\varepsilon} - w &= w_Q - \varepsilon\,\ell(Q)\imath(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q - w \\ &= (\eta_Q - 1)w - \varepsilon\,\ell(Q)\imath(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q \\ &= -\varepsilon\,\ell(Q)\imath(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q. \end{aligned}$$

Setting $u = (1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q$ and $\Upsilon = \Gamma$, we apply Lemma 3.1.28

$$\begin{aligned} \left|\fint_Q f^w_{Q,\varepsilon} - w\right| &= \left|\fint_Q \varepsilon\,\ell(Q)\imath(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right| \\ &= \varepsilon\,\ell(Q)\left|\fint Q\imath(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right| \\ &\lesssim \frac{\varepsilon\,\ell(Q)}{t^{\frac{\eta}{2}}}\left(\fint_Q \left|(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right| \, d\mu\right)^{\frac{\eta}{4}} \\ &\qquad\qquad \left(\fint_Q \left|\Gamma(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right|^2 \, d\mu\right)^{\frac{1}{2} - \frac{\eta}{4}} \\ &= \left(\frac{\varepsilon\,\ell(Q)}{t}\right)^{\frac{\eta}{2}}\left(\fint_Q \left|(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right| \, d\mu\right)^{\frac{\eta}{4}} \\ &\qquad\qquad \left(\fint_Q \left|\varepsilon\,\ell(Q)\imath\Gamma(1 + \varepsilon\,\ell(Q)\imath\Pi_B)^{-1} w_Q\right|^2 \, d\mu\right)^{\frac{1}{2} - \frac{\eta}{4}}. \end{aligned}$$

The proof is completed by noting $t \simeq \ell(Q)$ and invoking Proposition 2.5 and Lemma

4.2 of [8]. □

The proof of Proposition 3.1.31 then follows a procedure similar to that which is used to prove Lemma 5.12 in [8].

We note that our hypotheses (H1)-(H6), (H7-H) and (H8-H) remain unchanged upon replacing (Γ, B_1, B_2) by (Γ^*, B_2, B_1), $(\Gamma^*, {B_2}^*, {B_1}^*)$ and $(\Gamma, {B_1}^*, {B_2}^*)$. Thus, the hypothesis of Proposition 1.8.6 is satisfied and Theorem 3.1.1 is proved.

3.2 Exponential growth and GBG bundles

In this section, we obtain quadratic estimates on measure metric spaces on vector bundles which satisfy the generalised bounded geometry condition as described in §1.4.

These estimates generalise the quadratic estimates in [46], but our proof here employs some significantly different tools, even though we rely upon some of the technical results from this source.

First, we note that the proof here is carried out on a measure metric space where the measure does not arise from the metric in a canonical way. This is in contrast to the work in [46] where the estimates were obtained on Riemannian manifolds with the Riemannian metric and induced volume measure. We remark, however, that an adaptation of the proof in [46] to this more general setting would only require minor changes.

The significant difference in our proof is that we have allowed for our vector bundle to be non-trivial, whereas in [46], estimates were carried out on the bundle $\mathbb{C}^N$. This forces fundamental changes to the argument - the non-triviality of our vector bundle means that many tools of harmonic analysis, such as the notion of an integral average, is no longer canonical.

Our assumption of generalised bounded geometry for the vector bundle and the existence of a truncated dyadic structure are the two vehicles that allow us passage from Euclidean analysis to our geometric setting. We define our notion of an integral, and hence the average, only for cubes beneath a fundamental scale, by fixing GBG coordinates across the entire bundle. As a consequence, we are forced to formulate the geometric hypotheses of this section only on cubes beneath this scale. This is in contrast to the previous situation, both in §3.1 and in [46], [8] and [7] where the corresponding similar hypotheses were independent of coordinates or dyadic structure.

The formulation of generalised bounded geometry for arbitrary vector bundles arose from our analysis of these estimates on manifolds with curvature and injectivity radius bounds. Through careful analysis, we were able to extract this notion of generalised bounded geometry on a vector bundle to be the core property which we need in order to prove quadratic estimates.

The strength of our results here is perhaps best measured through the theorems which we prove using them. In Chapters 5 and 6, we use these quadratic estimates to prove Kato square root problems on manifolds in an *intrinsic* way. Such results are stronger (and capture) those in [46], where the triviality of the bundle requires the manifold be embedded in Euclidean space.

The estimates we obtain here are *local*, and this is forced on us by the fact that we are, in general, working with a non-trivial bundle and also because we allow for the measure to exhibit exponential growth.

Let us fix $\mathcal{V}$ to be a vector bundle over a measure metric space $\mathcal{X}$ satisfying the GBG criterion. Recall that in this setting, we are able to obtain a truncated dyadic structure as guaranteed by Theorem 1.3.4. Also recall $C_1, C_2, a_0, \delta, \eta, \mathscr{Q}^k, I_k$ and Q^k_α from the same theorem.

Throughout this section we fix $\mathrm{J} \in \mathbb{N}$ such that $C_1\delta^{\mathrm{J}} \leq \frac{\rho}{5}$, where ρ is the GBG radius from Definition 1.4.1. For $j \geq \mathrm{J}$, we write $\mathscr{Q}^j$ to denote the collection of all cubes Q^j_α, and set $\mathscr{Q} = \cup_{j \geq \mathrm{J}} \mathscr{Q}^j$. We call $\mathrm{t_S} = \delta^{\mathrm{J}}$ the *scale*. Furthermore, when $t \leq \mathrm{t_S}$, set $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$.

The existence of a truncated dyadic structure allows us to formulate the following system of coordinates on $\mathcal{V}$.

Definition 3.2.1 (GBG coordinates of $\mathcal{V}$)**.** *Let x_Q denote the* centre *of each $Q \in \mathscr{Q}^{\mathrm{J}}$. Then, we call the system of coordinates*

$$\mathscr{C} = \left\{\psi : B(x_Q, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_Q, \rho)) \ s.t.\ Q \in \mathscr{Q}^{\mathrm{J}}\right\}$$

the GBG coordinates. *We call the set*

$$\mathscr{C}_{\mathrm{J}} = \left\{ \psi_Q = \psi|_Q : Q \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(Q) \ s.t.\ Q \in \mathscr{Q}^{\mathrm{J}} \right\}$$

the dyadic GBG coordinates. *For any cube $Q \in \mathscr{Q}^j$, there is a unique cube $\widehat{Q} \in \mathscr{Q}^{\mathrm{J}}$ satisfying $Q \subset \widehat{Q}$ and we call this cube the* GBG cube of Q. *The* GBG coordinate system *of Q is then $\psi : B(x_{\widehat{Q}}, \rho) \times \mathbb{C}^N \to \pi_{\mathcal{V}}^{-1}(B(x_{\widehat{Q}}, \rho)$.*

Remark 3.2.2. *Since $\cup \mathscr{Q}^J$ is an almost-everywhere covering of $\mathcal{X}$, the GBG coordinate system of $\mathcal{V}$ defines an almost-everywhere continuous trivialisation of the vector bundle.*

We emphasise that throughout this section, for any cube $Q \in \mathscr{Q}^j$, we denote its GBG cube by $\widehat{Q}$.

A first and important consequence of having GBG coordinates is that it allows us to define the notion of a *constant section.* We emphasise that this is entirely seen in the eyes of our GBG coordinates. More precisely, given $w \in \mathbb{C}^N$, we define $\omega(x) = w$ whenever $x \in Q \in \mathscr{Q}^J = \mathscr{Q}_{\mathrm{ts}}$ in the GBG coordinates associated to Q. Thus, we call ω the *GBG constant section* associated to w. Note that $\omega \in \mathrm{L}^\infty(\mathcal{V})$, and that although ω corresponds to a constant vector $w \in \mathbb{C}^N$, ω typically has discontinuities across the boundaries of cubes $Q \in \mathscr{Q}^J$. We remark that this notion is crucial in later parts of this section. Next, we can define a notion of *cube integration* in the following way. Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$. Then, for any $Q \in \mathscr{Q}$, we can write

$$\int_Q u \ d\mu = \left(\int_Q u_i \ d\mu \right) e^i$$

where $u = u_i e^i$ in the GBG coordinates of Q. Note that $\int_Q u \ d\mu$ is a function from Q to $\mathbf{\Gamma}(\mathcal{V})$.

Pursuing a similar vein of thought, we define a *cube average.* Given a cube $Q \in \mathscr{Q}$ and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$, define $u_Q \in \mathrm{L}^\infty(\mathcal{V})$ by

$$u_Q(y) = \begin{cases} \frac{1}{\mu(Q)} \int_Q u \ d\mu & \text{when } y \in B(x_{\widehat{Q}}, \rho), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.3. *We remark that in the average of a function over a general measurable set of positive measure, we do not perform a cutoff as we do here. However, whenever we write u_Q with Q being a cube (even for a function), we shall always* *assume this definition.*

As before, we define the *dyadic averaging operator*. For each $t \in (0, \mathrm{t_S}]$, define the operator $\mathcal{A}_t : \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V}) \to \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ by

$$\mathcal{A}_t u(x) = \frac{1}{\mu(Q)} \int_Q u(y)\ d\mu(y)$$

whenever $x \in Q \in \mathscr{Q}_t$. We remark that $\mathcal{A}_t : \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$ is a bounded operator with norm bounded uniformly for all $t \leq \mathrm{t_S}$ by a bound depending on the constant C arising in the GBG criterion.

With the language of cube integration at hand, we formulate the following *local cancellation hypothesis.*

(H7-I) There exists $c > 0$ such that for all $Q \in \mathscr{Q}$,

$$\left| \int_Q \Gamma u\ d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_Q \Gamma^* v\ d\mu \right| \leq c\mu(Q)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma),\ v \in \mathcal{D}(\Gamma^*)$ satisfying $\operatorname{spt} u,\ \operatorname{spt} v \subset Q$.

Lastly, we make the following abstract Poincaré and coercivity hypotheses on the bundle (recalling that $\mathscr{Q}_t = \mathscr{Q}^j$ whenever $\delta^{j+1} < t \leq \delta^j$).

(H8-I) There exist $C_P, C_C, c, \tilde{c} > 0$ and an operator $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathscr{N})$, where $\mathscr{N}$ is a normed bundle over $\mathcal{X}$ with norm $|\cdot|_{\mathscr{N}}$ and $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$, satisfying:

-1 (Dyadic Poincaré)

$$\int_B |u - u_Q|^2\ d\mu \leq C_P(1 + r^\kappa e^{\lambda crt})(rt)^2 \int_{\tilde{c}B} \left(|\Xi u|^2_{\mathscr{N}} + |u|^2 \right)\ d\mu$$

for all balls $B = B(x_Q, rt)$ with $r \geq C_1/\delta$ where $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$, and

-2 (Coercivity)

$$\|\Xi u\|^2_{\mathrm{L}^2(\mathscr{N})} + \|u\|^2_{\mathrm{L}^2(\mathcal{V})} \leq C_C \|\Pi u\|^2_{\mathrm{L}^2(\mathcal{V})}.$$

for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

We remark that we have made a substantial departure from [8] and [46] in our formulation of (H7-I) and (H8-I). The main difference is that we have used the dyadic structure, rather than just the metric structure (balls), in their formulation. This cannot be avoided since we are forced to employ quantities which are defined through GBG coordinates.

We justify calling this an abstract Poincaré inequality for two reasons. First, the inequality is for sections on a vector bundle, not just for scalar-valued functions. Secondly, in the usual Poincaré inequality, the operator Ξ is simply ∇. We allow ourselves other possibilities in allowing the operator Ξ here as it may be useful in applications to situations where the vector bundle is in general non-flat and non-trivial.

We devote the rest of this section to prove the following.

Theorem 3.2.4. *Suppose that $(\mathcal{X}, d, \mu)$ is a complete measure metric space that is at most exponential in growth, $\mathcal{V}$ satisfies the generalised bounded geometry criterion, and that (Γ, B_1, B_2) satisfy (H1)-(H6), (H7-I) and (H8-I). Then, Π_B satisfies the quadratic estimate*

$$\int_0^\infty \|Q_t^B u\|^2 \, \frac{dt}{t} \simeq \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathrm{L}^2(\mathcal{V})$.

3.2.1 Carleson measure reduction

Similar to the argument in the polynomial case in §3.1, where Theorem 3.1.1 was obtained by invoking Proposition 1.8.6, we proceed to prove Theorem 3.2.4 by invoking Proposition 1.8.7. Our goal is to show that under (H1)-(H6), (H7-I) and (H8-I), the estimate

$$\int_0^{t_0} \|Q_t^B P_t u\| \, \frac{dt}{t} \lesssim \|u\|$$

holds for some $t_0 \in (0, \infty)$ and all $u \in \overline{\mathcal{R}(\Pi)}$. We call this the *main local quadratic estimate.*

The proof proceeds by reducing this main local quadratic estimate to a local *Carleson measure* estimate. Thus, we first recall the notion of a *local Carleson measure.* For some $t_0 \in (0, \mathrm{t_S}]$, set $\mathcal{X}_{+,t_0} = \mathcal{X} \times (0, t_0]$ to be the *truncated upper half space.* We emphasise that we restrict our considerations to $t \leq t_0$. The *Carleson box* over

$Q \in \mathscr{Q}_t$ is then defined as $\mathrm{R}_Q = \overline{Q} \times (0, \ell(Q)]$. A positive Borel measure ν on $\mathcal{X}_{+,t_0}$ is called a local Carleson measure if there exists $C > 0$ such that $\nu(\mathrm{R}_Q) \leq C\mu(Q)$ for all dyadic cubes $Q \in \mathscr{Q}_t$ for $t \leq t_0$. The *local Carleson norm* $\|\nu\|_{\mathcal{C},t_0}$ is defined by

$$\|\nu\|_{\mathcal{C},t_0} = \sup_{Q \in \mathscr{Q}_t,\ t \leq t_0} \frac{\nu(\mathrm{R}_Q)}{\mu(Q)}.$$

Let $\mathcal{C}_{t_0}$ denote the set of all local Carleson measures.

Let $t_1 = \frac{t_0}{2}$ and define the *truncated cone* at x to be

$$\Gamma_{t_1}(x) = \{(y,t) \in \mathcal{X}_{+,t_0} : y \in B(x,t)\}.$$

For a measurable function $f : \mathcal{X}_{+,t_0} \to \mathbb{C}$, we define the following maximal operator $\mathcal{M}^{\infty}_{\mathrm{loc}} f(x) = \sup_{(y,t) \in \Gamma_{t_1}(x)} |f(y,t)|$.

First, we have the following lemma.

Lemma 3.2.5. *For a Borel function $f : \mathcal{X}_{+,t_0} \to \mathbb{R}$, the set $\{x \in X : (\mathcal{M}^{\infty}_{\mathrm{loc}} f)(x) > \alpha\}$ is open and there $\mathcal{M}^{\infty}_{\mathrm{loc}}$ is Borel and hence measurable.*

Proof. The proof of this is contained in the proof of (1) of Theorem 4.3.3 in [44]. □

This allows us to define the following.

Definition 3.2.6 (Local Nirenberg functions). *Let $f : \mathcal{X}_{+,t_1} \to \mathbb{R}$ be Borel. Define $\|f\|_{\mathcal{N},t_1} = \|\mathcal{M}^{\infty}_{\mathrm{loc}} f\|_{\mathrm{L}^1}$. Then, we say that $f \in \mathcal{N}_{t_1}$ if $\|f\|_{\mathcal{N},t_1} < \infty$ and we equip the space $\mathcal{N}_{t_1}$ with the norm $\|\cdot\|_{\mathcal{N},t_1}$.*

Theorem 3.2.7 (Local Carleson's Theorem). *Let $f \in \mathcal{N}_{t_1}$ and $\nu \in \mathcal{C}_{t_0}$. Then,*

$$\iint_{\mathcal{X}_{+,t_1}} |f(x,t)|\ d\nu(x,t) \lesssim \|f\|_{\mathcal{N}} \|\nu\|_{\mathcal{C}_{t_0}}.$$

Proof. The proof of this is essentially contained in the proof of Theorem 4.3.3 in [44]. The passing from f to $|\mathcal{A}_t u|^2$ simply needs to be omitted. □

The reader will find a more elaborate description of Carleson measures in the classical setting in §II.2 of [52] by Stein. The local construction described here is a fraction of a larger theory explored by Morris in [44] and [46].

With a notion of a local Carleson measure at hand, we now illustrate how to reduce the main local quadratic estimate to a Carleson measure estimate. The key is the following proposition, which is a special case of Theorem 3.2.7.

Proposition 3.2.8. *For all $u \in \mathscr{H}$ and for all $\nu \in \mathcal{C}_{t_0}$,*

$$\iint_{\mathcal{X}_{+,t_0}} |\mathcal{A}_t u|^2 \ d\nu(x,t) \lesssim \|u\|^2 \|\nu\|_{\mathcal{C}}.$$

Remark 3.2.9. *Note that we integrate all the way to t_0, not just t_1. Carleson's Theorem 3.2.7 gives us this inequality up to t_1, and the nature of $\mathcal{A}_t$ allow us to deduce the same inequality from t_1 to t_0.*

Further, recall that whenever $w \in \mathbb{C}^N$, the associated GBG constant section is given by $\omega(x) = w$ whenever $x \in Q \in \mathscr{Q}_{\mathrm{t_S}}$ in the GBG coordinates associated to Q. As before, we define the *principal part* as $\gamma_t(x)w = (\Theta_t^B \omega)(x)$. With the aid of this notation, and for $t_0 \in (0, \mathrm{t_S}]$ to be chosen later, we split the main quadratic estimate in the following way:

$$\begin{aligned}\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \ \frac{dt}{t} \lesssim & \int_0^{t_0} \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 \ \frac{dt}{t} \\ &+ \int_0^{t_0} \|\gamma_t \mathcal{A}_t (P_t - I) u\|^2 \ \frac{dt}{t} \\ &+ \int_0^{t_0} \int_M |\mathcal{A}_t u|^2 \, |\gamma_t|^2 \ \frac{d\mu(x)\ dt}{t}.\end{aligned} \tag{Q$_{\mathrm{loc}}$}$$

As before, let us call the first two terms on the right of (Q$_{\mathrm{loc}}$) the *principal terms*. Then, Proposition 3.2.8 allows us to reduce estimating the last term to proving that

$$A \mapsto \int_A |\gamma_t(x)|^2 \, \frac{d\mu(x)\ dt}{t}$$

is a local Carleson measure. As before, we call this the *Carleson term*.

3.2.2 Estimation of principal terms

As the title suggests, in this section, we illustrate how to estimate the two principal terms of (Q$_{\mathrm{loc}}$). We proceed to do so by coupling the existence of exponential off-diagonal bounds with our dyadic Poincaré inequality and cancellation hypothesis. The estimates here are straightforward and are more or less adapted from [8], [46] and [4].

First, let us state and prove the following useful technical lemma.

Lemma 3.2.10. *Let $r > 0$ and suppose that $\{B_j = B(x_j, r)\}$ is a disjoint collection of balls. Then, whenever $\eta \geq 1$,*

$$\sum_j \chi_{\eta B_j} \lesssim \eta^{\kappa} \mathrm{e}^{4\lambda c \eta r}.$$

Proof. Fix $x \in \mathcal{X}$ and let $\mathscr{C}_x = \{x_j \in \mathcal{X} : x \in B(x_j, \eta r)\}$. It is easy to see that $\sum_j \chi_{\eta B_j}(x) = \operatorname{card} \mathscr{C}_x$. That $x_j \in \mathscr{C}_x$ is equivalent to saying that $d(x, x_j) < \eta r$, and therefore for any $y \in \eta B_j$, $d(x, y) \leq d(x, x_j) + d(x_j, y) < (\eta + 1)r$. That is, $B(x, (\eta + 1)r) \supset B(x_j, r)$ and by the disjointness of $\{B_j\}$, $\mu(B(x, (\eta + 1)r)) \geq \sum_{x_j \in \mathscr{C}_x} \mu(B(x_j, r))$.

Next, note that (Exp) implies that $\mu(\eta B_j) \lesssim \eta^{\kappa} \mathrm{e}^{\lambda c \eta r} \mu(B_j)$ and therefore,

$$\sum_{x_j \in \mathscr{C}_x} \mu(\eta B_j) \lesssim \eta^{\kappa} \mathrm{e}^{\lambda c \eta r} \mu(B(x, (\eta + 1)r)).$$

Thus, it is enough to compare $\mu(\eta B_j)$ to $\mu(B(x, (\eta + 1)r))$. So, take any $y \in B(x, (\eta + 1)r)$, and note that $d(x, y) \leq d(x, x_j) + d(x_j, y) < (2\eta + 1)r < 3\eta r$. Thus,

$$\mu(B(x, (\eta + 1)r)) \leq \mu(B(x_j, 3\eta r) \lesssim 3^{\kappa} \mathrm{e}^{3\lambda c \eta r} \mu(B(x_j, \eta r))$$

and the estimate

$$\operatorname{card} \mathscr{C}_x \ \mathrm{e}^{-3\lambda c \eta r} \lesssim \sum_{x_j \in \mathscr{C}_x} \frac{\mu(\eta B_j)}{\mu(B(x, (\eta + 1)r))} \lesssim \eta^{\kappa} e^{\lambda c \eta r}$$

completes the proof. $\square$

In the proof of the following estimate, we depart from adapting the argument in [8] or [46] and instead modify the argument of Auscher, Axelsson (Rosén) and McIntosh in [4]. Our argument involves slightly more work since we allow for exponential growth.

Proposition 3.2.11 (First principal term estimate)**.** *For all $u \in \mathcal{R}(\Pi)$,*

$$\int_0^{t_2} \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 \, \frac{dt}{t} \lesssim \|u\|^2$$

where $t_2 \leq \min\left\{\frac{\rho}{5}, \frac{C_\Theta}{4\lambda c(1+4\tilde{c})}\right\}$.

Proof. Let $v = P_t u$.

(i) First, we note that

$$\|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 = \sum_{Q \in \mathscr{Q}_t} \|\Theta_t^B (v - v_Q)\|^2_{\mathrm{L}^2(Q)}.$$

For each $Q \in \mathscr{Q}_t$, write $B_Q = B(x_Q, \frac{C_1}{\delta} t) \supset Q$ and $C_j(Q) = 2^{j+1} B_Q \setminus 2^j B_Q$. Then, for each such cube Q,

$$\begin{aligned}
\int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \, d\mu &= \int_Q \left|\Theta_t^B \left(\sum_{j=0}^{\infty} \chi_{C_j(Q)} (v - v_Q)\right)\right|^2 \, d\mu \\
&\leq \sum_{j=0}^{\infty} \int_Q \left|\Theta_t^B (\chi_{C_j(Q)} (v - v_Q))\right|^2 \, d\mu \\
&\lesssim \sum_{j=0}^{\infty} \left\langle \frac{t}{d(Q, C_j(Q))} \right\rangle^M \\
&\qquad \exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \int_{C_j(Q)} |v - v_Q|^2 \, d\mu
\end{aligned}$$

(ii) Next, note that by (4.1) in [46]

$$2^j \frac{C_1}{\delta} t \leq d(x_Q, C_j(Q)) \leq d(Q, C_j(Q)) + \operatorname{diam} Q$$

which implies that

$$\left\langle \frac{t}{d(Q, C_j(Q))} \right\rangle^M \lesssim 2^{-M(j+1)}.$$

Next, for $j \geq 1$,

$$d(Q, C_j(Q)) \geq 2^j \frac{C_1}{2\delta} t$$

and therefore,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \leq \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^{j+1}\right).$$

For $j = 0$,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) = 1 = \exp\left(\frac{C_\Theta C_1}{4\delta}\right) \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^0\right).$$

Fix $t' > 0$ to be chosen later. Then, for all $t \le t'$,

$$\exp\left(-C_\Theta \frac{d(Q, C_j(Q))}{t}\right) \lesssim \exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right)$$

for all $j \geq 0$.

(iii) Combining the estimates in (i) and (ii),

$$\int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \, d\mu \lesssim \sum_{j=0}^{\infty} 2^{-M(j+1)} \exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right) \int_{C_j(Q)} |v - v_Q|^2 \, d\mu.$$

Since $v = P_t u \in \mathcal{D}(\Pi^2) = \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$, we conclude from (H8)-1 that

$$\int_{C_j(Q)} |v - v_Q|^2 \, d\mu \lesssim \left(1 + \left(\frac{C_1}{\delta}\right)^{\kappa} 2^{\kappa(j+1)} \exp\left(\frac{\lambda c C_1}{\delta} 2^{j+1} t\right)\right) \left(\frac{C_1}{\delta}\right)^2 2^{2(j+1)} t^2 \int_{\tilde{c} 2^{j+1} B_Q} \left(|\Xi v|^2 + |v|^2\right) d\mu.$$

Therefore,

$$\begin{aligned}
&\sum_{Q \in \mathscr{Q}_t} \int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \, d\mu \\
&\quad \lesssim \sum_{Q \in \mathscr{Q}_t} \sum_{j=0}^{\infty} 2^{-M(j+1)} \exp\left(-\frac{C_\Theta C_1}{4\delta} 2^{j+1}\right) \\
&\qquad \left(1 + \left(\frac{C_1}{\delta}\right)^{\kappa} 2^{\kappa(j+1)} \exp\left(\frac{\lambda c C_1}{\delta} 2^{j+1} t\right)\right) \left(\frac{C_1}{\delta}\right)^2 2^{2(j+1)} t^2 \\
&\qquad \int_{\tilde{c} 2^{j+1} B_Q} \left(|\Xi v|^2 + |v|^2\right) d\mu \\
&\quad \lesssim \sum_{j=0}^{\infty} 2^{-(M-\kappa-2)(j+1)} \\
&\qquad \left[\exp\left(-\frac{C_\Theta C_1}{4\delta t'} 2^{j+1} t\right) + \exp\left(-\frac{C_1}{\delta}\left(\frac{C_\Theta}{4t'} - \lambda c\right) 2^{j+1} t\right)\right] \\
&\qquad t^2 \int_{\mathcal{X}} \sum_{Q \in \mathscr{Q}_t} \chi_{\tilde{c} 2^{j+1} B_Q} \left(|\Xi v|^2 + |v|^2\right) d\mu.
\end{aligned}$$

(iv) Set $\eta = \tilde{c} 2^{j+1}$ and $r = \frac{C_1}{\delta} t$ and invoke Lemma 3.2.10 to conclude that

$$\chi_{\tilde{c} 2^{j+1} B_Q} \lesssim 2^{\kappa(j+1)} \exp\left(\frac{4\lambda c \tilde{c} C_1}{\delta} 2^{j+1} t\right).$$

Combining this with (iii), we have

$$\sum_{Q \in \mathscr{Q}_t} \int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \ d\mu$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-(M-2\kappa-2)(j+1)}$$
$$\left[\exp\left(-\frac{C_1}{\delta} \left(\frac{C_\Theta}{4t'} - 4\lambda c\tilde{c} \right) 2^{j+1} t \right) \right.$$
$$\left. + \exp\left(-\frac{C_1}{\delta} \left(\frac{C_\Theta}{4t'} - \lambda c - 4\lambda c\tilde{c} \right) 2^{j+1} t \right) \right]$$
$$t^2 \left(\|\Xi v\|^2 + \|v\|^2 \right)$$

(v) Now, we choose $t' \leq \frac{\varrho}{5}$ so that

$$-\frac{C_1}{\delta} \left(\frac{C_\Theta}{4t'} - 4\lambda c\tilde{c} \right) \leq 0 \quad \text{and} \quad -\frac{C_1}{\delta} \left(\frac{C_\Theta}{4t'} - \lambda c - 4\lambda c\tilde{c} \right) \leq 0.$$

That is,

$$\frac{C_\Theta}{4\lambda c(1 + 4\tilde{c})} \leq t'.$$

We can, therefore, set $t' = t_2$ and then,

$$\sum_{Q \in \mathscr{Q}_t} \int_Q \left|\Theta_t^B (v - v_Q)\right|^2 \ d\mu \lesssim t^2 \sum_{j=0}^{\infty} 2^{-(M-2\kappa-2)(j+1)} \|\Pi v\|^2$$

by invoking (H8)-2. By choosing $M > 2\kappa + 2$, and substituting $v = P_t u$, we have

$$\int_0^{t_2} \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 \ \frac{dt}{t} \lesssim \int_0^{t_2} \|t \Pi P_t u\|^2 \ \frac{dt}{t} \leq \int_0^{\infty} \|t \Pi P_t u\|^2 \ \frac{dt}{t} \lesssim \|u\|.$$

□

Next, as in [8] and [46], we note the following consequence of (H7-I).

Lemma 3.2.12. *Let* $\Upsilon = \Gamma,\ \Gamma^*$ *or* Π. *Then,*

$$\left| \fint_Q \Upsilon u \ d\mu \right| \lesssim \frac{1}{\ell(Q)^\eta} \left(\fint_Q |u| \ d\mu \right)^{\frac{\eta}{2}} \left(\fint_Q |\Upsilon u| \ d\mu \right)^{1-\frac{\eta}{2}} + \fint_Q |u|^2$$

for all $u \in \mathcal{D}(\Upsilon)$ *and* $Q \in \mathscr{Q}_t$ *where* $t \leq \mathrm{t_S}$.

The proof of this lemma is the same as that of the proof of Lemma 5.9 in [46]. Thus, we deduce the following.

Proposition 3.2.13 (Second principal term estimate). *For all $u \in \mathrm{L}^2(\mathcal{V})$, we have*

$$\int_0^{\mathrm{t_S}} \|\gamma_t \mathcal{A}_t (P_t - I)u\|^2 \frac{dt}{t} \lesssim \|u\|^2.$$

The proof of this proposition is similar to the proof of Proposition 5.10 in [46] with the principle difference being that we only consider $t \leq \mathrm{t_S}$. This concludes our proofs of the principal term estimates.

3.2.3 Carleson measure estimate

Due to Propositions 3.2.11 and 3.2.13, the only remaining task is to estimate the final term of ($\mathrm{Q_{loc}}$). We follow in the footsteps of [5] and [8] remarking that, in a sense, it is to preserve the main thrust of this Carleson argument that we have formulated geometric structures and made various geometric assumptions in our analysis. More precisely, we show that the argument runs as before with some changes that are possible as a consequence of the our geometric assumptions.

In §5 of [46], Morris illustrates how to prove

$$A \mapsto \int_A |\gamma_t(x)|^2 \, d\mu(x) \frac{dt}{t}$$

is a local Carleson measure. At the heart of this proof lies a certain test function. Here, we illustrate how to use the GBG condition to set up a suitable substitute test function $\mathrm{f}^w_{Q,\varepsilon}$. The main complication is to choose a cutoff in a way that we stay inside GBG coordinates of each cube.

Let $\tau > 1$ be chosen later. We note that we can find a bounded Lipschitz function $\eta_Q : \mathcal{X} \to [0,1]$ such that $\eta = 1$ on $B(x_Q, \tau C_1 \, \ell(Q))$ and $\eta = 0$ on $\mathcal{X} \setminus B(x_Q, 2\tau C_1 \, \ell(Q))$ and satisfying the "gradient" bound $\operatorname{Lip} \eta(x) \lesssim \frac{1}{\ell(Q)}$. The constant here depends on τ.

Let $\widehat{Q}$ be the GBG cube with centre $x_{\widehat{Q}}$. We want to use η to perform a cutoff inside the ball $B(x_Q, \rho)$. That is, we want the ball $B(x_Q, 2\tau C_1 \, \ell(Q)) \subset B(x_Q, \rho)$. A simple calculation then yields that we need to choose $\tau < 3$. Thus, for the sake of the argument, let us fix $\tau = 2$.

Next, let $v \in \mathcal{L}(\mathbb{C}^N)$ with $|v| = 1$. Let $\hat{w}, w \in \mathbb{C}^N$ such that $|\hat{w}| = |w| = 1$ and $v^*(\hat{w}) = w$. Let $\tilde{w} = \eta\, w$. Certainly, we have that $\tilde{w} \in \mathrm{L}^2(\mathcal{V})$ since $\tilde{w} = 0$ outside of $B(x_{\widehat{Q}}, \rho)$. Now, fix $\varepsilon > 0$ and define

$$\mathrm{f}^w_{Q,\varepsilon} = \tilde{w} - \imath\varepsilon\,\ell(Q)\Gamma R^B_{\varepsilon\,\ell(Q)}\tilde{w} = (1 + \imath\varepsilon\,\ell(Q)\Gamma^*_B R^B_{\varepsilon\,\ell(Q)})\tilde{w}.$$

This allows us to prove a lemma similar to that of Lemma 5.12 in [46].

Lemma 3.2.14. *There exists $c > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq \mathrm{t_S}$,*

$$\|\mathrm{f}^w_{Q,\varepsilon}\| \leq c\mu(Q)^{\frac{1}{2}}, \quad \iint_{\mathrm{R}_Q} \left|\Theta^B_t \mathrm{f}^w_{Q,\varepsilon}\right|^2 \, d\mu\frac{dt}{t} \leq c\frac{\mu(Q)}{\varepsilon^2} \quad \text{and} \quad \left|\fint_Q \mathrm{f}^w_{Q,\varepsilon} - w \, d\mu\right| \leq c\varepsilon^{\frac{\eta}{2}}.$$

The proof of this lemma proceeds exactly as the proof of Lemma 5.12 in [46]. The last estimate relies upon Lemma 3.2.12.

Setting $\varepsilon = \left(\frac{1}{2c}\right)^{\frac{\eta}{2}}$ we obtain the same conclusion as the author of [46] that

$$\mathrm{Re}\left\langle w, \fint_Q f^w_Q \right\rangle \geq \frac{1}{2}$$

where we have set $f^w_Q = f^w_{Q,\varepsilon}$. Furthermore, a stopping time argument as in Lemma 5.11 in [8] yields the following.

Lemma 3.2.15. *Let $t_3 = \min\left\{\frac{\varrho}{5}, \frac{C_\Theta}{4a^3\lambda}\right\}$. There exists $\alpha, \beta > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq t_3$, there exists a collection of subcubes $\{Q_k\} \subset \cup_{t \leq t_3}\mathscr{Q}_t$ of Q such that $E_Q = Q \setminus \cup_k Q_k$ satisfies $\mu(E_Q) \geq \beta\mu(Q)$ and $E^*_Q = \mathrm{R}_Q \setminus \cup_k \mathrm{R}_{Q_k}$ satisfies*

$$\mathrm{Re}\left\langle w, \fint_{Q'} f^w_Q \right\rangle \geq \alpha \quad \text{and} \quad \fint_{Q'} |f^w_Q| \leq \frac{1}{\alpha}$$

whenever $Q' \in \cup_{t \leq t_3}\mathscr{Q}_t$ with $Q' \subset Q$ and $\mathrm{R}_{Q'} \cap E^*_Q = \varnothing$.

Let $\sigma > 0$ to be chosen later and let $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$ and define

$$K_{\nu,\sigma} = \left\{\nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left|\frac{\nu'}{|\nu'|} - \nu\right| \leq \sigma\right\}.$$

Proposition 3.2.16. *Let $t_3 = \min\left\{\frac{\varrho}{5}, \frac{C_\Theta}{4a^3\lambda}\right\}$. There exists $\sigma, \beta, c > 0$ such that for all $Q \in \mathscr{Q}_t$ with $t \leq t_3$, and $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $|\nu| = 1$, there exists a collection of* *subcubes* $\{Q_k\} \subset \cup_{t \le t_3} \mathscr{Q}_t$ *of* Q *such that* $E_Q = Q \setminus \cup_k Q_k$ *satisfies* $\mu(E_Q) \ge \beta\mu(Q)$ *and* $E_Q^* = \mathrm{R}_Q \setminus \cup_k \mathrm{R}_{Q_k}$ *satisfies*

$$\iint_{(x,t)\in E_Q^*,\ \gamma_t(x)\in K_{\nu,\sigma}} |\gamma_t(x)|^2 \ d\mu(x)\frac{dt}{t} \le c\mu(Q).$$

Then, the argument immediately following Proposition 5.11 in [46] illustrates that

$$\iint_{\mathrm{R}_Q} |\gamma_t(x)|^2 \ d\mu(x)\frac{dt}{t} \lesssim \mu(Q)$$

and this is the required Carleson measure estimate.

On choosing $t_0 = \min\{t_2, t_3\}$ and applying Proposition 3.2.8, we find that we have bounded all three terms on the right of ($\mathrm{Q}_{\mathrm{loc}}$), and hence deduced the estimate

$$\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \ \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in \overline{\mathcal{R}(\Pi)}$.

The invariance of (H1)-(H6), (H7-I) and (H8-I) upon replacing (Γ, B_1, B_2) by (Γ^*, B_2, B_1), (Γ^*, B_2^*, B_1), and (Γ^*, B_1^*, B_2^*) proves Theorem 3.2.4 via Proposition 1.8.7.

3.3 Alternative reduction of the local quadratic estimate

In §3.2.1, we have followed the paradigm set forth in [8] to reduce estimating

$$\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

by a three way split as in ($\mathrm{Q}_{\mathrm{loc}}$). In particular, the middle term here relies upon the hypothesis (H7-I). This cancellation assumption as well as the Poincaré assumption (H8-I) may prove to be difficult to establish on geometries lacking sufficient smoothness. Instead, we propose to replace (H7-I) by an assumption of L^p off-diagonal decay which we believe to be operator theoretic and possibly weaker than (H7-I).

We remark here that recent work (unpublished and ongoing) by Frey, McIntosh and Portal may provide a way of also eliminating the Poincaré assumption (H8-I)-1 by replacing the principal part approximation argument using (H6).

This new reduction we present here requires analysis by a new maximal function suited to the study of exponential spaces. Thus, the first part of our presentation deals with establishing some basic facts about this tool. We couple this with local Carleson-Nirenberg theory to reduce the local quadratic estimate to a *two term* split while retaining the main Carleson measure argument.

Our work here is motivated by a calculation of Hytönen privately communicated via McIntosh.

3.3.1 Exponential maximal function

The usual maximal function is well suited for the study of spaces that are at most polynomial. Exponential growth makes it impossible to establish its boundedness. Thus, we define the following maximal function that is better suited to this setting.

Definition 3.3.1 (Exponential maximal function)**.** *For $\beta > 0$, define*

$$\mathcal{M}^\beta f(x) = \sup_{B \ni x} \mathrm{e}^{-\beta \,\mathrm{rad}(B)} \fint_B |f| \ d\mu$$

and for $m \in \mathbb{R}^+$,

$$\mathcal{M}^\beta_m f(x) = \sup_{B \ni x,\, \mathrm{rad}(B) \leq m} \mathrm{e}^{-\beta \,\mathrm{rad}(B)} \fint_B |f| \ d\mu$$

whenever $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$.

We remark that the exponential decay term β typically arises naturally, for instance, through off-diagonal estimates of an operator. In the following maximal theorem, we recall that λ is the exponential parameter appearing in the growth assumption (Exp).

Theorem 3.3.2. *Whenever $\beta > 3\lambda$, $\mathcal{M}^\beta_m$ and $\mathcal{M}^\beta$ are lower-semicontinuous and there exists $C > 0$ such that whenever $\xi > 0$,*

$$\mu \left\{ \mathcal{M}^\beta_m f > \xi \right\} \leq \frac{C}{\xi} \int_{\mathcal{M}^\beta_m f > \xi} |f| \ d\mu \quad \text{and} \quad \mu \left\{ \mathcal{M}^\beta f > \xi \right\} \leq \frac{C}{\xi} \int_{\mathcal{M}^\beta f > \xi} |f| \ d\mu.$$

Proof. Fix x such that $\mathcal{M}^{\beta}_m f(x) > \xi$. Then, there exists B with $\mathrm{rad}(B) \leq m$ such that

$$\frac{\mathrm{e}^{-\beta\,\mathrm{rad}(B)}}{\mu(B)} \int_B |f| > \xi.$$

It is also easy to see that $B \subset \left\{\mathcal{M}^{\beta}_m f > \xi\right\}$ and so $\mathcal{M}^{\beta}_m f$ is lower semicontinous. Similarly for $\mathcal{M}^{\beta}$.

Next, let $\{B_j\} \subset \left\{B_x : B_x \subset \left\{\mathcal{M}^{\beta}_m f > \xi\right\}\right\}$ be the disjoint countable subcollection guaranteed by Proposition 3.18 in [44]. Then,

$$\begin{aligned}
\mu\left\{\mathcal{M}^{\beta}_m f > \xi\right\} &\leq \sum_j \mu(4B_j) \leq A4^{\kappa} \sum_j \mathrm{e}^{3\xi\,\mathrm{rad}(B_j)} \mu(B_j) \\
&\leq \frac{C}{\xi} \sum_j \mathrm{e}^{3\xi\,\mathrm{rad}(B_j)} \mathrm{e}^{-\beta\,\mathrm{rad}(B_j)} \int_{B_j} |f|\ d\mu \leq \frac{C}{\xi} \sum_j \int_{B_j} |f|\ d\mu \leq \frac{C}{\xi} \int_{\mathcal{M}^{\beta}_m f > \xi} |f|\ d\mu
\end{aligned}$$

where $C = A4^{\kappa}$.

Now, note that for $m \leq n$, $\left\{\mathcal{M}^{\beta}_m f > \xi\right\} \subset \left\{\mathcal{M}^{\beta}_n f > \xi\right\} \subset \left\{\mathcal{M}^{\beta} f > \xi\right\}$ and

$$\left\{\mathcal{M}^{\beta} f > \xi\right\} = \cup_{m=1}^{\infty} \left\{\mathcal{M}^{\beta}_m f > \xi\right\} = \lim_{m\to\infty} \left\{\mathcal{M}^{\beta}_m f > \xi\right\}.$$

So, by monotone convergence,

$$\begin{aligned}
\mu\left\{\mathcal{M}^{\beta} f > \xi\right\} = \lim_{m\to\infty} \mu\left\{\mathcal{M}^{\beta}_m f > \xi\right\} &\leq \lim_{m\to\infty} \frac{C}{\xi} \int_{\mathcal{M}^{\beta}_m f > \xi} |f|\ d\mu \\
&\leq \lim_{m\to\infty} \frac{C}{\xi} \int_{\mathcal{M}^{\beta} f > \xi} |f|\ d\mu = \frac{C}{\xi} \int_{\mathcal{M}^{\beta} f > \xi} |f|\ d\mu.
\end{aligned}$$

□

Corollary 3.3.3. *For $f \in \mathrm{L}^p(\mathcal{V})$ where $1 < p < \infty$, we have that*

$$\|\mathcal{M}^{\beta}_m f\| \lesssim \|f\|_p \quad \text{and} \quad \|\mathcal{M}^{\beta} f\| \lesssim \|f\|_p.$$

3.3.2 Off-diagonal assumptions in L^p

In what is to follow, we need to obtain pointwise control on $x \mapsto |\mathcal{A}_t P_t u(x)|$. In order to do this, we introduce the following L^p off-diagonal hypothesis.

(H9) (L^p off-diagonal bounds) Let U_t be either R^B_t, P^B_t, Q^B_t or Θ^B_t for $t \in \mathbb{R}^+$. There exists $C_\Theta > 0$ and $p < 2$ such that, for every $M > 0$ there exists $c > 0$ with

$$\|\chi_E U_t u\|_p \leq c \left\langle \frac{|t|}{d(E,F)} \right\rangle^M \exp\left(-C_\Theta \frac{d(E,F)}{t}\right) \|\chi_F u\|_p$$

whenever E, F are Borel, spt $u \subset F$.

For the following proposition and the rest of the section, let $a = \max\left\{1, \frac{a_0}{\delta}\right\}$.

Proposition 3.3.4. *If (H9) holds, whenever $t_2 < \min\left\{\frac{\varrho}{5}, \frac{C_\Theta}{24a\lambda}\right\}$, and for p coming from (H9), we have that for all $t \in (0, t_2]$, $|\mathcal{A}_t P_t u(x)|^p \lesssim \mathcal{M}^\beta(|u|^p)(y)$ where $x \in Q \in \mathscr{Q}_t$ and $y \in B(x_Q, \frac{4C_1}{\delta}t)$.*

Proof. Fix $Q \in \mathscr{Q}_t$ and $x \in Q$.

(i) We first note that $|\mathcal{A}_t u(x)| \lesssim \fint_Q |u| \ d\mu$ since

$$\begin{aligned} |\mathcal{A}_t u| &= \left|\left(\fint_Q u_i \ d\mu\right) e^i\right|_{\mathrm{h}} \simeq \left|\left(\fint_Q u_i \ d\mu\right) e^i\right|_\delta \\ &= \left(\sum_i \left|\fint_Q u_i \ d\mu\right|^2\right)^{\frac{1}{2}} \simeq \sum_i \left|\fint_Q u_i \ d\mu\right| \\ &\leq \sum_i \fint_Q |u_i| \ d\mu \ \simeq \fint_Q \left(\sum_i |u_i|^2\right)^{\frac{1}{2}} \ d\mu \\ &= \fint_Q |u|_\delta \ d\mu \simeq \fint_Q |u| \ d\mu. \end{aligned}$$

(ii) Now, for $1 < p < \infty$ and taking $\frac{1}{p'} + \frac{1}{p} = 1$ we have by Hölder's inequality,

$$\begin{aligned} \fint_Q |u| \ d\mu &= \int_Q \mu(Q)^{-\frac{1}{p}} |u| \, \mu(Q)^{-\frac{1}{p'}} \\ &\leq \left(\int_Q \mu(Q)^{-1} |u|^p\right)^{\frac{1}{p}} \left(\int_Q \mu(Q)^{-1}\right)^{\frac{1}{p'}} = \mu(Q)^{-\frac{1}{p}} \|u\|_{\mathrm{L}^p(Q)} \end{aligned}$$

(iii) Now, fix a particular p from (H9). Then, again by Hölder's inequality,

$$\begin{aligned} &\mu(Q)^{-\frac{1}{p}} \|u\|_{\mathrm{L}^p(Q)} \\ &\qquad \lesssim \mu(Q)^{-\frac{1}{p}} \sum_{R \in \mathscr{Q}_t} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t}\right) \|u\|_{\mathrm{L}^p(R)} \end{aligned}$$

$$\leq \left(\sum_{R \in \mathscr{Q}_t} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t} \right) \right)^{\frac{1}{p'}}$$

$$\left(\frac{1}{\mu(Q)} \sum_{R \in \mathscr{Q}_t} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t} \right) \|u\|^p_{\mathrm{L}^p(R)} \right)^{\frac{1}{p}}.$$

(iv) We bound the first term. First, note that

$$\begin{aligned}
&\sum_{R \in \mathscr{Q}_t} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t} \right) \\
&\quad = \sum_{R \in \mathscr{Q}_t} \frac{\mu(R)}{\mu(Q)} \frac{\mu(Q)}{\mu(R)} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t} \right) \\
&\quad \leq \sum_{R \in \mathscr{Q}_t} \frac{\mu(R)}{\mu(Q)} \left\langle \frac{t}{d(Q,R)} \right\rangle^{M-\kappa} \exp\left(-C_\Theta \frac{d(Q,R)}{t} + a\lambda t \frac{d(Q,R)}{t} \right) \\
&\quad = \sum_{R \in \mathscr{Q}_t} \frac{\mu(R)}{\mu(Q)} \left\langle \frac{t}{d(Q,R)} \right\rangle^{M-\kappa} \exp\left(-(C_\Theta - a\lambda t) \frac{d(Q,R)}{t} \right)
\end{aligned}$$

by invoking Lemma 4.4 in [46] to the quantity $\mu(Q)\mu(R)^{-1}$.

So, choose $M > 2\kappa$ and $t \leq \min\left\{\frac{\rho}{5}, \frac{C_\Theta}{2a\lambda}\right\}$. Thus, $t < C_\Theta(2a\lambda)^{-1}$ and since $a \geq 1$, we have that $\lambda t < C_\Theta - a\lambda t$. By Lemma 4.5 in [46],

$$\sup_{R \in \mathscr{Q}_t} \sum_{R \in \mathscr{Q}_t} \frac{\mu(R)}{\mu(Q)} \left\langle \frac{t}{d(Q,R)} \right\rangle^{M-\kappa} \exp\left(-(C_\Theta - a\lambda t) \frac{d(Q,R)}{t} \right) < \infty.$$

(v) We note that as a consequence from 4.1 in [46],

$$\exp\left(\frac{-C_\Theta}{a} \frac{d(Q,x)}{t} \right) \geq \exp(-2C_\Theta) \exp\left(-C_\theta \frac{d(Q,R)}{t} \right)$$

for all $x \in R$ and by similar argument to the one found in the proof of Proposition 5.8 in [46] and

$$\begin{aligned}
&\frac{1}{\mu(Q)} \sum_{R \in \mathscr{Q}_t} \left\langle \frac{t}{d(Q,R)} \right\rangle^M \exp\left(-C_\Theta \frac{d(Q,R)}{t} \right) \|u\|^p_{\mathrm{L}^p(R)} \\
&\qquad \lesssim \mu(Q) \sum_{R \in \mathscr{Q}_t} \int_R \left\langle \frac{t}{d(Q,x)} \right\rangle^M \exp\left(-\frac{C_\Theta}{a} \frac{d(Q,x)}{t} \right) |u(x)|^p \, d\mu(x).
\end{aligned}$$

(vi) We consider this expression in dyadic annuli. Let $B = B\left(x_Q, \frac{2C_1}{\delta} t\right) \supset Q$.

Then, for $j \geq 1$ and $x \in 2^{j+1}B \setminus 2^j B$, we have that

$$2^j \frac{2C_1}{\delta} t \leq d(x_Q, x) \leq d(Q, x) + \operatorname{diam} Q \leq d(Q, x) + \frac{C_1}{\delta}$$

and therefore,

$$2^j \frac{C_1}{\delta} t \leq d(Q, x) \quad \text{and} \quad -\frac{d(Q, x)}{t} \leq -2^{j+1} \frac{C_1}{2\delta}.$$

This yields

$$\left\langle \frac{t}{d(Q, x)} \right\rangle^M \lesssim 2^{-Mj} \lesssim 2^{-(M-\kappa)j} 2^{-\kappa(j+1)}$$

by writing $Mj = (M - \kappa)j - \kappa - \kappa(j + 1)$ and

$$\exp\left(-\frac{C_\Theta}{\alpha} \frac{d(Q, x)}{t}\right) \lesssim \exp\left(-\frac{C_\Theta}{\alpha} 2^{j+1} \frac{C_1}{2\delta}\right)$$

Now, for the case of $j = 0$, we have that

$$\left\langle \frac{t}{d(Q, x)} \right\rangle \leq 1 \lesssim 2^{-(M-\kappa)j} 2^{-\kappa(j+1)}$$

and

$$\begin{aligned} \exp\left(-\frac{C_\Theta}{\alpha} \frac{d(Q, x)}{t}\right) &\lesssim \exp\left(\frac{C_\Theta}{a} \frac{C_1}{\delta}\right) \exp\left(-\frac{C_\Theta}{\alpha} 2^{j+1} \frac{C_1}{2\delta}\right) \\ &\lesssim \exp\left(-\frac{C_\Theta}{\alpha} 2^{j+1} \frac{C_1}{2\delta}\right). \end{aligned}$$

Combining these estimates, for $j \geq 0$ and $x \in 2^{j+1}B \setminus 2^j B$ we have

$$\begin{aligned} &\left\langle \frac{t}{d(Q, x)} \right\rangle^M \exp\left(-\frac{C_\Theta}{a} \frac{d(Q, x)}{t}\right) \\ &\quad \lesssim \left[2^{-\kappa(j+1)} \exp\left(-\frac{C_\Theta}{\alpha} \frac{C_1}{4\delta} 2^{j+1}\right)\right] \left[2^{-(M-\kappa)j}\right] \left[\exp\left(-\frac{C_\Theta}{\alpha} \frac{C_1}{4\delta} 2^{j+1}\right)\right]. \end{aligned}$$

(vii) Next, we use the condition (Exp) to note that

$$\begin{aligned} \mu(2^{j+1}B) &= \mu\left(B\left(x_Q, 2^{j+1} \frac{2C_1}{\delta} t\right)\right) \\ &\leq c \left(2^{j+1} \frac{2C_1}{a_0 \delta}\right)^\kappa \exp\left(\lambda \frac{2C_1}{a_0 \delta} a_0 t\right) \mu(B(x_Q, a_0 t)) \\ &\lesssim 2^{\kappa(j+1)} \exp\left(\lambda \frac{2C_1}{\delta} t\right) \mu(B(x_Q, a_0 t)). \end{aligned}$$

Thus,

$$\frac{2^{-\kappa(j+1)}\exp\left(-\frac{C_\Theta}{\alpha}\frac{C_1}{4\delta}2^{j+1}\right)}{\mu(Q)} \lesssim \frac{\exp\left(-\frac{C_\Theta}{\alpha}\frac{C_1}{4\delta}2^{j+1}+\lambda\frac{2C_1}{\delta}2^{j+1}t\right)}{\mu(2^{j+1}B)} \lesssim \frac{1}{\mu(2^{j+1}B)}$$

by choosing $t \leq t_2 \leq \min\left\{\frac{\varrho}{5}, \frac{C_\Theta}{8\lambda a}\right\}$.

(viii) Next, note that

$$-\frac{C_\Theta}{a}\frac{C_1}{4\delta}2^{j+1} = -\frac{C_\Theta}{8a}\frac{1}{t}2^{j+1}\frac{2C_1}{\delta}t = -\frac{C_\Theta}{8a}\frac{1}{t_2}\operatorname{rad}(2^{j+1}B).$$

Letting $\beta = \frac{C_\Theta}{8a}\frac{1}{t_2}$ and choosing $t_2 < \min\left\{\frac{\varrho}{5}, \frac{C_\Theta}{24\lambda a}\right\}$, gives us $\beta > 3\lambda$. Thus,

$$\begin{aligned}\mu(Q)\sum_{R\in\mathscr{Q}_t}\int_R \left\langle\frac{t}{d(Q,x)}\right\rangle^M \exp\left(-\frac{C_\Theta}{a}\frac{d(Q,x)}{t}\right)|u(x)|^p \ d\mu(x) \\ \lesssim \sum_{j=0}^\infty 2^{-(M-\kappa)j}\frac{1}{2^{j+1}B}\exp(-\beta\operatorname{rad}(2^{j+1}B))\int_{2^{j+1}\setminus 2^jB}|u|^p \ d\mu \\ \lesssim \sum_{j=0}^\infty 2^{-(M-\kappa)j}\exp(-\beta\operatorname{rad}(2^{j+1}B))\fint_{2^{j+1}B}|u|^p \ d\mu\end{aligned}$$

Since the case $j=0$ is the ball $2B$, we have

$$\exp(-\beta\operatorname{rad}(2^{j+1}B))\fint_{2^{j+1}B}|u|^p \ d\mu \leq \mathcal{M}^\beta(|u|^p)(y)$$

for any $y \in 2B$.

$\square$

Corollary 3.3.5. $(x,t) \mapsto |\mathcal{A}_tP_tu(x)|^2 \in \mathcal{N}_{t_1}$.

Proof. Fix $x \in \mathcal{M}$ and let $(y,t) \in \Gamma_{t_1}(x)$ such that $y \in Q$. Let y_Q be the centre of this cube. Then,

$$d(x,y_Q) \leq d(x,y) + d(y,y_Q) \leq t + \frac{C_1}{\delta}t < 4\frac{C_1}{\delta}.$$

Thus,

$$\sup_{(y,t)\in\Gamma_{t_1}(x)}|\mathcal{A}_tP_tu|^2 \lesssim \left(\mathcal{M}^\beta(|u|^p)\right)^{\frac{2}{p}}.$$

Note then that $2/p > 1$ and so by the maximal Theorem 3.3.2,

$$\int_{\mathcal{M}} \left(\mathcal{M}^{\beta}(|u|^{p})\right)^{\frac{2}{p}} \ d\mu \lesssim \int_{\mathcal{M}} (|u|^{p})^{\frac{2}{p}} = \|u\|^{2} < \infty.$$

$\square$

Corollary 3.3.6. *For every $\nu \in \mathcal{C}_{t_0}$,*

$$\iint_{\mathcal{M}\times(0,t_0]} |A_t P_t u|^2 \ d\nu(x,t) \lesssim \|u\|^2 \|\nu\|_{\mathcal{C}_{t_0}}.$$

Proof. The estimate

$$\iint_{\mathcal{M}\times(0,t_1]} |A_t P_t u|^2 \ d\nu(x,t) \lesssim \|u\|^2 \|\nu\|_{\mathcal{C}_{t_0}}$$

comes from the application of Carleson's Theorem 3.2.7 and the Maximal Theorem 3.3.2. Then,

$$\iint_{\mathcal{M}\times(t_1,t_0]} |A_t P_t u|^2 \ d\nu(x,t) \lesssim \|P_t u\|^2 \|\nu\|_{\mathcal{C}_{t_0}}$$

as noted in the proof of Theorem 4.3 in [46] and we have that $\|P_t u\| \lesssim \|u\|$ uniformly in t. $\square$

As a consequence of this corollary, we split the main local quadratic estimate in the following way:

$$\int_0^{t_0} \|\Theta_t^B P_t u\|^2 \ \frac{dt}{t} \lesssim \int_0^{t_0} \|\Theta_t^B P_t u - \gamma_t \mathcal{A}_t P_t u\|^2 \ \frac{dt}{t} + \int_0^{t_0} \int_M |\mathcal{A}_t P_t u|^2 \, |\gamma_t|^2 \ \frac{d\mu(x) \ dt}{t},$$

with $t_0 < \infty$ to be chosen later. Then, obtaining the main local quadratic estimate from §3.2.1 is still reduced to proving

$$A \mapsto \int_A |\gamma_t(x)|^2 \, \frac{d\mu(x) \ dt}{t}$$

is a local Carleson measure.

4

Subelliptic problems on Lie groups

In this chapter, we present solutions to Kato square root problems for perturbations of subelliptic operators on Lie groups. We first obtain homogeneous estimates on nilpotent Lie groups. The more general theorem solves a similar problem for all Lie groups, but the best estimates we obtain are inhomogeneous.

We mention that the results presented here are considerably strengthened in [12]. Although we do not provide a detailed account of these results, we list an important result for elliptic operators on polynomial Lie groups. We omit the proof since it requires substantial facts from the structure theory of Lie groups. At the heart of these more general theorems lie the unique role of the first-order perspective which allows us to reduce subelliptic problems to strongly elliptic ones. Thus, our focus will be to present the core ideas rather than a detailed account of the most general theorems.

4.1 Subelliptic operators

Let $(\mathcal{G}, d, \mu)$ be a connected Lie group, with d the sub-Riemannian distance associated to an algebraic basis $\mathfrak{a}$ of dimension $m \leq \dim \mathcal{G}$, equipped with the left-invariant Haar measure μ. Recall the frame $\{D_i\}$ which defines $\mathcal{A}$ from §1.6. The D_i are regarded as differential operators by their action on functions.

In the L^2 theory, let us write A_i as the *infinitesimal generator* of the group $t \mapsto$

$\mathrm{L}_{\exp(ta_i)}$. First, we have the following proposition.

Proposition 4.1.1. *Whenever $u \in \mathrm{C}^\infty_\mathrm{c}(\mathcal{G})$, $D_i u = A_i u$.*

Proof. By the definition of D_i, we have that

$$(D_i u)(g) = \frac{d}{dt} t|_{t=0} u(\exp(-ta_i)g).$$

But since $\mathrm{C}^\infty_\mathrm{c}(\mathcal{G}) \subset \mathrm{L}^2(\mathcal{G})$, $(A_i u)(g)$ is given by the same expression. $\square$

Furthermore, the following classical result of Stone in [53] illustrates some important properties of A_i.

Proposition 4.1.2. *The operators A_i are densely-defined, closed and skew-adjoint.*

Let us denote $\cap_{i=1}^m \mathcal{D}(A_i)$ by $\mathrm{W}^{1,2}(\mathcal{G})''$ and equip it with the norm $\|u\|_{\mathrm{W}^{1,2''}} = \|u\| + \sum_{i=1}^m \|A_i u\|$. Then, $\mathrm{C}^\infty_\mathrm{c}(\mathcal{G}) \subset \mathrm{W}^{1,2}(\mathcal{G})''$. By the closedness of each A_i, it is easy to see that this is a complete space with respect to the $\mathrm{W}^{1,2}(\mathcal{G})''$ norm and that it is dense in $\mathrm{L}^2(\mathcal{G})$. We consider the operator $\Delta = -\sum_{i=1}^m A_i^2$ as the *sub-Laplacian*, and the following proposition justifies this nomenclature. For a proof, see §3.1 in the paper [23] by Driver, Gross and Saloff-Coste.

Proposition 4.1.3. *The sub-Laplacian is self-adjoint.*

The Kato problems we consider are related to L^∞ perturbations of this operator. The first such operator we define is a homogeneous version.

Definition 4.1.4 (Homogeneous perturbed sub-Laplacian)**.** *Let $a, a_{ij} \in \mathrm{L}^\infty(\mathcal{G})$. Then, we define*

$$\mathrm{L}_A^\mathrm{H} = -a \sum_{i,j=1}^m A_i a_{ij} A_j$$

with domain $\mathcal{D}(\mathrm{L}_A^\mathrm{H}) = \{u \in \mathrm{L}^2(\mathcal{G}) : u \in \mathrm{W}^{1,2}(\mathcal{G})'',\ a_{ij}A_j u \in \mathrm{W}^{1,2}(\mathcal{G})''\}$.

The *homogeneous Kato problem for subelliptic operators* is then to determine the conditions under which

$$\begin{cases} \mathcal{D}(\sqrt{\mathrm{L}_A^\mathrm{H}}) = \mathrm{W}^{1,2}(\mathcal{G})'' \\ \|\sqrt{\mathrm{L}_A^\mathrm{H}} u\| \simeq \sum_{i=1}^m \|A_i u\|, \quad u \in \mathrm{W}^{1,2}(\mathcal{G})''. \end{cases} \tag{H}$$

An inhomogeneous version of this operator is the following.

Definition 4.1.5 (Inhomogeneous perturbed sub-Laplacian)**.** *Let* $a, a_{ij}, c_i, c_i', c_0 \in \mathrm{L}^\infty(\mathcal{G})$*. Then,*

$$\mathrm{L}_a^{\mathrm{I}} = \mathrm{L}_A^{\mathrm{H}} + a(\sum_{i=1}^m c_i A_i + A_i c_i' + c_0 \mathrm{I}).$$

with domain $\mathcal{D}(\mathrm{L}_A^{\mathrm{I}}) = \left\{ u \in \mathrm{W}^{1,2}(\mathcal{G})'' : \mathrm{L}_A^{\mathrm{H}} u,\ A_i c_i' u \in \mathrm{L}^2(\mathcal{G}) \right\}$.

The corresponding inhomogeneous Kato square root problem on Lie groups is then the following.

$$\begin{cases} \mathcal{D}(\sqrt{\mathrm{L}_A^{\mathrm{I}}}) = \mathrm{W}^{1,2}(\mathcal{G})'' \\ \|\sqrt{\mathrm{L}_A^{\mathrm{I}}} u\| \simeq \|u\| + \sum_{i=1}^m \|A_i u\|, \quad u \in \mathrm{W}^{1,2}(\mathcal{G})''. \end{cases} \tag{IH}$$

4.2 Geometric framework

In this section, we rephrase the subelliptic problems that we described in §4.1 in the language of the vector bundle machinery from §1.6. We will see that this allows a *divergence-form* perspective of the operators $\mathrm{L}_A^{\mathrm{H}}$ and $\mathrm{L}_A^{\mathrm{I}}$, and more importantly, allow for a first-order analysis of (H) and (IH).

First, define the sub-connection $\nabla : \mathrm{C}^\infty(\mathcal{A}) \to \mathrm{C}^\infty(\mathcal{A}^* \otimes \mathcal{A})$ by

$$\nabla X = \nabla(X^i D_i) = (\nabla X_i) \otimes D_i = D_j X_i\ D^j \otimes D_i.$$

Recall that with respect to the frame D_i, the metric on $\mathcal{A}$ is one that makes D_i orthonormal. With a slight abuse of notation, we can consider ∇ as a closed, densely-defined operator by taking the L^2 closure of the original sub-connection as an operator with domain $\mathrm{C}_{\mathrm{c}}^\infty(\mathcal{A})$. The closure exists since by Proposition 4.1.1, $D_i = A_i$ on $\mathrm{C}_{\mathrm{c}}^\infty(\mathcal{G})$, and by Proposition 4.1.2, D_i on $\mathrm{C}_{\mathrm{c}}^\infty(\mathcal{G})$ is skew-symmetric. We have the following proposition highlighting the connection between this closed, densely-defined ∇ and the operators A_i. It is an easy consequence of Lemma 2.4 in [26].

Proposition 4.2.1. *We have that* $\mathrm{C}_{\mathrm{c}}^\infty(\mathcal{G})$ *is dense in* $\mathrm{W}^{1,2}(\mathcal{G})''$*. Hence,* $\mathrm{W}^{1,2}(\mathcal{G})' = \mathrm{W}_0^{1,2}(\mathcal{G})'' = \mathrm{W}^{1,2}(\mathcal{G})''$*, and* $\nabla u = A_i u^j\ D^i$*. Furthermore,* $\|\cdot\|_{\mathrm{W}^{1,2\prime\prime}} = \|\cdot\|_{\mathrm{W}^{1,2\prime}}$*. Similar conclusions hold for* $\mathcal{G}$ *replaced with* $\mathcal{A}$*, and with* $\nabla v = A_i v^j\ D_j \otimes D^i$*.*

Let us denote the adjoint of ∇ by $-\operatorname{div}$. Then, by considering the complex form $J[u,v] = \langle \nabla u, \nabla v\rangle$ with $u, v \in \mathrm{W}^{1,2}(\mathcal{G})'$, we have that $\Delta = -\operatorname{div}\nabla$. Furthermore, $\mathrm{L}^{\mathrm{H}}_A = -a \operatorname{div} A\nabla$, where $a, A = (a_{ij}) \in \mathrm{L}^\infty$. For the case $a \equiv 1$, we obtain the divergence form characterisation of $\mathrm{L}^{\mathrm{H}}_A$ as promised by considering the complex energy $J_A[u,v] = \langle A\nabla u, \nabla v\rangle$ with $u, v \in \mathrm{W}^{1,2}(\mathcal{G})'$. To realise the operator $\mathrm{L}^{\mathrm{I}}_A$ in this way, we write $S = (\mathrm{I}, \nabla)$ and consider the form $J_B[u,v] = \langle BSu, Sv\rangle$, where B is an appropriate L^∞ matrix containing the coefficients A, c_i, c_i' and c_0.

Using this language, we point out the following regularity result.

Proposition 4.2.2. *We have that $\mathcal{D}(\Delta) = \mathrm{W}^{2,2}(\mathcal{G})'$ and that $\|\nabla^2 u\| \lesssim \|\Delta u\| + \|u\|$ for all $u \in \mathcal{D}(\Delta)$. If further $\mathcal{G}$ is nilpotent, then $\|\nabla^2 u\| \simeq \|\Delta u\|$.*

Proof. The proof for the general case is given by Theorem 7.2 in [27] and the estimate for the nilpotent case is due to Lemma 4.2 in [29]. $\square$

4.3 Homogeneous estimates and nilpotent Lie groups

Set $\mathcal{X} = \mathcal{G}$, and for the remainder of this section, assume that $\mathcal{G}$ is nilpotent. Define: $\Gamma : \mathcal{D}(\Gamma) \subset \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{A}^*) \to \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{A}^*)$ by

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}$$

so that

$$\Gamma^* = \begin{pmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & -\operatorname{div} \\ \nabla & 0 \end{pmatrix}.$$

Let $A = (a_{ij})$ and define

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

At this point, since the bundle $\mathcal{A}$ is trivial and the metric on it also trivial, we identify it with its trivialisation after complexification. That is, we write D_i as the complex basis e_i by identifying the space $\mathcal{A}$ with $\mathbb{C}^m$.

On setting $\mathscr{H} = \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathbb{C}^m) \cong \mathrm{L}^2(\mathcal{G}, \mathbb{C}^{m+1})$, we obtain a solution to the Kato square root problem (H).

Theorem 4.3.1. *Let $(\mathcal{G}, d, \mu)$ be a connected, nilpotent Lie group and suppose that $A \in \mathrm{L}^\infty$ satisfying: there exists $\kappa_1,\ \kappa_2 > 0$ such that*

$$\mathrm{Re}\, \langle au, u\rangle \geq \kappa_1 \|u\|^2 \quad \textit{and} \quad \mathrm{Re}\, \langle A\nabla v, \nabla v\rangle \geq \kappa_2 \|\nabla v\|^2$$

for every $u \in \mathrm{L}^2(\mathcal{G})$ and $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then,

(i) Π_B has a bounded holomorphic functional calculus, and

(ii) $\mathcal{D}(\sqrt{\mathrm{L}_A^{\mathrm{H}}}) = \mathrm{W}^{1,2}(\mathcal{G})'$ and $\|\sqrt{\mathrm{L}_A^{\mathrm{H}}} u\| \simeq \|\nabla u\|$ for all $u \in \mathrm{W}^{1,2}(\mathcal{G})'$.

Proof. We show that (H1)-(H6), (H7-H) and (H8-H) are satisfied for (Γ, B_1, B_2) in order to apply Theorem 3.1.1 and Theorem 1.8.2.

The proof of (H1) is immediate since $\mathrm{C}_{\mathrm{c}}^\infty(\mathcal{G})$ is dense in $\mathrm{L}^2(\mathcal{G})$ and ∇ is a closed, densely defined operator The nilpotency of Γ is trivial.

For (H2), let $u \in \mathcal{R}(\Gamma^*)$. So, $u = (u_1, u_2) = (-\operatorname{div} v_2, 0)$ for some $v = (v_1, v_2) \in \mathcal{D}(\Gamma^*)$. Then,

$$\mathrm{Re}\, \langle B_1 u, u\rangle = \mathrm{Re}\, \langle a \operatorname{div} v_2, \operatorname{div} v_2\rangle \geq \kappa_1 \|\operatorname{div} v_2\|^2 = \kappa_1 \|u\|^2.$$

Similarly, let $u \in \mathcal{R}(\Gamma)$, so that $u = (u_1, u_2) = (0, \nabla v_1)$ where $v = (v_1, v_2) \in \mathcal{D}(\Gamma)$. Then,

$$\mathrm{Re}\, \langle B_2 u, u\rangle = \mathrm{Re} \int_{\mathcal{G}} \langle A\nabla v_1, \nabla v_1\rangle \ d\mu \geq \kappa_2 \|\nabla v_1\|^2 = \kappa_2 \|u\|^2.$$

The hypothesis (H3) follows trivially since $B_2 B_1 = B_1 B_2 = 0$ and (H4) and (H5) are satisfied by definition.

To prove (H6), we let η be a bounded Lipschitz function. Since Lipschitz functions are almost everywhere differentiable, we have that

$$[\Gamma, \eta]u = \Gamma(\eta u) - \eta \Gamma u = \nabla(\eta u_1) - \eta \nabla u_1 = u_1 \nabla \eta + \eta \nabla u_1 - \eta \nabla u_1 = u_1 \nabla \eta.$$

Thus, $|[\Gamma, \eta](x)| \leq |\nabla \eta|(x)$ and $|\nabla \eta(x)| \leq m \operatorname{Lip} \eta(x)$ for almost all x.

We prove (H7-H). Fix $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G})$ with spt $u, v \subset B$ for some open ball B. Then,

$$\int_B \Gamma u \, d\mu = \sum_k \left(\int_B A_k u_1 \, d\mu \right) e_k$$

and $\int_B A_k u_1 \, d\mu = 0$ by the left invariance of $d\mu$.

Now, pick a function $\zeta \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G})$ with $\zeta = 1$ on B and $\zeta = 0$ outside $2B$. Then,

$$\begin{aligned}\int_B \Gamma^* u \, d\mu &= \int_B - \operatorname{div} v_2 \, d\mu = \int_{\mathcal{G}} - \operatorname{div} v_2 \zeta \, d\mu \\ &= \int_{\mathcal{G}} \langle v_2, \nabla \zeta \rangle \, d\mu = \int_B \langle v_2, \nabla \zeta \rangle \, d\mu = 0.\end{aligned}$$

The same conclusions hold for $u \in \mathcal{D}(\Gamma)$ or $u \in \mathcal{D}(\Gamma^*)$ with spt $u \subset B$ respectively by invoking Proposition 4.2.1.

Lastly, we show that (H8-H) is satisfied. We note that by the nilpotency of $\mathcal{G}$ we conclude that the following Poincaré inequality holds

$$\int_B |f - f_B|^2 \, d\mu \lesssim r^2 \int_B |\nabla f|^2 \, d\mu$$

for $f \in \mathrm{C}^{\infty}(B)$ where $B = B(x,r)$ and $r > 0$ by (P.1) on p118 of [50].

Next, define $\Xi : \mathbb{C}^{m+1} \to \mathbb{C}^{(m+m^2)}$ by $\Xi u = (\nabla u_1, \nabla u_2)$ for $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G} \oplus \mathcal{A})$. Let $B = B(x,r)$ for $r > 0$. Then,

$$\begin{aligned}\int_B |u - u_B|^2 \, d\mu &= \int_B |u_1 - (u_1)_B|^2 \, d\mu + \int_B |u_2 - (u_2)_B|^2 \, d\mu \\ &\lesssim r^2 \int_B |\nabla u_1|^2 \, d\mu + r^2 \int_B \sum_{j=1}^n |\nabla (u_2)_j|^2 \, d\mu \\ &= r^2 \int_B |\Xi u|^2 \, d\mu.\end{aligned}$$

As before, we obtain $\mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G} \oplus \mathcal{A}^*)$ is dense in $\mathcal{D}(\Pi)$ by Proposition 4.2.1 and therefore, the above inequality holds for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

Now, let $u \in \mathcal{R}(\Pi) \cap \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G} \oplus \mathcal{A}^*)$. Thus, $u = \Pi v$, for $v \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{G} \oplus \mathcal{A}^*)$. Writing $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we have that $\Pi u = (- \operatorname{div} u_2, \nabla u_1) = (\Delta v_1, \nabla u_1)$.

Thus, $\|\Pi u\|^2 = \|\Delta v_1\|^2 + \|\nabla u_1\|^2$. Also,

$$\|\Xi u\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2 = \|\nabla u_1\|^2 + \|\nabla^2 v_1\|^2 \simeq \|\nabla u_1\|^2 + \|\Delta v_1\|^2$$

where the last equivalence follows by Proposition 4.2.2. Thus, we have that $\|\Xi u\| \lesssim \|\Pi u\|$ and this holds for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$ by a density argument.

Theorem 3.1.1 guarantees that Π_B has a bounded holomorphic functional calculus. This proves (i). Then, by Theorem 1.8.2, and by considering $(u, 0) \in \mathcal{D}(\Pi_B)$ for $u \in \mathrm{W}^{1,2}(\mathcal{G})$, we conclude (ii). $\square$

Remark 4.3.2. *On setting $\mathfrak{a} = \{e_i : 1 \leq 1 \leq n\}$ where each e_i is the standard basis for $\mathbb{R}^n$, $\mathcal{G} = \mathbb{R}^n$ and $\mu = \mathscr{L}$, we recover a solution to the classical Kato square root problem on $\mathbb{R}^n$ as a consequence of this theorem.*

We end this section by stating the following theorem from [12] concerning strongly elliptic operators on polynomial Lie groups. The proof of the theorem is omitted as it goes beyond the scope of this thesis.

Theorem 4.3.3. *Let $(\mathcal{G}, d, \mu)$ be a connected Lie group satisfying (Pol) and that the algebraic basis defining our ∇ is a vector space basis (i.e. $m = \dim(\mathcal{G})$). Suppose further that $A \in \mathrm{L}^\infty$ satisfies: there exists $\kappa_1,\ \kappa_2 > 0$ such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \quad \textit{and} \quad \mathrm{Re}\,\langle A\nabla v, \nabla v\rangle \geq \kappa_2 \|\nabla v\|^2$$

for every $u \in \mathrm{L}^2(\mathcal{G})$ and $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then,

(i) Π_B has a bounded holomorphic functional calculus, and

(ii) $\mathcal{D}(\sqrt{\mathrm{L}_A^{\mathrm{H}}}) = \mathrm{W}^{1,2}(\mathcal{G})'$ and $\|\sqrt{\mathrm{L}_A^{\mathrm{H}}}u\| \simeq \|\nabla u\|$ for all $u \in \mathrm{W}^{1,2}(\mathcal{G})'$.

4.4 Inhomogeneous estimates on general Lie groups

Following [44] or [7], define $S : \mathrm{L}^2(\mathcal{G}) \to \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{A}^*)$ by $S = (\mathrm{I}, \nabla)$. Then, $S^* = [\mathrm{I}\ \ -\mathrm{div}]$ and we define $\Gamma : \mathcal{D}(\Gamma) \subset \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{A}^*)$ by

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}$$

so that

$$\Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi^* = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}.$$

Now, for $c_i, c_i', c_0, a_{ij} \in \mathrm{L}^\infty(\mathcal{G})$, define

$$\tilde{A}_{00} = c_0,\ \tilde{A}_{10} = (c_1, \ldots, c_m),\ \tilde{A}_{01} = (c_1', \ldots, c_m')^{\mathrm{tr}},\ \tilde{A}_{11} = (a_{ij}),$$

and $A = (\tilde{A}_{ij})$. Then, we can write

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

On setting $\mathscr{H} = \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathcal{G}) \oplus \mathrm{L}^2(\mathbb{C}^m) \cong \mathrm{L}^2(\mathbb{C}^{m+2})$, we solve the inhomogeneous problem (IH).

Theorem 4.4.1. *Let $(\mathcal{G}, d, \mu)$ be a connected Lie group, and $A \in \mathrm{L}^\infty$ such that there exists $\kappa_1,\ \kappa_2 > 0$ satisfying*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \quad \textit{and} \quad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|_{\mathrm{W}^{1,2\prime}}$$

for all $u \in \mathrm{L}^2(\mathcal{G})$ and $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Then,

(i) Π_B has a bounded holomorphic functional calculus, and

(ii) $\mathcal{D}(\sqrt{\mathrm{L}_A^{\mathrm{I}}}) = \mathrm{W}^{1,2}(\mathcal{G})'$ and $\|\sqrt{\mathrm{L}_A^{\mathrm{I}}}u\| \simeq \|u\|_{\mathrm{W}^{1,2\prime}}$.

Proof. We show that (H1)-(H6), (H7-I) and (H8-I) hold for (Γ, B_1, B_2) to apply Theorem 3.2.4 and Theorem 1.8.2.

The proofs of the hypotheses (H1),(H3)-(H6) can be proved in a similar way as in the proof of these hypotheses in Theorem 4.3.1.

We show that (H2) is satisfied. It is easy to see that $\mathrm{Re}\,\langle B_1 u, u\rangle \geq \kappa_1 \|u\|^2$ for $u \in \mathcal{R}(\Gamma^*)$. Fix $u \in \mathcal{R}(\Gamma)$ so that $u = (0, Sv) = (0, v_1, \nabla v_1)$ for some $v \in \mathrm{W}^{1,2}(\mathcal{G})'$. Thus,

$$\mathrm{Re}\,\langle B_2 u, u\rangle = \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|Sv\|^2 = \kappa_2 \|u\|^2.$$

Next, we show that (H7-I) holds. Fix B, an open ball and Let $u = (u_1, u_2, u_3) \in \mathcal{D}(\Gamma)$

and $v = (v_1, v_2, v_3) \in \mathcal{D}(\Gamma^*)$ such that $\operatorname{spt} u, \operatorname{spt} v \subset B$. Then,

$$\left| \int_B \Gamma u \, d\mu \right| = \left| \int_B u_1 \, d\mu \right| + \left| \int_B \nabla u_1 \, d\mu \right|.$$

The second term is zero for the same reasons as in (H6) in the homogeneous case and the remaining term is controlled by $\mu(B)^{\frac{1}{2}} \|u\|$ by Cauchy-Schwartz. Similarly,

$$\begin{aligned}\left| \int_B \Gamma^* v \, d\mu \right| = \left| \int_B v_2 - \operatorname{div} v_3 \, d\mu \right| \leq \left| \int_B v_2 \, d\mu \right| + \left| \int_B \operatorname{div} v_3 \, d\mu \right| \\ = \left| \int_B v_2 \, d\mu \right| \leq \mu(B)^{\frac{1}{2}} \|v\|\end{aligned}$$

Lastly, we show that (H8-I) is satisfied. Define $\Xi u = (\nabla u_1, \nabla u_2, \nabla u_3)$. The existence of a local Poincaré inequality is guaranteed by Proposition 2.4 in [28] and thus for functions $f \in \mathrm{C}^\infty(B)$, we have

$$\int_B |f - f_B|^2 \, d\mu \lesssim r^2 \int_B |\nabla f|^2 \, d\mu$$

for all balls $B = B(x, r)$ with $r \leq 1$. Therefore, an easy calculation yields

$$\int_B |u - u_B|^2 \, d\mu \lesssim r^2 \int_B (|\Xi u|^2 + |u|^2) \, d\mu$$

for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$ by a density argument.

To prove coercivity, fix $u \in \mathcal{R}(\Pi) \cap \mathrm{C}_c^\infty(\mathcal{G} \oplus \mathcal{G} \oplus \mathcal{A})$. Thus, $u = (u_1, u_2, u_3) = \Pi v = (v_2 - \operatorname{div} v_3, v_1, \nabla v_1)$ for some $v \in \mathrm{C}_c^\infty(\mathcal{G} \oplus \mathcal{G} \oplus \mathcal{A})$. We note then that $\Pi u = (u_2 - \operatorname{div} u_3, u_1, \nabla u_1) = (v_1 + \Delta v_1, u_1, \nabla u_1)$ and therefore,

$$\|\Pi u\|^2 = \|(I + \Delta) u\|^2 + \|u_1\|^2 + \|\nabla u_1\|^2.$$

Similarly, we calculate

$$\|u\|^2 + \|\Xi u\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2 + \|\nabla u_3\|^2 + \|u_1\|^2 + \|v_1\|^2 + \|\nabla v_1\|^2.$$

Trivially, we have that $\|\nabla u_1\|^2 + \|u_1\|^2 \leq \|\Pi u\|^2$. The positivity of $(I + \Delta)$ gives $\|v_1\|^2 + \|\nabla v_1\|^2 \lesssim \|(I + \Delta) v_1\|^2$. For the same reason, $\|\nabla u_2\| = \|\nabla v_1\| \lesssim \|(I + \Delta) v_1\|$. Lastly,

$$\|\nabla u_3\|^2 = \|\nabla^2 v_1\|^2 \lesssim (\|\Delta v_1\|^2 + \|v_1\|^2) \lesssim \|(\mathrm{I} + \Delta) v_1\|^2$$

by Proposition 4.2.2. Thus, $\|u\|^2 + \|\Xi u\|^2 \lesssim \|\Pi u\|^2$ for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$. □

Remark 4.4.2. *Although we prove this theorem by resorting to the quadratic estimates obtained through Theorem 3.2.4, the solution to this problem is really a consequence of the work of Morris in [46]. His work precedes that of ours, with the primary difference being that his estimates are restricted to the situation of trivial bundles on manifolds. With an almost negligible modification of his argument to make it valid on measure metric spaces, his technology is sufficiently powerful to prove the above theorem.*

4.5 Stability

Let $\tilde{a}$ and $\tilde{A}$ satisfy the same conditions as a and A as in the set up of either the homogeneous or inhomogeneous problem. Then, write

$$\tilde{B}_1 = \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A} \end{pmatrix}.$$

By application of Theorem 1.8.5, we obtain the following stability theorem.

Theorem 4.5.1 (Lipschitz estimate)**.** *Assume the hypotheses of Theorem 4.4.1 are satisfied. Fix $\eta_i < \kappa_i$ and suppose that $\tilde{B}_i$ satisfy $\|\tilde{B}_i\|_\infty \leq \eta_i$ for $i = 1, 2$ and set $0 < \hat{\omega}_i < \frac{\pi}{2}$ by $\cos \hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_\infty + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Then, for all $\hat{\omega} < \mu < \frac{\pi}{2}$,*

$$\|f(\Pi_B) - f(\Pi_{B+\tilde{B}})\| \lesssim (\|\tilde{B}_1\|_\infty + \|\tilde{B}_2\|_\infty)\|f\|_\infty$$

for all $f \in \mathrm{Hol}^\infty(\mathrm{S}^o_\mu)$, and

$$\int_0^\infty \|\psi(t\Pi_B)v - \psi(t\Pi_{B+\tilde{B}})v\|^2 \frac{dt}{t} \lesssim (\|\tilde{B}_1\|^2_\infty + \|\tilde{B}_2\|^2_\infty)\|v\|^2,$$

for all $\psi \in \Psi(\mathrm{S}^o_\mu)$ and all $v \in \mathscr{H}$. The implicit constants depend in particular on on B_i and η_i. The same conclusions hold for the operator Π_B in the setting of Theorem 4.3.1 under the hypothesis of that theorem.

Proof. First, we note that $\tilde{B}_i$ trivially satisfies conditions (i)-(iii) of Theorem 1.8.5. Also, the operator $\Pi_{B_i + \zeta \tilde{B}_i}$ satisfies (H1)-(H6), (H7-I) and (H8-I) with constants $\kappa_i - \eta_i$ in place of κ_i. Thus, we obtain uniform quadratic estimates (Qest) for $\Pi_{B_i + \zeta \tilde{B}_i}$ for the range of ζ required in (iv) of Theorem 1.8.5. Hence, the conclusion is obtained by invoking Theorem 1.8.5. $\square$

These Lipschitz estimates imply the following stability of square root theorems as an immediate consequence.

Theorem 4.5.2 (Stability of the inhomogenous square root). *Assume the hypotheses of Theorem 4.4.1 and fix $\eta_i < \kappa_i$. If $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, then*

$$\|\sqrt{\mathrm{L}^{\mathrm{I}}_A}\, u - \sqrt{\mathrm{L}^{\mathrm{I}}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2\prime}}$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. The implicit constant depends in particular on A, a and η_i.

Proof. Let $\tilde{B}_i$ be given as in the hypothesis of Theorem 4.5.1, so that $\|\tilde{B}_1\|_\infty = \|\tilde{a}\|_\infty \leq \eta_1$ and $\|\tilde{B}_2\|_\infty = \|\tilde{A}\|_\infty \leq \eta_2$. By Theorem 5.4.1,

$$\|\operatorname{sgn}(\Pi_B)v - \operatorname{sgn}(\Pi_{B+\tilde{B}})v\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|v\|$$

for all $v \in \mathscr{H}$, in particular for all $v = \begin{pmatrix} 0 \\ Su \end{pmatrix}$ with $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. Note that $v = \Pi_B \begin{pmatrix} u \\ 0 \end{pmatrix} = \Pi_{B+\tilde{B}} \begin{pmatrix} u \\ 0 \end{pmatrix}$ so $\operatorname{sgn}(\Pi_B)v = \begin{pmatrix} \sqrt{\mathrm{L}^{\mathrm{I}}_A}\, u \\ 0 \end{pmatrix}$ and $\operatorname{sgn}(\Pi_{B+\tilde{B}})v = \begin{pmatrix} \sqrt{\mathrm{L}^{\mathrm{I}}_{A+\tilde{A}}}\, u \\ 0 \end{pmatrix}$. Thus, on substitution, we obtain the desired result. $\square$

The homogeneous result below is then proved in a similar manner.

Theorem 4.5.3 (Stablity of the homogenous square root). *Assume the hypotheses of Theorem 4.3.1 and fix $\eta_i < \kappa_i$. If $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, then*

$$\|\sqrt{\mathrm{L}^{\mathrm{H}}_A}\, u - \sqrt{\mathrm{L}^{\mathrm{H}}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|\nabla u\|$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{G})'$. The implicit constant depends in particular on A, a and η_i.

5

Divergence-form operators on vector bundles and manifolds

In this chapter, we prove the Kato square root problem for divergence-form operators on vector bundles under an appropriate set of assumptions. We show that under Ricci curvature bounds and lower bounds on the injectivity radius, these assumptions are satisfied for the bundle of functions and hence, obtain a solution to the Kato square root problem for functions as a highlight theorem. A similar theorem is obtained on finite rank tensors, under the additional assumption that Hessian (on the tensors) can be controlled in terms of the Laplacian.

The proof of the theorem on vector bundles proceeds by demonstrating that under the assumptions we make, the hypotheses (H1)-(H6), (H7-I) and (H8-I) are satisfied. We begin our presentation with a discussion of Poincaré inequalities on vector bundles with generalised bounded geometry.

5.1 Poincaré Inequalities

In this short section we show that, under appropriate geometric conditions, we can bootstrap the Poincaré inequality on functions to the dyadic Poincaré inequality on a vector bundle. As in [46], we define the following.

Definition 5.1.1 (Local Poincaré inequality)**.** *We say that $\mathcal{M}$ satisfies a* local

Poincaré inequality *if there exists* $c \geq 1$ *such that for all* $f \in \mathrm{W}^{1,2}(\mathcal{M})$,

$$\|f - f_B\|_{\mathrm{L}^2(B)} \leq c \ \operatorname{rad}(B)\|f\|_{\mathrm{W}^{1,2}(B)} \tag{$\mathrm{P}_{\mathrm{loc}}$}$$

for all balls B *in* $\mathcal{M}$ *such that* $\operatorname{rad}(B) \leq 1$.

Remark 5.1.2. *Note that we allow the Sobolev norm* $\|\cdot\|_{\mathrm{W}^{1,2}(B)}$ *over the ball on the right of the inequality, rather than simply* $\|\nabla \cdot\|$.

The following proposition illustrates that under appropriate gradient bounds on the GBG coordinate basis for a vector bundle satisfying GBG, we can obtain a dyadic Poincaré inequality on the bundle.

Proposition 5.1.3. *Suppose that* $\mathcal{M}$ *satisfies both (Exp) and* (P_{loc}) *and let* $\mathcal{V}$ *be a vector bundle which has generalised bounded geometry. Furthermore, suppose that that there exists* $C_G > 0$ *such that in each GBG chart with basis denoted by* $\{e^i\}$ *we have* $|\nabla e^i| \leq C_G$ *for each* i. *Then, for all* $u \in \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$, $t \leq \mathrm{t_S}$, $Q \in \mathscr{Q}_t$, *and* $r \geq \frac{C_1}{\delta}$,

$$\int_B |u - u_Q|^2 \ d\mu \lesssim (1 + e^{\lambda rt})(rt)^2 \int_B (|\nabla u|^2 + |u|^2) \ d\mu$$

where $B = B(x_Q, rt)$.

Proof. First, consider the case $(rt) \geq \frac{\rho}{5}$. Then

$$\|u - u_Q\|^2_{\mathrm{L}^2(B)} \lesssim \|u\|^2_{\mathrm{L}^2(B)} + \|u_Q\|^2_{\mathrm{L}^2(B)}.$$

Recalling that $\widehat{Q}$ is the GBG cube of Q,

$$\begin{aligned}\int_B |u_Q|^2 \ d\mu &= \int_B \left| \left(\fint_Q u_i \ d\mu \right) \chi_{B(x_{\widehat{Q}}, \rho)} e^i \right|^2 \ d\mu \simeq \sum_i \int_B \left| \fint_Q u_i \ d\mu \right|^2 \chi_{B(x_{\widehat{Q}}, \rho)} d\mu \\ &\leq \sum_i \int_B \left(\fint_Q |u_i|^2 \ d\mu \right) \ d\mu \leq \frac{\mu(B)}{\mu(Q)} \int_B |u|^2 \ d\mu \lesssim (1 + r^\kappa e^{\lambda rt}) \|u\|^2_{\mathrm{L}^2(B)}.\end{aligned}$$

Thus, $\|u - u_Q\|^2_{\mathrm{L}^2(B)} \lesssim (1 + r^\kappa e^{\lambda rt})(rt)^2 \|u\|^2_{\mathrm{L}^2(B)}$ since $rt \geq \frac{\rho}{5}$.

Next, suppose that $(rt) < \frac{\rho}{5}$. It is easy to see that we have $B(x_Q, rt) \subset B(x_{\widehat{Q}}, \rho)$ and so,

$$\int_B |u - u_Q|^2 \ d\mu \simeq \sum_i \int_B \left| u_i - \left(\fint_Q u_i \ d\mu \right) \chi_{B(x_{\widehat{Q}}, \rho)} \right|^2 \ d\mu$$

$$\leq \sum_i \int_B |u_i - (u_i)_B|^2 \, d\mu + \sum_i \int_B \left|(u_i)_B - \left(\fint_Q u_i \, d\mu\right)\right|^2 \, d\mu$$

For the first term, we invoke ($\mathrm{P}_{\mathrm{loc}}$) so that

$$\sum_i \int_B |u_i - (u_i)_B|^2 \, d\mu \lesssim (rt)^2 \int_B \left(\sum_i |\nabla u_i|^2 + |u|^2\right) \, d\mu.$$

Now, for the second term,

$$\begin{aligned}\sum_i \int_B \left|(u_i)_B - \left(\fint_Q u_i \, d\mu\right)\right|^2 \, d\mu &\leq \sum_i \frac{\mu(B)}{\mu(Q)} \int_Q |u_i - (u_i)_B|^2 \, d\mu \\ &\lesssim (1 + r^\kappa e^{\lambda rt})(rt)^2 \int_B \left(\sum_i |\nabla u_i|^2 + |u|^2\right) \, d\mu\end{aligned}$$

Next, we note that $\nabla u = \nabla(u_i) \otimes e^i + u_i \otimes \nabla e^i$ and therefore, by the hypothesis $|\nabla e^i| \leq C_G$,

$$\sum_i |\nabla u_i|^2 \simeq \left|\nabla u_i \otimes e^i\right|^2 \leq |\nabla u|^2 + |u_i|^2 \left|\nabla e^i\right|^2 \lesssim |\nabla u|^2 + |u| \, .$$

The proof is complete by combining these estimates. □

5.2 The Kato square root problem on vector bundles

Let us now present a solution to the Kato square root problem on vector bundles under an appropriate set of assumptions. First, we describe a setup of operators which is a generalisation of §1 of [46] (and before that from [7]), making the necessary changes to facilitate the fact that we are working, in general, on a non-trivial bundle.

Let $\mathscr{H} = \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Let us consider $\nabla : \mathrm{W}^{1,2}(\mathcal{V}) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ as a closed, densely-defined operator. Furthermore, let us denote its adjoint by $-\operatorname{div} : \mathcal{D}(\operatorname{div}) \subset \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$.

With this notation at hand, let $A_{00} \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$, $A_{01} \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}, \mathcal{V}))$, $A_{10} \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}, \mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$, $A_{11} \in \mathrm{L}^\infty(\mathcal{L}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}))$. Then, define $A \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V} \oplus (\mathrm{T}^*\mathcal{M} \otimes$

$\mathcal{V})))$ by

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}.$$

Furthermore, let $a \in \mathrm{L}^\infty(\mathcal{L}(\mathcal{V}))$. Set $B_1, B_2 : \mathscr{H} \to \mathscr{H}$ by

$$B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Moreover, set

$$S = \begin{pmatrix} \mathrm{I} \\ \nabla \end{pmatrix}$$

so that

$$S^* = \begin{pmatrix} \mathrm{I} & -\operatorname{div} \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \ \text{and } \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix},$$

and define the following divergence-form operator $\mathrm{L}_A : \mathcal{D}(\mathrm{L}_A) \subset \mathrm{L}^2(\mathcal{V}) \to \mathrm{L}^2(\mathcal{V})$ by

$$\mathrm{L}_A u = aS^*ASu = -a \operatorname{div}(A_{11}\nabla u) - a \operatorname{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$$

We apply Theorem 3.2.4 to prove the Kato square root problem on vector bundles.

Theorem 5.2.1 (Kato square root problem for vector bundles)**.** *Suppose that $\mathcal{M}$ is a complete, smooth Riemannian manifold and both $\mathcal{V}$ and $\mathrm{T}^*\mathcal{M}$ have generalised bounded geometry (so that they are equipped with GBG coordinate systems), and that*

(i) $\mathcal{M}$ satisfies (P_{loc}) and (Exp),

(ii) the connection ∇ and the metric h are compatible,

(iii) there exists $C > 0$ such that in each GBG coordinate chart $\{e^j\}$ for $\mathcal{V}$ and $\{dx^i\}$ for $\mathrm{T}^*\mathcal{M}$, we have that $|\nabla e^j|, |\nabla dx^i|, |\partial_k \mathrm{h}^{ij}|, |\partial_k \mathrm{g}^{ij}| \leq C$ a.e.,

(iv) there exist $\kappa_1, \kappa_2 > 0$ such that

$$\operatorname{Re}\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and} \qquad \operatorname{Re}\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{V})$ and $v \in \mathrm{W}^{1,2}(\mathcal{V})$, and

(v) we have that $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{V})$, and there exist $C' > 0$ such that

$$\|\nabla^2 u\| \leq C' \|(\mathrm{I} + \Delta)u\|$$

whenever $u \in \mathcal{D}(\Delta)$.

Then,

(i) Π_B *has a bounded holomorphic functional calculus, and*

(ii) $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{V})$ *with* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{V})$.

Proof. We show that (Γ, B_1, B_2) satisfy (H1)-(H6), (H7-I) and (H8-I) and invoke Theorem 3.2.4 and Theorem 1.8.2.

Nilpotency of Γ is immediate. That Γ is densely-defined and closed follows easily from Proposition 2.2.12. This settles (H1). Also, (H3) is an easy calculation and (H4)-(H5) are immediate. The fact that (H6) is satisfied is an immediate consequence of the Leibniz property of the connection. The conditions (i) and (ii) allow us to invoke Proposition 5.1.3, thus proving (H8-I)-1. It remains to demonstrate (H2), (H7-I), and (H8-I)-2 hold with $\Xi : \mathcal{D}(\Xi) \subset \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \to \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V}) \oplus \mathrm{L}^2(\mathcal{T}^{(0,2)}\mathcal{M} \otimes \mathcal{V})$ defined by $\Xi(u_1, u_2, u_3) = (\nabla u_1, \nabla u_2, \nabla u_3)$. The domain of Ξ is $\mathcal{D}(\Xi) = \mathrm{W}^{1,2}(\mathcal{V}) \oplus \mathrm{W}^{1,2}(\mathcal{V}) \oplus \mathrm{W}^{1,2}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$.

Fix $u \in \mathcal{R}(\Gamma^*)$. That is, $u = (S^* v, 0)$ for $v \in \mathrm{L}^2(\mathcal{V}) \oplus \mathrm{L}^2(\mathrm{T}^* M \otimes \mathcal{V})$. Thus,

$$\operatorname{Re} \langle B_1 u, u\rangle = \operatorname{Re} \langle a S^* v, S^* v\rangle \geq \kappa_1 \|S^* v\|^2 = \kappa_1 \|u\|^2.$$

Next, let $u \in \mathcal{R}(\Gamma)$. Thus, $u = (0, Sv)$ for $v \in \mathrm{L}^2(\mathcal{V})$. Therefore,

$$\operatorname{Re} \langle B_2 u, u\rangle = \operatorname{Re} \langle A S v, S v\rangle \geq \kappa_2 \|u\|^2$$

which settles (H2).

We verify (H7-I). First, let $u = (u_1, u_2, u_3) \in \mathcal{D}(\Gamma)$ with $\operatorname{spt} u \subset Q$ and $v = (v_1, v_2, v_3) \in \mathcal{D}(\Gamma^*)$ with $\operatorname{spt} v \subset Q$. Then, $\Gamma u = (0, S u_1) = (0, u_1, \nabla u_1)$, $\Gamma^* v = (S^*(v_2, v_3), 0) = (v_2 - \operatorname{div} v_3, 0, 0)$ and we have that

$$\left|\int_Q \Gamma u \; d\mu\right| = \left|\int_Q u_1 \; d\mu\right| + \left|\int_Q \nabla u_1 \; d\mu\right|$$

and

$$\left|\int_Q \Gamma^* v \; d\mu\right| = \left|\int_Q v_2 - \operatorname{div} v_3 \; d\mu\right| \leq \left|\int_Q v_2 \; d\mu\right| + \left|\int_Q \operatorname{div} v_3 \; d\mu\right|.$$

By Cauchy-Schwartz,

$$\left|\int_Q u_1 \; d\mu\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u_1\| \leq \mu(Q)^{\frac{1}{2}} \|u\|,$$

and by a similar computation,

$$\left|\int_Q v_2 \; d\mu\right| \lesssim \mu(Q)^{\frac{1}{2}} \|v\|.$$

To conveniently deal with the two remaining estimates, we omit the indices in u_1 and v_3 and note that it remains to prove

$$\text{(a)} \quad \left|\int_Q \nabla u\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \text{(b)} \quad \left|\int_Q \operatorname{div} v\right| \lesssim \mu(Q)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\nabla)$, $v \in \mathcal{D}(\operatorname{div})$ with spt u, spt $v \subset Q$.

Before continuing, we remark that every function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ can be written as $f = f_i e^i = \mathrm{h}(f, \mathrm{h}_{ki} e^k) e^i$ in $B(x_{\widehat{Q}}, \rho)$. Here $\mathrm{h}_{ij} = \mathrm{h}(e_i, e_j)$, where $\{e_i\}$ is the dual basis of $\{e^i\}$, and we use the same notation h to denote the induced inner product on $\mathcal{V}^*$ by requiring that $\mathrm{h}_{ij}\mathrm{h}^{jk} = \delta_i^k$. We also remark that every function $F \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$ can be written as $F = F_{ij} \; dx^i \otimes e^j = \mathrm{g} \otimes \mathrm{h}(F, \mathrm{g}_{ai} h_{bj} \; dx^a \otimes e^b) \; dx^i \otimes e^j$ in $B(x_{\widehat{Q}}, \rho)$ where $\mathrm{g}_{ij} = \mathrm{g}(\partial_i, \partial_j)$ on $\mathrm{T}\mathcal{M}$.

Turning to the proof of (a), let $u \in \mathrm{W}^{1,2}(\mathcal{V})$ with spt $u \subset Q$. Choose $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$ such that spt $\psi \subset B(x_{\widehat{Q}}, \rho)$ and $\psi = 1$ on Q, and extend $\psi \mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b$ to be zero outside of $B(x_{\widehat{Q}}, \rho)$. Then, by the above remark with $F = \nabla u$, we have the following identity on Q.

$$\begin{aligned}
\int_Q \nabla u &= \int_Q \mathrm{g} \otimes \mathrm{h}(\nabla u, \psi \mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b) \; d\mu \; dx^i \otimes e^j \\
&= \int_{\mathcal{M}} \mathrm{g} \otimes \mathrm{h}(\nabla u, \psi \mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b) \; d\mu \; dx^i \otimes e^j \\
&= \int_{\mathcal{M}} \mathrm{h}(u, -\operatorname{tr} \nabla(\psi \mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b)) \; d\mu \; dx^i \otimes e^j \\
&= \int_Q \mathrm{h}(u, -\operatorname{tr} \nabla(\mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b)) \; d\mu \; dx^i \otimes e^j \\
&= \int_Q -\mathrm{h}(u, \mathrm{g}_{ai}\mathrm{h}_{bj} \operatorname{tr} \nabla(dx^a \otimes e^b) + \operatorname{tr}((\nabla \mathrm{g}_{ai}\mathrm{h}_{bj}) \otimes dx^a \otimes e^b)) \; d\mu \; dx^i \otimes e^j \; .
\end{aligned}$$

We have used Proposition 2.2.1 (since $\mathrm{g}_{ai}\mathrm{h}_{bj} \; dx^a \otimes e^b$ are smooth), the product rule for ∇, and the linearity of tr. We note that by an easy calculation, we have

$|\mathrm{tr}(X)| \lesssim |X|$ for all $x \in \mathcal{M}$ whenever $X \in \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}^*\mathcal{M} \otimes \mathcal{V})$. Furthermore, the bound on the metric in each GBG chart implies bounds on $|\mathrm{h}_{ai}|$ and $|\mathrm{g}_{ai}|$, and the bounds in (iv) imply bounds on $|\partial_k \mathrm{h}_{ai}|$ and $|\partial_k \mathrm{g}_{bj}|$. Also, $|e^b|$ and $|dx^a|$ are bounded by the GBG hypothesis, $|\nabla e^b|$ and $|\nabla dx^i|$ by (iv), and so we conclude that

$$\left|\mathrm{g}_{ai}\mathrm{h}_{bj} \,\mathrm{tr}\, \nabla (dx^a \otimes e^b) + \mathrm{tr}((\nabla \mathrm{g}_{ai}\mathrm{h}_{bj}) \otimes dx^a \otimes e^b)\right| \lesssim 1.$$

On combining these estimates, and applying the Cauchy-Schwartz inequality, we conclude that

$$\left|\int_Q \nabla u\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\|$$

as required.

To verify (b), let $v \in \mathcal{D}(\mathrm{div})$ with $\mathrm{spt}\, v \subset Q$, and apply the above remark with $f = \mathrm{div}\, v$ to obtain by a similar argument that

$$\int_Q \mathrm{div}\, v = \int_Q \mathrm{h}(\mathrm{div}\, v, \mathrm{h}_{ki}\ e^k)\ d\mu\ e^i = \int_Q \mathrm{g} \otimes \mathrm{h}(v, \nabla(\mathrm{h}_{ki}\ e^k))\ d\mu\ e^i.$$

Reasoning as before, we have that $\left|\nabla(\mathrm{h}_{ki}\ e^k)\right|$ is bounded and by the Cauchy-Schwartz inequality, we conclude that

$$\left|\int_Q \mathrm{div}\, v\right| \leq \mu(Q)^{\frac{1}{2}} \|v\|$$

as required. This completes the proof of (H7-I).

To show (H8-I) let $\Xi(u_1, u_2, u_3) = (\nabla u_1, \nabla u_2, \nabla u_3)$. Upon noting that

$$\left|\nabla(dx^i \otimes e^j)\right| \leq \left|\nabla dx^i\right| \left|e^j\right| + \left|\nabla e^j\right| \left|dx^i\right| \lesssim 1,$$

we apply Proposition 5.1.3 which proves (H8-I)-1.

It remains to show (H8-I)-2. Fix $v = (v_1, v_2, v_3) \in \mathcal{R}(\Pi) \cap \mathcal{D}(\Pi)$. Thus, there is $u = (u_1, u_2, u_3) \in \mathcal{D}(\Pi)$ such that $v = \Pi u = (u_2 - \mathrm{div}\, u_3, u_1, \nabla u_1)$. A calculation the shows that

$$\|v\|^2 + \|\Xi v\|^2 = \|u_1\|^2 + 2\|\nabla u_1\|^2 + \|\nabla^2 u_1\| + \left(\|v_1\|^2 + \|\nabla v_1\|^2\right).$$

Also,

$$\|\Pi v\|^2 = \|(\mathrm{I} + \Delta)u_1\|^2 + \|v_1\|^2 + \|\nabla v_1\|^2.$$

But note that

$$\|(\mathrm{I}+\Delta)u_1\|^2 = \langle(\mathrm{I}+\Delta)u_1, (\mathrm{I}+\Delta)u_1\rangle = \|u_1\|^2 + 2\|\nabla u_1\|^2 + \|\Delta u_1\|^2.$$

Combining these estimates with (iv), we have that $\|v\|^2 + \|\Xi v\|^2 \lesssim \|\Pi v\|$ which proves (H8-I)-2.

Now, by invoking Theorem 3.2.4, we conclude that Π_B has a bounded holomorphic functional calculus. By Theorem 1.8.2, we conclude that $\|\sqrt{{\Pi_B}^2}v\| \simeq \|\Pi_B v\|$ for $v \in \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{{\Pi_B}^2})$. Fix $u \in \mathrm{W}^{1,2}(\mathcal{V})$. Then $v = (u, 0, 0) \in \mathcal{D}(\Pi_B)$ and

$$\|\sqrt{\mathrm{L}_A}u\| = \|\sqrt{{\Pi_B}^2}v\| \simeq \|\Pi_B v\| = \|v\|_{\mathrm{W}^{1,2}}$$

which finishes the proof. □

Remark 5.2.2. *(i) Instead of taking ∇ to be a connection, we could have considered a* sub-connection*, on some sub-bundle $\mathcal{E} \subset \mathrm{T}\mathcal{M}$. As described in Chapter 4, this is related to the study of square roots of subelliptic operators associated with sub-Laplacians on Lie groups. However, it is not clear whether a sub-connection is automatically densely-defined and closable.*

(ii) If we assume that the dx^i arise from coordinate systems on $\mathcal{M}$, then $|\nabla dx^i| \lesssim 1$ is automatic since ∇dx^i is written in terms of Christoffel symbols which only depend on the bounded quantities g_{ab}, g^{ab} and $\partial_c \mathrm{g}_{ab}$.

5.3 Manifolds with injectivity and Ricci bounds

In this section, we apply Theorem 5.2.1 to establish the Kato square root estimates for manifolds which have injectivity radius bounds and Ricci bounds. Our approach is to use the existence of *harmonic coordinates* under the assumptions we make which allow us to control the metric and its derivatives. We then use these coordinates to show that the bundle of (p, q) tensors satisfy the GBG criterion, as described in the following proposition.

Proposition 5.3.1 (Existence of GBG coordinates for finite rank tensors)**.** *Under the assumptions that $\mathrm{inj}(M) \geq \kappa > 0$ and $|\mathrm{Ric}| \leq \eta$, there exist GBG coordinates for $\mathcal{T}^{(p,q)}\mathcal{M}$. Furthermore, in each basis $\{e^i\}$ in each GBG coordinate system for $\mathcal{T}^{(p,q)}\mathcal{M}$, we have that $|\nabla e^i| \leq C_{p,q}$.*

Proof. First, we note that the Ricci bounds imply that there exists $\eta \in \mathbb{R}$ such that that $\mathrm{Ric} \geq \eta \mathrm{g}$. Thus, as in the proof of Theorem 1.1 in [46], we conclude that $\mathcal{M}$ satisfies (Exp).

Fix $A = 2$ and $\alpha = \frac{1}{2}$. The previous theorem guarantees the existence of harmonic coordinates for these choices. Thus, this yields GBG coordinates for $\mathrm{T}\mathcal{M}$ with $2^{-1} \leq \mathrm{g} \leq 2$.

It is an easy calculation to show that $G \simeq \mathrm{I}$ implies $G^{-1} \simeq \mathrm{I}$ with the same constants for a positive definite matrix G. Thus, we obtain GBG coordinates for $\mathrm{T}^*\mathcal{M}$ with $2^{-1} \leq \mathrm{g} \leq 2$ where we denote the metric on $\mathrm{T}^*\mathcal{M}$ also by g.

Next, if two inner products u, v on vector spaces U, V satisfy $C_1^{-1} \leq u \leq C_1$ and $C_2^{-1} \leq v \leq C_2$, then $(C_1C_2)^{-1} \leq u \otimes v \leq C_1C_2$ on $U \otimes V$. Thus, by induction, we have that $2^{-pq} \leq \mathrm{g} \leq 2^{pq}$ for the metric g on $\mathcal{T}^{(p,q)}\mathcal{M}$.

For the gradient bounds, first consider $\mathrm{T}\mathcal{M} = \mathcal{T}^{(1,0)}\mathcal{M}$. Since we assume our connection is Levi-Civita, we can write the Christoffel symbols Γ_{ij}^k purely in terms of $\partial_a \mathrm{g}_{bc}$ and g^{ab}. The Cauchy-Schwartz inequality allows us to bound the g^{ab} and the bounds on $\partial_a \mathrm{g}_{bc}$ comes from the theorem. Thus, $\left|\Gamma_{ij}^k\right| \leq C$ and so $|\nabla e_i| \leq C$. An inductive argument then yields the result for $\mathcal{T}^{(p,q)}\mathcal{M}$ with the constant dependent on p and q. $\square$

With this result, we can apply the general Theorem 5.2.1 to obtain the following solution to the Kato square root problem on finite rank tensors.

Theorem 5.3.2 (Kato square root problem on finite rank tensors)**.** *Let $(\mathcal{M}, \mathrm{g})$ be a smooth, complete Riemannian manifold. Suppose that $|\mathrm{Ric}| \leq C$, $\mathrm{inj}(M) \geq \kappa > 0$, and the following ellipticity condition holds: there exist $\kappa_1, \kappa_2 > 0$ such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \text{and} \qquad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{T}^{(p,q)}\mathcal{M})$ and $v \in \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$. Suppose further that $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{V})$ and that there exists $C' > 0$ such that

$$\|\nabla^2 u\| \leq C' \|(\mathrm{I} + \Delta)u\| \tag{R}$$

whenever $u \in \mathcal{D}(\Delta)$. Then, $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$ and $\|\sqrt{\mathrm{L}_A}u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ for all $u \in \mathrm{W}^{1,2}(\mathcal{T}^{(p,q)}\mathcal{M})$.

Proof. We apply Theorem 5.2.1. The Ricci bounds imply that there exists $\eta \in \mathbb{R}$ such that that $\mathrm{Ric} \geq \eta \mathrm{g}$. This shows condition (i) as in the proof of Theorem 1.1 in [46]. Conditions (ii) and (iv) are a consequence of Proposition 5.3.1, and (iii) holds because the connection is Levi-Cevita. □

The Riesz transform condition (R) is satisfied automatically for $(0,0)$ tensors, or in other words, for scalar-valued functions, as we now show.

Theorem 5.3.3 (Kato square root problem for functions)**.** *Let* $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold with* $|\mathrm{Ric}| \leq C$ *and* $\mathrm{inj}(M) \geq \kappa > 0$*. Suppose that the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \textit{and}$$
$$\mathrm{Re}(\langle A_{11}\nabla v, \nabla v\rangle + \langle A_{10} v, \nabla v\rangle + \langle A_{01}\nabla v, v\rangle + \langle A_{00} v, v\rangle) \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$*. Then,* $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathcal{D}(\nabla) = \mathrm{W}^{1,2}(\mathcal{M})$ *and* $\|\sqrt{\mathrm{L}_A} u\| \simeq \|\nabla u\| + \|u\| = \|u\|_{\mathrm{W}^{1,2}}$ *for all* $u \in \mathrm{W}^{1,2}(\mathcal{M})$*.*

Proof. Recall the Bochner-Lichnerowicz-Weitzenböck identity

$$\langle \Delta(\nabla f), \nabla f\rangle = \frac{1}{2}\Delta(|\nabla f|^2) + \left|\nabla^2 f\right|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

for all $f \in \mathrm{C}^\infty(\mathcal{M})$. When $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$, we get $\|\nabla^2 f\| \lesssim \|(\mathrm{I} + \Delta) f\|$ by integrating this identity.

By Theorem 2.2.6 and Theorem 2.4.1 we conclude that $\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$ is dense in both $\mathcal{D}(\Delta)$ and $\mathrm{W}^{2,2}(\mathcal{M})$. Thus, $\mathcal{D}(\Delta) \subset \mathrm{W}^{2,2}(\mathcal{M})$ and $\|\nabla^2 u\| \lesssim \|(\mathrm{I} + \Delta)u\|$ for $u \in \mathcal{D}(\Delta)$. Hence the hypotheses of Theorem 5.3.2 hold and the theorem is proved. □

5.4 Lipschitz Estimates and Stability

In §1.8, we assert that obtaining a bounded holomorphic functional calculus gives us holomorphic dependency results for free. The purpose of this short section is to present the corresponding stability consequences from the quadratic estimates we establish in §5.2 and §5.3.

Let $\tilde{a}$ and $\tilde{A}$ satisfy the same conditions as specified for a and A prior to Theorem 5.2.1, and set

$$\tilde{B}_1 = \begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A} \end{pmatrix}.$$

We obtain the following Lipschitz estimates as a consequence of Theorem 1.8.5. The proof of this theorem is similar to the proof of Theorem 4.5.1.

Theorem 5.4.1 (Lipschitz estimate)**.** *Assume the hypotheses of Theorem 5.2.1 and fix $\eta_i < \kappa_i$. Suppose that $\tilde{B}_i$ satisfy $\|\tilde{B}_i\|_\infty \leq \eta_i$ for $i = 1, 2$ and set $0 < \hat{\omega}_i < \frac{\pi}{2}$ by $\cos \hat{\omega}_i = \frac{\kappa_i - \eta_i}{\|B_i\|_\infty + \eta_i}$ and $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_2)$. Then, for all $\hat{\omega} < \mu < \frac{\pi}{2}$,*

$$\|f(\Pi_B) - f(\Pi_{B+\tilde{B}})\| \lesssim (\|\tilde{B}_1\|_\infty + \|\tilde{B}_2\|_\infty)\|f\|_\infty$$

for all $f \in \mathrm{Hol}^\infty(\mathrm{S}^{\mathrm{o}}_\mu)$, and

$$\int_0^\infty \|\psi(t\Pi_B)v - \psi(t\Pi_{B+\tilde{B}})v\|^2 \frac{dt}{t} \lesssim (\|\tilde{B}_1\|_\infty^2 + \|\tilde{B}_2\|_\infty^2)\|v\|^2,$$

for all $\psi \in \Psi(\mathrm{S}^{\mathrm{o}}_\mu)$ and all $v \in \mathscr{H}$. The implicit constants depend in particular on on B_i and η_i.

We use the coefficients $\tilde{a}$ and $\tilde{A}$ to perturb the coefficients a and A. Then, we construct the following perturbed operator $\mathrm{L}_{A+\tilde{A}}$ defined similar to L_A given by $\mathrm{L}_{A+\tilde{A}} u = (a + \tilde{a})S^*(A + \tilde{A})Su$ for $u \in \mathcal{D}(\mathrm{L}_{A+\tilde{A}})$.

Theorem 5.4.2 (Stability of the square root)**.** *Assume the hypotheses of Theorem 5.2.1 and fix $\eta_i < \kappa_i$. If $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$, then*

$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{V})$. The implicit constant depends in particular on A, a and η_i.

Proof. The proof is essentially the same as that of the proof of Theorem 4.5.2 except that in place of invoking Theorem 4.5.1, we invoke Theorem 5.4.1. □

We point out that the conclusions of both these theorems hold if, instead of assuming the hypotheses of Theorem 5.2.1, we assume the hypotheses of Theorem 5.3.2. This yields Lipschitz estimates and the stability result for (p, q) tensors. We conclude this section by highlighting the stability of the square root for scalar-valued functions as a corollary.

Corollary 5.4.3 (Stability of the square root for functions)**.** *Let* $(\mathcal{M}, \mathrm{g})$ *be a smooth, complete Riemannian manifold. Suppose that* $|\mathrm{Ric}| \leq C$, $\mathrm{inj}(M) \geq \kappa > 0$ *and the following ellipticity condition holds: there exist* $\kappa_1, \kappa_2 > 0$ *such that*

$$\mathrm{Re}\,\langle au, u\rangle \geq \kappa_1 \|u\|^2 \qquad \textit{and} \qquad \mathrm{Re}\,\langle ASv, Sv\rangle \geq \kappa_2 \|v\|^2_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{L}^2(\mathcal{M})$ *and* $v \in \mathrm{W}^{1,2}(\mathcal{M})$*. Fix* $\eta_i < \kappa_i$*. If* $\|\tilde{a}\|_\infty \leq \eta_1$, $\|\tilde{A}\|_\infty \leq \eta_2$*, then*

$$\|\sqrt{\mathrm{L}_A}\, u - \sqrt{\mathrm{L}_{A+\tilde{A}}}\, u\| \lesssim (\|\tilde{a}\|_\infty + \|\tilde{A}\|_\infty)\|u\|_{\mathrm{W}^{1,2}}$$

for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$*. The implicit constant depends on* $C, \kappa, \kappa_i, A, a$ *and* η_i*.*

6

Inhomogeneous Hodge-Dirac operators on manifolds

In this chapter, we prove the Kato square root problem for perturbations of inhomogeneous Hodge-Dirac operators on manifolds under a set of natural curvature assumptions. We first introduce a kind of “twisted” inhomogeneity on *spacetime manifolds* $\mathcal{M} \times \mathbb{S}^1$ and $\mathcal{M} \times \mathbb{R}$. In the last section, we consider a more standard inhomogeneous setup. We begin our presentation by establishing some facts we need about product manifolds.

6.1 Products of manifolds

Let $(\mathcal{M}, \mathrm{g}^{\mathcal{M}}, \nabla^{\mathcal{M}})$ and $(\mathcal{N}, \mathrm{g}^{\mathcal{N}}, \nabla^{\mathcal{N}})$ be two Riemannian manifolds, where the triple represents the manifold, the metric and Levi-Cevita connection respectively.

Define $\mathcal{P} = \mathcal{M} \times \mathcal{N}$. We define the *product metric* g on $\mathcal{P}$ by writing $\mathrm{g}(x, y) = \mathrm{g}^{\mathcal{M}}(x) + \mathrm{g}^{\mathcal{N}}(y)$ after extending $\mathrm{g}^{\mathcal{M}}$ to $\mathrm{T}\mathcal{P}$ by letting it vanish on on $\mathrm{T}\mathcal{N}$ and $\mathrm{g}^{\mathcal{N}}$ to $\mathrm{T}\mathcal{P}$ by letting vanish on $\mathrm{T}\mathcal{M}$. This ensures that $\mathrm{T}\mathcal{P} = \mathrm{T}\mathcal{M} \oplus^{\perp} \mathrm{T}\mathcal{N}$.

Let (x, y) in $\mathcal{P}$ and $U \times V \subset \mathcal{M} \times \mathcal{N} = \mathcal{P}$ be a coordinate system near (x, y). Let $\mathrm{span}\,\{\partial_{x^i}\} = \mathrm{T}U$ and $\mathrm{span}\,\{\partial_{y^j}\} = \mathrm{T}V$. We extend $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{N}}$ to $\mathrm{T}\mathcal{P}$ in the following way. First, let $\nabla_{\partial_{x^i}} \partial_{y^j} = \nabla_{\partial_{y^j}} \partial_{x^i} = 0$. For $X, Y \in \mathrm{C}^{\infty}(\mathrm{T}\mathcal{P})$, we note that the metric induces the splitting $X = X_{\mathcal{M}} + X_{\mathcal{N}}$ and $Y = Y_{\mathcal{M}} + Y_{\mathcal{N}}$

where $X_{\mathcal{M}}, Y_{\mathcal{M}} \in \mathrm{C}^\infty(\mathrm{T}\mathcal{M})$ and $X_{\mathcal{N}}, Y_{\mathcal{N}} \in \mathrm{C}^\infty(\mathrm{T}\mathcal{N})$. We shall use this notation throughout this section. Next, define

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\mathcal{M}}_{X_{\mathcal{M}}} Y_{\mathcal{M}} + X^i_{\mathcal{M}} \partial_{x^i}(Y^j_{\mathcal{N}})\partial_{y^j} \quad \text{and} \quad \nabla^{\mathcal{N}}_X Y = \nabla^{\mathcal{N}}_{X_{\mathcal{N}}} Y_{\mathcal{N}} + X^j_{\mathcal{N}} \partial_{y^j}(Y^i_{\mathcal{M}})\partial_{x^i}.$$

Indeed, these are readily checked to be connections on $\mathcal{P}$ since ∂_{x^i} is independent of y and ∂_{y^i} is independent of x, even though the coefficients of these vectorfields are dependent on both x and y. Finally, we define

$$\nabla_X Y = \nabla^{\mathcal{M}}_X Y + \nabla^{\mathcal{N}}_X Y.$$

We have the following important proposition about our setup.

Proposition 6.1.1. *We have that $(\mathcal{P}, \mathrm{g}, \nabla)$ is a smooth Riemannian manifold and ∇ is the Levi-Cevita connection with respect to* g.

Proof. The smoothness of $\mathcal{P}$ is easy: at each $(x, y) \in \mathcal{P}$, the coordinate charts are of the form $U \times V$ with U being coordinates for $\mathcal{M}$ at x and V coordinates for $\mathcal{N}$ at y. Also, it is easy to see that the metric g is smooth since each $\mathrm{g}^{\mathcal{M}}$ and $\mathrm{g}^{\mathcal{N}}$ is smooth.

We show that ∇ is Levi-Cevita with respect to g. Fix $(x, y) \in \mathcal{P}$ and let $U \times V$ be a coordinate chart. Now, $\nabla_{\partial_{x^i}} \partial_{x^j} = \nabla^{\mathcal{M}}_{\partial_{x^i}} \partial_{x^j}$, $\nabla_{\partial_{y^i}} \partial_{y^j} = \nabla^{\mathcal{N}}_{\partial_{y^i}} \partial_{y^j}$, and $\nabla_{\partial_{x^i}} \partial_{y^j} = 0$. This shows that in each case, the Christoffel symbols of $\Gamma^k_{i,j} = \nabla_{\partial_i} \partial_j \partial_k$ are symmetric in i, j, which is equivalent to the statement that ∇ is torsion-free.

To prove metric compatibility, we note that $\nabla \mathrm{g}(x, y) = \nabla \mathrm{g}^{\mathcal{M}}(x) + \nabla \mathrm{g}^{\mathcal{N}}(y)$. Computing this locally,

$$\begin{aligned}
\nabla \mathrm{g}^{\mathcal{M}} &= \nabla(\mathrm{g}^{\mathcal{M}}_{ij}\ dx^i \otimes dx^j) \\
&= dx^k \otimes \nabla_{\partial_{x^k}}(\mathrm{g}^{\mathcal{M}}_{ij}\ dx^i \otimes dx^j) + dy^l \otimes \nabla_{\partial_{y^l}}(\mathrm{g}^{\mathcal{M}}_{ij}\ dx^i \otimes dx^j) \\
&= \nabla^{\mathcal{M}} \mathrm{g}^{\mathcal{M}} + dy^l \otimes (\partial_{y^l} \mathrm{g}^{\mathcal{M}}_{ij}\ dx^i \otimes dx^j) + dy^l \otimes (\mathrm{g}^{\mathcal{M}}_{ij}\ \nabla_{\partial_{y^l}}(dx^i \otimes dx^j).
\end{aligned}$$

By the metric compatibility of $\nabla^{\mathcal{M}}$ and $\mathrm{g}^{\mathcal{M}}$, we have the first term is zero. The second term vanishes since $\mathrm{g}^{\mathcal{M}}_{ij}$ is only dependent on x. The last term also vanishes since

$$(\nabla_{\partial_{y^l}} dx^i)(\partial_{x^j}) = \partial_{y^l}(dx^i(\partial_{x^j}) - dx^i(\nabla_{\partial_{y^l}} \partial_{x^j}) = 0.$$

Thus, since the Levi-Cevita connection is the unique connection satisfying the properties torsion-free and metric compatible, we have that ∇ is the Levi-Cevita connection of g. $\square$

Let the induced Riemannian volume measures of $\mathcal{M}$ and $\mathcal{N}$ be denoted by μ and ν and the induced Riemannian volume measure of $\mathcal{P}$ be denoted by σ. The product of the measures μ and ν are then denoted by $\mu\ \nu(A \times B) = \mu(A)\nu(B)$. The following proposition describes σ in terms of μ and ν

Proposition 6.1.2. *The volume measure is given by $\sigma = \mu\ \nu$.*

Proof. As previously, assume local coordinates at (x, y) and write g as a matrix. Then, by orthogonality of $\mathrm{T}^*\mathcal{M}$ and $\mathrm{T}^*\mathcal{N}$, we can write g as the block matrix

$$\mathrm{g} = \begin{pmatrix} \mathrm{g}^{\mathcal{M}} & 0 \\ 0 & \mathrm{g}^{\mathcal{N}} \end{pmatrix}.$$

Thus, $\det \mathrm{g} = \det \mathrm{g}^{\mathcal{M}} \det \mathrm{g}^{\mathcal{N}}$ which establishes the claim. $\square$

Let $\mathcal{R}^{\mathcal{M}}$ and $\mathcal{R}^{\mathcal{N}}$ denote the curvature of $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{N}}$ respectively and $\mathcal{R}$ the curvature of ∇. We denote the Riemann curvature tensor by $\mathrm{Rm}^{\mathcal{M}}$, $\mathrm{Rm}^{\mathcal{N}}$ and Rm respectively. Similarly, let the Ricci curvatures be denoted by $\mathrm{Ric}^{\mathcal{M}}$, $\mathrm{Ric}^{\mathcal{N}}$ and Ric.

Proposition 6.1.3. *Let $X, Y, Z, W \in \mathrm{C}^\infty(\mathrm{T}\mathcal{P})$. Then,*

$$\mathcal{R}(X, Y)Z = \mathcal{R}^{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}})Z_{\mathcal{M}} + \mathcal{R}^{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}})Z_{\mathcal{N}}.$$

Moreover,

$$\mathrm{Rm}(X, Y, Z, W) = \mathrm{Rm}^{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}}, Z_{\mathcal{M}}, W_{\mathcal{M}}) + \mathcal{R}^{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}}, Z_{\mathcal{N}}, W_{\mathcal{N}}),$$

and

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}^{\mathcal{M}}(X_{\mathcal{M}}, Y_{\mathcal{M}}) + \mathrm{Ric}^{\mathcal{N}}(X_{\mathcal{N}}, Y_{\mathcal{N}}).$$

Proof. For local coordinates near (x, y), we have

$$\mathcal{R}(\partial_{y^i}, \partial_{x^i})\partial_{x^k} = \nabla_{\partial_{y^i}}(\nabla_{\partial_{x^j}}\partial_{x^k})\nabla_{\partial_{x^j}}(\nabla_{\partial_{y^i}}\partial_{x^k}) = 0,$$

and similarly, $\mathcal{R}(\partial_{x^i}, \partial_{y^i})\partial_{x^k} = \mathcal{R}(\partial_{x^i}, \partial_{x^j})\partial_{y^k} = 0$. Also, $\mathcal{R}$ is function linear and so, $\mathcal{R}(X_{\mathcal{M}}, Y_{\mathcal{N}})Z = \mathcal{R}(X_{\mathcal{M}}, Y)Z_{\mathcal{N}} = \mathcal{R}(X, Y_{\mathcal{M}})Z_{\mathcal{N}} = 0$. Moreover, this equation is invariant under interchanging $\mathcal{M}$ and $\mathcal{N}$. This proves the first claim. For the second claim, a contraction through the metric keeping in mind that $\mathrm{T}^*\mathcal{M} \oplus^\perp \mathrm{T}^*\mathcal{N}$ establishes the claim. The last claim then follows immediately, by tracing the curvature tensor. $\square$

6.2 Spacetime manifolds and inhomogeneous operators

In this section, we fix $\mathcal{N}$ to be a connected 1-manifold. By the classification of connected 1-manifolds, the only non-compact 1-manifold is $\mathbb{R}$ and the only compact one is $\mathbb{S}^1$. We use the notation $\mathcal{M}_{\mathrm{st}} = \mathcal{M} \times \mathcal{N}$ to denote the *spacetime* manifold. That is, we interpret the coordinates of $\mathcal{N}$ as time. We emphasise that, despite referring to $\mathcal{M}_{\mathrm{st}}$ as a spacetime manifold, the metric on $\mathcal{M}_{\mathrm{st}}$ is Riemannian, not Lorentzian.

Let $\mathbf{\Omega}_1^p(\mathcal{M}_{\mathrm{st}})$ be the space of forms which can be written locally as $\omega_{i_1\dots i_p}(x,t)\ dx^{i_1\cdots i_p}$ and $\mathbf{\Omega}_2^p(\mathcal{M}_{\mathrm{st}})$ as the space of forms locally expressed as $\omega_{i_1\dots i_{p-1}}(x,t)\ ds\wedge dx^{i_1\cdots i_{p-1}}$. An immediate consequence of the orthogonality of the product space is the orthogonal splitting $\mathbf{\Omega}^p(\mathcal{M}_{\mathrm{st}}) = \mathbf{\Omega}_1^p(\mathcal{M}_{\mathrm{st}}) \oplus^\perp \mathbf{\Omega}_2^p(\mathcal{M}_{\mathrm{st}})$ and $\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}) = \mathbf{\Omega}_1(\mathcal{M}_{\mathrm{st}}) \oplus^\perp \mathbf{\Omega}_2(\mathcal{M}_{\mathrm{st}})$. By $\mathbf{P}_{\mathbf{\Omega}_1(\mathcal{M}_{\mathrm{st}})}$, let us denote the projection $\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})$ to $\mathbf{\Omega}_1(\mathcal{M}_{\mathrm{st}})$ and by $\mathbf{P}_{\mathbf{\Omega}_2(\mathcal{M}_{\mathrm{st}})}$, the projection from $\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})$ to $\mathbf{\Omega}_2(\mathcal{M}_{\mathrm{st}})$.

In addition to the fact that $\mathbb{R}$ and $\mathbb{S}^1$ are complete and connected, they are also Lie groups. Since they are 1-dimensional, $\mathrm{T}^*\mathcal{N}$ orients $\mathcal{N}$ and we can find a unique (up to orientation) $\tau \in \mathrm{C}^\infty(\mathrm{T}^*\mathcal{N})$ such that $|\tau| = 1$. We use the same notation to denote the embedding of τ into $\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})$. We call τ the *timekeeper*, and emphasise that it is constant in the direction of $\mathcal{M}$.

The following coordinates which we define near some $t \in \mathcal{N}$ is often useful in calculations to follow.

Definition 6.2.1 (Neighbourhood normal coordinates)**.** *Let $t \in \mathcal{N}$. We define coordinates $\{s\}$ near t in a neighbourhood $\mathcal{U}_{\mathcal{N}}$. That such a neighbourhood exists is an easy fact. If $\mathcal{N} = \mathbb{R}$, then we let s be the usual coordinates, given by $s \mapsto s+t$ for $s \in \mathbb{R}$. These coordinates are, in fact, global (so that $\mathcal{U}_{\mathbb{R}} = \mathbb{R}$). For the case of $\mathcal{N} = \mathbb{S}^1 \subset \mathbb{R}^2$, we write $t = (\sin t', \cos t')$ and so we we let these coordinates be $s \mapsto (\sin(s+t'), \cos(s+t'))$ for $s \in (0, 2\pi)$. Then, $\mathcal{U}_{\mathbb{S}^1} = \{(\sin(s+t'), \cos(s+t')) : s \in (0,2\pi)\}$.*

Proposition 6.2.2. *The neighbourhood normal coordinates are normal coordinates in $\mathcal{U}_{\mathcal{N}}$, and in particular at t. Furthermore, $\tau = ds$ or $\tau = -ds$ inside $\mathcal{U}_{\mathcal{N}}$.*

Proof. Since both $\mathbb{R}$ (under addition) and $\mathbb{S}^1$ are Lie groups, we restrict computations to the identity. First, it is easy to see that these coordinates are given by exponentiation. For the case of $\mathbb{R}$ this is trivial, and for the case of $\mathbb{S}^1$, this is best seen by considering the coordinates under the map $\Phi : \mathbb{S}^1 \to \{z \in \mathbb{C} : |z| = 1\}$ given by $\Phi(t_1, t_2) = t_1 + \imath t_2$. For the same reason, we are able to restrict our attention to a single point on $\mathbb{R}$ and $\mathbb{S}^1$ since we can obtain general points via translation.

In these coordinates, note that $|ds| = 1$. Writing $\tau = \tilde{\tau}\, ds$ near t, and then applying this second fact, we find that $|\tilde{\tau}| = 1$. We obtain the conclusion since τ is smooth. $\square$

We now define the following inhomogeneous spacetime Dirac operators.

Definition 6.2.3 (Inhomogenous interior, exterior differential and Dirac operator)**.** *Let $\alpha > 0$ and $\omega \in \mathbf{\Omega}^p(\mathcal{M}_{\mathrm{st}})$. Then, define $\mathrm{d}_\alpha : \mathbf{\Omega}^p(\mathcal{M}_{\mathrm{st}}) \to \mathbf{\Omega}^{p+1}(\mathcal{M}_{\mathrm{st}})$ and $\delta_\alpha : \mathbf{\Omega}^p(\mathcal{M}_{\mathrm{st}}) \to \mathbf{\Omega}^{p-1}(\mathcal{M}_{\mathrm{st}})$ by*

$$\mathrm{d}_\alpha \omega = \mathrm{d}\,\omega + \alpha\tau \wedge \omega \quad \textit{and} \quad \delta_\alpha\omega = \delta\omega + \alpha\tau \llcorner \omega.$$

The inhomogeneous Hodge-Dirac operator $\mathrm{D}_\alpha : \mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}) \to \mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})$ is then defined as $\mathrm{D}_\alpha = \mathrm{d}_\alpha + \delta_\alpha$.

We emphasise the fact that in this definition, the exterior derivative acts in both space and *time*. That is, locally, $\mathrm{d}\,\omega = \partial_i\omega(x,t)\ dx^i + \partial_t\omega(x,t)\ ds$ for $\omega \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M}_{\mathrm{st}})$. It is this fact that makes these operators non-trivial.

In the L^2 theory, it is not difficult to see that d_α and δ_α are adjoints of each other, and the operator D_α is self-adjoint. Furthermore, since these operators are respectively d and δ with a lower order term, we have the following density result. It is an easy consequence of Theorems 2.3.1 and 2.3.2.

Proposition 6.2.4. *We have that $\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))$ is dense in $\mathcal{D}(\mathrm{d}_\alpha)$, $\mathcal{D}(\delta_\alpha)$, $\mathcal{D}(\mathrm{D}_\alpha)$ and $\mathcal{D}(\mathrm{D}^2_\alpha)$.*

This density result is useful in many of the calculations we carry out in this section. We end this section with the following propositions.

Proposition 6.2.5. *The operators* d_α *and* δ_α *are nilpotent.*

Proof. Let $\omega \in \mathrm{C}^\infty_\mathrm{c}(\mathbf{\Omega}(\mathcal{M}_\mathrm{st}))$. Then,

$$\begin{aligned}\mathrm{d}^2_\alpha\,\omega = \mathrm{d}_\alpha(\mathrm{d}\,\omega + \alpha\tau\wedge\omega) = \mathrm{d}(\mathrm{d}\,\omega + \alpha\tau\wedge\omega) + \alpha\tau\wedge(\mathrm{d}\,\omega + \alpha\tau\wedge\omega)\\ = \mathrm{d}^2\,\omega + \alpha\,\mathrm{d}(\tau\wedge\omega) + \alpha\tau\wedge\mathrm{d}\,\omega + \alpha^2\tau\wedge\tau\wedge\omega.\end{aligned}$$

The nilpotency of d and the properties of $\wedge$ means that $\mathrm{d}^2\,\omega 0$ and $\tau\wedge\tau\wedge\omega = 0$. By the properties of d, we have that $\mathrm{d}(\tau\wedge\omega) + \tau\wedge\mathrm{d}\,\omega = (\mathrm{d}\,\tau)\wedge\omega$. But, $\mathrm{d}\,\tau = \mathrm{d}^{\mathcal{N}}\,\tau$, where $\mathrm{d}^{\mathcal{N}}$ is the exterior derivative of $\mathcal{N}$, since τ is constant in the direction of $\mathcal{M}$. Since $\mathcal{N}$ is a 1-manifold, $\mathrm{d}^{\mathcal{N}}\,\tau = 0$. This proves that $\mathrm{d}^2_\alpha = 0$. That $\delta^2_\alpha = 0$ can be obtained by duality. □

Proposition 6.2.6. $\mathrm{D}^2_\alpha = \mathrm{D}^2 + \alpha^2\mathrm{I}$.

Proof. Let $\omega \in \mathrm{C}^\infty_\mathrm{c}(\mathcal{M}_\mathrm{st})$, and note that by the nilpotency of d_α and δ_α, we have $\mathrm{D}^2_\alpha\,\omega = \mathrm{d}_\alpha\,\delta_\alpha\omega + \delta_\alpha\,\mathrm{d}_\alpha\,\omega$. Then,

$$\mathrm{d}_\alpha\,\delta_\alpha = \mathrm{d}_\alpha(\delta\omega + \alpha\tau \llcorner \omega) = \mathrm{d}\,\delta\omega + \alpha\,\mathrm{d}(\tau \llcorner \omega) + \alpha\tau\wedge\delta\omega + \alpha^2\tau\wedge\tau \llcorner \omega,$$

and similarly,

$$\delta_\alpha\,\mathrm{d}_\alpha\,\omega = \delta_\alpha(\mathrm{d}\,\omega + \alpha\tau\wedge\omega) = \delta\,\mathrm{d}\,\omega + \alpha\delta(\tau\wedge\omega) + \alpha\tau \llcorner \mathrm{d}\,\omega + \alpha^2\tau \llcorner \tau\wedge\omega.$$

Now, fix $(x,t) \in \mathcal{M}_\mathrm{st}$, and let $\{x^i\}$ be normal coordinates for $\mathcal{M}$ at x and s neighbourhood normal coordinates at t. In light of Proposition 6.2.2, in order to make future calculations easier and without the loss of generality, we assume that near t, $\tau = ds$. Furthermore, in these coordinates, let

$$\mathbf{P}_{\mathbf{\Omega}_1(\mathcal{M}_\mathrm{st})}\,\omega = \omega_{i_1\dots i_p}\;dx^{i_1\dots i_p} \quad\text{and}\quad \mathbf{P}_{\mathbf{\Omega}_2(\mathcal{M}_\mathrm{st})}\,\omega = {}^\mathrm{c}\omega_{i_1\dots i_{p-1}}\;ds\wedge dx^{i_1\dots i_{p-1}}.$$

First, we show that $\tau \llcorner \tau\wedge\omega + \tau\wedge\tau \llcorner \omega = \omega$. Since $\tau = ds$ and by orthogonality of $\mathrm{T}^*\mathcal{M}$ and $\mathrm{T}^*\mathcal{N}$, we have that $\tau\wedge\omega = \omega_{i_1\dots i_p}\;ds\wedge dx^{i_1\dots i_p}$ and $\tau \llcorner \omega = {}^\mathrm{c}\omega_{i_1\dots i_{p-1}}\;dx^{i_1\dots i_{p-1}}$. Thus,

$$\begin{aligned}\tau\wedge\tau \llcorner \omega &= {}^\mathrm{c}\omega_{i_1\dots i_{p-1}}\;ds\wedge dx^{i_1\dots i_{p-1}} = \mathbf{P}_{\mathbf{\Omega}_2(\mathcal{M}_\mathrm{st})}\,\omega,\ \text{and}\\ \tau \llcorner \tau\wedge\omega &= \omega_{i_1\dots i_p}\wedge\;dx^{i_1\dots i_p} = \mathbf{P}_{\mathbf{\Omega}_1(\mathcal{M}_\mathrm{st})}\,\omega.\end{aligned}$$

Thus, $\tau \llcorner \tau \wedge \omega + \tau \wedge \tau \llcorner \omega = \omega$.

At this point, we point out that

$$\mathrm{D}_{\alpha}^2 \omega = \mathrm{D}^2 \omega + \alpha^2 \omega + \alpha \left[\mathrm{d}(\tau \llcorner \omega) + \tau \llcorner \mathrm{d}\,\omega + \delta(\tau \wedge \omega) + \tau \wedge \delta\omega\right].$$

Thus, it remains to show that the term in the square bracket is zero.

At (x,t),

$$\mathrm{d}(\tau \llcorner \omega) = \partial_{x^k}\omega_{i_1\ldots i_p} dx^k \wedge dx^{i_1\ldots i_p} + \partial_{x^k}{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} dx^k \wedge dx^{i_1\ldots i_p} \\ + \partial_s{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} ds \wedge dx^{i_1\ldots i_p}.$$

Now,

$$\mathrm{d}\,\omega = \partial_{x^k}\omega_{i_1\ldots i_p} dx^k \wedge dx^{i_1\ldots i_p} + \partial_s\omega_{i_1\ldots i_p} ds \wedge dx^{i_1\ldots i_p} \\ + \partial_{x^k}{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} dx^k \wedge ds \wedge dx^{i_1\ldots i_p}.$$

Thus,

$$\tau \llcorner \mathrm{d}\,\omega = \partial_s\omega_{i_p\ldots i_p} dx^{i_1\ldots i_p} - \partial_{x^k}{}^{\mathrm{c}}\omega_{i_1\ldots i_p} dx^k \wedge dx^{i_1\ldots i_{p-1}},$$

and therefore,

$$\mathrm{d}(\tau \llcorner \omega) + \tau \llcorner \mathrm{d}\,\omega = \partial_s\omega_{i_1\ldots i_p} dx^{i_1\ldots i_p} + \partial_s{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} ds \wedge dx^{i_1\ldots i_{p-1}}.$$

Similarly,

$$\delta(\tau \wedge \omega) = -\partial_{x^k}\omega_{i_1\ldots i_p} dx^k \llcorner ds \wedge dx^{i_1\ldots i_p} - \partial_s\omega_{i_1\ldots i_p} dx^{i_1\ldots i_p},$$

and

$$\delta\omega = -\partial_{x^k}\omega_{i_1\ldots i_p} dx^k \llcorner dx^{i_1\ldots i_p} - \partial_s{}^{\mathrm{c}}\omega_{i_1\ldots i_p} dx^k \llcorner ds \wedge dx^{i_1\ldots i_p} \\ - \partial_{x^k}{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} dx^k \llcorner ds \wedge dx^{i_1\ldots i_{p-1}}.$$

Therefore,

$$\tau \wedge \delta\omega = -\partial_{x^k}\omega_{i_1\ldots i_p} ds \wedge dx^k \llcorner dx^{i_1\ldots i_p} - \partial_s{}^{\mathrm{c}}\omega_{i_1\ldots i_{p-1}} ds \wedge dx^{i_1\ldots i_{p-1}}.$$

Since we are working in normal coordinates at (x,t), and since $dx^k \neq ds$, $ds \wedge$

$dx^k \llcorner dx^{i_1 \dots i_p} = -dx^k \llcorner ds \wedge dx^{i_1 \dots i_p}$. Therefore,

$$\delta(\tau \wedge \omega) + \tau \wedge \delta\omega = -\partial_s \omega_{i_1 \dots i_p} dx^{i_1 \dots i_p} - \partial_s{}^{\mathrm{c}} \omega_{i_1 \dots i_{p-1}} ds \wedge dx^{i_1 \dots i_p}.$$

Combining these estimates show that $\mathrm{d}(\tau \llcorner \omega) + \tau \llcorner \mathrm{d}\omega + \delta(\tau \wedge \omega) + \tau \wedge \delta\omega = 0$ and this completes the proof. □

6.3 Square roots of perturbed inhomogeneous operators on spacetime manifolds

In this section, we define perturbed versions of the inhomogeneous Dirac operator and present the main results pertaining to a version of the Kato square root problem for such operators. First, we introduce a notion of curvature for forms as defined by Rosenberg on page 70 in [48].

Definition 6.3.1 (Curvature endomorphism on forms)**.** *Let $\{\theta^i\}$ represent a local orthonormal frame for $\mathbf{\Omega}^1(\mathcal{M})$. Then, the curvature endomorphism is given by*

$$\mathrm{R}\,\omega = -\,\mathrm{Rm}_{ijkl}\,\theta^i \wedge (\theta^j \llcorner (\theta^k \wedge (\theta^l \llcorner \omega))),$$

where Rm_{ijkl} is the components of the Riemannian curvature operator in this frame.

It is easy to check that the expression for R is independent of frame. A more interesting fact is the relationship of this operator to Ricci curvature as captured in the following proposition. Note that $\flat : \mathrm{T}^*\mathcal{M} \to \mathrm{T}\mathcal{M}$ is the *flat isomorphism* through the metric, given by $\eta^\flat = \langle \eta, \cdot \rangle$ for $\eta \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M})$.

Proposition 6.3.2. *For $\eta,\ \zeta \in \mathbf{\Omega}^1(\mathcal{M})$, we have that $\langle \mathrm{R}\,\eta, \zeta \rangle = \mathrm{Ric}(\eta^\flat, \zeta^\flat)$.*

Proof. Fix $x \in \mathcal{M}$ and let $\theta^i = dx^i$, the orthonormal frame induced by normal coordinates at x. Then,

$$\begin{aligned} -\,\mathrm{R}\,\eta &= \mathrm{Rm}_{ijkl}\, dx^i \wedge (dx^j \llcorner (dx^k \wedge (dx^l \llcorner \eta_m dx^m))) \\ &= \mathrm{Rm}_{ijkl}\, dx^i \wedge (dx^j \llcorner (dx^k \wedge \eta_m \delta^{lm})) = \mathrm{Rm}_{ijkl}\, dx^i \wedge (dx^j \llcorner \eta_m \delta^{lm}\ dx^k) \\ &= \mathrm{Rm}_{ijkl}\, dx^i \wedge \eta_m \delta^{lm} \delta^{jk} = \mathrm{Rm}_{ijkl}\, \eta_m \delta^{lm} \delta^{jk}\ dx^i. \end{aligned}$$

By the symmetries of Rm, we have that $\mathrm{Ric}_{il} = \mathrm{Rm}_{ijkl}\,\delta^{jk}$, and hence, $\mathrm{R}\,\eta = \mathrm{Ric}_{il}\,\eta_m \delta^{lm}\ dx^i$. Therefore,

$$\langle \mathrm{R}\,\eta, \zeta\rangle = \left\langle \mathrm{Ric}_{il}\,\eta_m \delta^{lm} dx^i, \zeta_a\ dx^a \right\rangle = \mathrm{Ric}_{il}\,\eta_m \zeta_a \delta^{lm}\delta^{ia}.$$

The proof is complete on noting that $\eta^\flat = \eta_m \delta^{lm}\partial_l$ and $\zeta^\flat = \zeta_a \delta^{ia}\partial_i$ and by the symmetry of the Ricci tensor. □

This proposition allows us to view R as a generalisation of Ricci curvature to the exterior algebra.

Let us now return to considering perturbations of our inhomogeneous Dirac operators. Let $B \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})))$ and suppose that B^{-1} exists. For $\alpha > 0$, define the operator D_B with domain $\mathcal{D}(\mathrm{D}_B) \subset \mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))$ by

$$\mathrm{D}_B\, u = (\mathrm{d}_\alpha + B^{-1} \updelta_\alpha B) u.$$

With this and on noting that $\mathrm{Ric}_{\mathcal{M}}$ and $\mathrm{R}_{\mathcal{M}}$ denote the Ricci and curvature endomorphisms of $\mathcal{M}$, we obtain the following Kato square root theorem.

Theorem 6.3.3 (Kato square root problem on spacetime manifolds)**.** *Let $(\mathcal{M}, \mathrm{g})$ be a smooth, complete Riemannian manifold, and suppose there exists $\eta, \kappa > 0$ such that $|\mathrm{Ric}_{\mathcal{M}}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Further suppose that there exists $\zeta \in \mathbb{R}$ such that $\mathrm{g}(\mathrm{R}_{\mathcal{M}}\,\omega, \omega) \geq \zeta\,|\omega|^2$ for $\omega \in \mathbf{\Omega}_x(\mathcal{M}_{\mathrm{st}})$ and all $x \in \mathcal{M}$, and that $B \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}})))$ satisfies the following ellipticity bound: there exists $\kappa' > 0$*

$$\mathrm{Re}\,\langle B\omega, \omega\rangle_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))} \geq \kappa' \|\omega\|^2_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))}$$

for all $\omega \in \mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))$. Then, for $\alpha > 0$, $\mathcal{D} = \mathcal{D}(\sqrt{\mathrm{D}_B^2}) = \mathcal{D}(\mathrm{D}_B) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\updelta B)$ (where $\mathrm{D}_B = \mathrm{d} + B^{-1}\updelta B$) and

$$\|\sqrt{\mathrm{D}_B^2}\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))} \simeq \|\,\mathrm{D}_B\,\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))} \simeq \|\,\mathrm{d}_\alpha\,\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))} + \|\updelta_\alpha B\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}_{\mathrm{st}}))}$$

for all $\omega \in \mathcal{D}$.

Proof. We proceed to prove this theorem by showing that (H1)-(H6), (H7-I) and H8-I) are satisfied in order to invoke Theorem 3.2.4 and Theorem 1.8.2.

We define $\Gamma = \mathrm{d}_\alpha$. Then (H1) is immediate from Proposition 6.2.5. Set $B_2 = B$ and note that the ellipticity bound on B implies that B^{-1} exists and satisfies a similar bound. Thus, we can write $B_1 = B^{-1}$. Then, (H2) is immediate, and coupling the fact that $B_1 B_2 = B_2 B_1 = \mathrm{I}$ with the nilpotency of d_α and δ_α, gives (H3).

We prove (H4). First, we show that $\mathcal{M}_{\mathrm{st}}$ possesses a uniform bound on its Ricci curvature and a uniform lower bound on its injectivity radius. For the curvature bounds, we note that by Proposition 6.1.3, $|\mathrm{Rm}_{\mathcal{M}_{\mathrm{st}}}|^2 = |\mathrm{Rm}_{\mathcal{M}}|^2 + |\mathrm{Rm}_{\mathcal{N}}|^2$. Since $\mathcal{N}$ is a 1-manifold, $\mathrm{Rm}_{\mathcal{N}} = 0$ and so $|\mathrm{Ric}_{\mathcal{M}_{\mathrm{st}}}| \lesssim \eta$. Therefore, $\mathcal{M}_{\mathrm{st}}$ satisfies (Exp).

Next, the injectivity radius of both $\mathbb{R}$ and $\mathbb{S}^1$ are uniformly bounded below, and therefore, it is easy to see that there exists a $\kappa' > 0$ (namely taking the minimum of κ and the injectivity radius of $\mathcal{N}$) so that $\mathrm{inj}(\mathcal{M}_{\mathrm{st}}) > \kappa'$. Thus, we are able to assert the existence of GBG coordinates for tensors through Proposition 5.3.1. Since we can isometrically embed $\boldsymbol{\Omega}(\mathcal{M}_{\mathrm{st}})$ into $\oplus_{p=0}^n \mathcal{T}^{(0,p)}(\mathcal{M}_{\mathrm{st}})$, we conclude that $\boldsymbol{\Omega}(\mathcal{M}_{\mathrm{st}})$ satisfies the GBG condition and furthermore, admits GBG coordinates. Thus, $\mathcal{M}_{\mathrm{st}}$ satisfies (H4).

We note that (H5) is immediate from the assumptions so we prove (H6). Fix $u \in \mathcal{D}(\Gamma)$ and ξ a bounded Lipschitz function. It is an easy fact that $\xi\mathcal{D}(\Gamma) \subset \mathcal{D}(\Gamma)$. Then, $[\Gamma, \xi\mathrm{I}]\, u = \Gamma(\xi u) - \xi\Gamma u$. We compute

$$\Gamma(\xi u) = \mathrm{d}_\alpha(\xi u) = \mathrm{d}(\xi u) + \tau \wedge (\xi u) = (d\xi)u + \xi \,\mathrm{d}\, u + \xi\tau \wedge u = d\xi u + \xi\Gamma u.$$

Thus, combining these estimates show that $[\Gamma, \xi\mathrm{I}]\, u = (d\xi)u$ and thus, and (H6) is proved.

To prove (H7-I), we proceed as in the proof of Theorem 5.2.1. Fix $u \in \mathcal{D}(\Gamma)$ with $\mathrm{spt}\ u \subset Q$ and so we have that

$$\left|\int_Q \Gamma u\right| \leq \left|\int_Q \mathrm{d}\, u\right| + \alpha \left|\int_Q \tau \wedge u\right|.$$

An easy calculation yields that

$$\left|\int_Q \tau \wedge u\right| \leq \int_Q |\tau \wedge u| \leq \int_Q |\tau|\, |u| = \int_Q |u| \lesssim \|u\| \mu(Q)^{\frac{1}{2}}.$$

For the remaining inequality, let $dx^I = dx^{i_1 \dots i_p}$ and $dx^K = dx^{j_1 \dots j_p}$ be p-forms with multi-indices I and K for notational simplicity. Similarly, let ∂_I and ∂_K denote the corresponding dual p-vectors. Furthermore, we note that without the loss of generality, we can assume that u is a p-form. Then, arguing as in the proof of (H7-I) in Theorem 5.2.1,

$$\int_Q \mathrm{d}\, u = \int_Q \left\langle \mathrm{d}\, u, \mathrm{g}_{KI} dx^K \right\rangle \, dx^I = \int_Q \left\langle u, \delta(\mathrm{g}_{KI} dx^K) \right\rangle \, dx^I,$$

where $\mathrm{g}_{KI} = \mathrm{g}(\partial_K, \partial_I)$. Thus,

$$\left| \int_Q \mathrm{d}\, u \right| \lesssim \sum_I \left| \int_Q \left\langle u, \delta(\mathrm{g}_{KI} dx^K) \right\rangle \right| \lesssim \int_Q |u| \sum_I \left| \delta(\mathrm{g}_{KI} dx^K) \right|.$$

Also,

$$\delta(\mathrm{g}_{KI} dx^K) = dx^l \llcorner \nabla_l(\mathrm{g}_{KI} dx^K) = \partial_l(\mathrm{g}_{KI}) dx^l \llcorner dx^K + \mathrm{g}_{KI} dx^l \llcorner \nabla_l dx^K.$$

The bounds on $\partial_k \mathrm{g}_{ij}$ guaranteed to us by Proposition 5.3.1 imply similar bounds on $\partial_i \mathrm{g}_{KI} = \partial_i \det((\mathrm{g}(dx^{i_r}, dx^{j_s})))$. Thus, we have that $\sum_I \left| \delta(\mathrm{g}_{KI} dx^K) \right| \lesssim 1$. Combining these estimates, we have that

$$\left| \int_Q \Gamma u \right| \lesssim \|u\| \mu(Q)^{\frac{1}{2}}.$$

Next, fix $v \in \mathcal{D}(\Gamma^*)$ with spt $v \subset Q$. First, we deal with the lower order term:

$$\left| \int_Q \alpha \tau \llcorner v \right| \lesssim \alpha \int_Q |\tau| \, |v| \leq \alpha \|v\| \mu(Q)^{\frac{1}{2}}.$$

Then, as before, we compute

$$\left| \int_Q \delta v \right| = \left| \int_Q \left\langle \delta v, \mathrm{g}_{KI} dx^K \right\rangle \, dx^I \right| = \left| \int_Q \left\langle v, \mathrm{d}(\mathrm{g}_{KI} dx^K) \right\rangle \, dx^I \right| \lesssim \sum_I \int_Q |v| \left| \mathrm{d}(\mathrm{g}_{KI} dx^K \right|.$$

We have that $\mathrm{d}(\mathrm{g}_{KI} dx^K) = \partial_l \mathrm{g}_{KI} dx^l \wedge dx^K$ which is bounded. Thus,

$$\left| \int_Q \delta v \right| \lesssim \|v\| \mu(Q)^{\frac{1}{2}}.$$

On combining these estimates,

$$\left| \int_Q \Gamma^* v \right| = \left| \int_Q \delta_\alpha v \right| \leq \left| \int_Q \delta v \right| + \left| \int_Q \alpha \tau \llcorner v \right| \lesssim \|u\| \mu(Q)^{\frac{1}{2}}$$

and this concludes the proof of (H7-I).

To prove (H8-I), set $\Xi = \nabla$. We note that since $|\mathrm{Ric}_{\mathcal{M}_{\mathrm{st}}}| \leq \eta$, the hypothesis of Proposition 5.1.3 is satisfied and hence, we obtain the Poincaré inequality that for all $u \in \mathrm{W}^{1,2}(\mathbf{\Omega}(\mathcal{M})) \subset \mathcal{D}(\mathrm{D})$, $t \leq \mathrm{t_S}$, $Q \in \mathscr{Q}_t$ and $r \geq \frac{C_1}{\delta}$,

$$\int_B |u - u_Q|^2 \ d\mu \lesssim (1 + e^{\lambda rt}(rt)^2 \int_B (|\nabla u|^2 + |u|^2) \ d\mu.$$

Thus, (H8-I)-1 is proved.

To prove (H8-I)-2, fix $u \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{M})$. By Theorem 2.38 in [48], we have that $\mathrm{D}^2 u = \Delta u + \mathrm{R}\, u$, where $\Delta = -\operatorname{tr} \nabla^2$ is the usual connection Laplacian. Then,

$$\begin{aligned} \|\nabla u\|^2 = \int_{\mathcal{M}} \langle \nabla u, \nabla u \rangle \ d\mu = \int_{\mathcal{M}} \langle \Delta u, u \rangle \ d\mu \\ = \int_{\mathcal{M}} (\left\langle \mathrm{D}^2 u, u \right\rangle - \langle \mathrm{R}\, u, u \rangle) \ d\mu \end{aligned}$$

While we assume lower bounds on $\langle \mathrm{R}_{\mathcal{M}}\, \omega, \omega \rangle$ for $\omega \in \mathbf{\Omega}^p(\mathcal{M})$, we need to show that these bounds hold for R. Note as before, that $\mathrm{Rm} = \mathrm{Rm}_{\mathcal{M}}$, and that $\omega \in \mathbf{\Omega}^p(\mathcal{M}_{\mathrm{st}})$ can be orthogonally decomposed into $\omega = \omega^1 + \omega^2$ where $\omega^1(x,t) = \omega^1_{i_1 \dots i_p}(x,t)\ \theta^{i_1 \dots i_p}$ and $\omega^2(x,t) = \omega^2_{i_1 \dots i_{p-1}}(x,t)\ ds \wedge \theta^{i_1 \dots i_{p-1}}$. Fix $t \in \mathcal{N}$, and note that $\mathrm{R}\, \omega^1(x,t) \in \mathbf{\Omega}^p_1(\mathcal{M}_{\mathrm{st}})$ and $\mathrm{R}\, \omega^2(x,t) = ds \wedge \mathrm{R}(\omega^2_{i_1 \dots i_p}(x,t)\ \theta^{i_1 \dots i_{p-1}}) \in \mathbf{\Omega}^p_2(\mathcal{M}_{\mathrm{st}})$, where the second equality follows from the fact that $\{\theta^i, ds\}$ is an orthonormal frame. Let $\omega^{2\prime}(x,t) = \omega^2_{i_1 \dots i_{p-1}}\ \theta^{i_1 \dots i_{p-1}}$ for each t, $\omega^{2\prime}(x,t) \in \mathbf{\Omega}^p(\mathcal{M})$. Thus,

$$\begin{aligned} \langle \mathrm{R}\, \omega(x,t), \omega(x,t) \rangle = \left\langle \mathrm{R}\, \omega^1(x,t), \omega^1(x,t) \right\rangle + \left\langle ds \wedge \mathrm{R}\, \omega^{2\prime}(x,t), ds \wedge \omega^{2\prime}(x,t) \right\rangle \\ \geq \zeta \left|\omega^1(x,t)\right| + |ds|\, \zeta \left|\omega^{2\prime}(x,t)\right| = \zeta\, |\omega(x,t)| \end{aligned}$$

by the fact that $|ds| = 1$.

Thus, we have that $\|\nabla u\|^2 \leq \|Du\|^2 - \zeta \|u\|^2$. Furthermore,

$$\begin{aligned} \|\mathrm{D}_\alpha u\|^2 = \int_{\mathcal{M}} \langle \mathrm{D}_\alpha u, \mathrm{D}_\alpha u \rangle \ d\mu = \int_{\mathcal{M}} \left\langle \mathrm{D}_\alpha^2 u, u \right\rangle \ d\mu \\ = \int_{\mathcal{M}} \left\langle \mathrm{D}^2 + \alpha^2 u, u \right\rangle \ d\mu = \|\mathrm{D}\, u\|^2 + \alpha^2 \|u\|^2 \end{aligned}$$

by Proposition 6.2.6. It is easy to see that if $\zeta \geq 0$, then it is immediate that $\|\nabla u\|^2 + \|u\| \leq \|\mathrm{D}_\alpha\|^2$. For the case that $\zeta < 0$, fix $C > 1$ to be chosen later so that

$$\|\nabla u\|^2 + \|u\|^2 \leq \|\mathrm{D}\, u\|^2 + (1 - \zeta)\|u\|^2$$

$$\leq C\|\mathrm{D}\,u\|^2 + (1-\zeta)\|u\|^2 = C\|\mathrm{D}_\alpha\,u\|^2 + (1-\zeta - C\alpha)\|u\|^2.$$

Then, we choose $C > 1$ so that $1 - \zeta - C\alpha \leq 0$. That is,

$$C > \max\left\{1, \frac{1-\zeta}{\alpha}\right\}.$$

Then, for any $\zeta \in \mathbb{R}$, we have that

$$\|\nabla u\|^2 + \|u\|^2 \leq C\|\mathrm{D}_\alpha\,u\|^2 = C\|\Pi u\|^2$$

which proves (H8-I)-2.

Since we have demonstrated (H1)-(H6), (H7-I) and (H8-I) are satisfied for our operators $(\Gamma, B_1, B_2) = (\mathrm{d}_\alpha, B_1, B_2)$, we obtain quadratic estimates (Qest) for our operator Π_B via Theorem 3.2.4. Therefore, we invoke Theorem 1.8.2 to conclude that $\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})$ and $\|\Gamma u\| + \|\Gamma_B^*\| \simeq \|\Pi_B u\| \simeq \|\sqrt{\Pi_B^2} u\|$. But note that $\Gamma = \mathrm{d}_\alpha$, $\Gamma_B^* = B^{-1}\delta_\alpha B$, $\Pi_B = \mathrm{D}_B$ and so $\mathcal{D}(\Gamma) = \mathcal{D}(\mathrm{d}_\alpha) = \mathcal{D}(\mathrm{d})$, $\mathcal{D}(\Gamma_B^*) = \mathcal{D}(B^{-1}\delta_\alpha B) = \mathcal{D}(\delta_\alpha B) = \mathcal{D}(\delta B)$. Furthermore, the ellipticity of B implies that $\|B^{-1}u\| \lesssim \|u\|$, and therefore, $\|B^{-1}\delta_\alpha B u\| \simeq \|\delta_\alpha B u\|$. This completes the proof of the theorem. $\square$

We say that a manifold $\mathcal{M}$ has a *straight line* if there exists a smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for $t, s \in \mathbb{R}$. The *Splitting Theorem* of Cheeger and Gromoll in [18] then asserts that for manifolds that contain a *straight line* and satisfy $\mathrm{Ric} \geq 0$, there exists a submanifold $\mathcal{S}$ with $\mathrm{Ric}_{\mathcal{S}} \geq 0$ such that $\mathcal{M} = \mathcal{S} \times \mathbb{R}$ isometrically. Thus, for such manifolds, there is an *internal* timekeeper and yields the following theorem.

Theorem 6.3.4 (Kato square root problem for spacetime manifolds with a straight line)**.** *Let $(\mathcal{M}, \mathrm{g})$ be a smooth complete Riemannian manifold that has a line and suppose there exists $\eta, \kappa > 0$ such that $\mathrm{Ric} \geq 0$, $|\mathrm{Ric}| \leq \eta$, and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Further suppose that there exists $\zeta \in \mathbb{R}$ such that $\mathrm{g}(\mathrm{R}\,\omega, \omega) \geq \zeta\,|\omega|^2$ for $\omega \in \mathbf{\Omega}_x(\mathcal{M})$ and all $x \in \mathcal{M}$, and that $B \in \mathrm{L}^\infty(\mathcal{L}(\mathbf{\Omega}(\mathcal{M})))$ satisfies the ellipticity condition: there exists $\kappa' > 0$ such that*

$$\mathrm{Re}\,\langle B\omega, \omega\rangle_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))} \geq \kappa' \|\omega\|^2_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))}$$

for all $\omega \in \mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))$. Then, $\mathcal{D} = \mathcal{D}(\sqrt{\mathrm{D}_B^2}) = \mathcal{D}(\mathrm{D}_B) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\delta B)$ and

$$\|\sqrt{\mathrm{D}_B^2}\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))} \simeq \|\mathrm{D}_B\,\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))} \simeq \|\mathrm{d}_\alpha\,\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))} + \|\delta_\alpha B\omega\|_{\mathrm{L}^2(\mathbf{\Omega}(\mathcal{M}))}$$

for all $\omega \in \mathcal{D}$.

Proof. By the Splitting Theorem, let $\mathcal{M} = \mathcal{S} \times \mathbb{R}$. We put $\mathcal{S}$ in place of $\mathcal{M}$ in Theorem 6.3.3, but we make the following alteration to the proof that $\boldsymbol{\Omega}(\mathcal{M}_{\mathrm{st}})$ satisfies GBG. Instead of using curvature bounds on $\mathcal{S}$ to get curvature bounds on $\mathcal{M} = \mathcal{S}_+$, we have the necessary curvature bounds on $\mathcal{M}$ by assumption. The conclusion is then immediate. □

6.4 Square roots of perturbed inhomogeneous Hodge-Laplacians

In this section, we consider a more standard problem for perturbations of inhomogeneous Hodge-Laplacians. The setup we present here is due to a suggestion by McIntosh.

Fix $\beta \in \mathbb{C} \setminus \{0\}$. Let $\mathrm{d}_\beta : \boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M}) \to \boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M})$ be defined on L^2 by

$$\mathrm{d}_\beta = \begin{pmatrix} \mathrm{d} & 0 \\ \beta & -\mathrm{d} \end{pmatrix}.$$

Then,

$$\delta_\beta = \begin{pmatrix} \delta & \overline{\beta} \\ 0 & -\delta \end{pmatrix},$$

and we define the inhomogeneous Hodge-Dirac by $\mathrm{D}_\beta = \mathrm{d}_\beta + \delta_\beta$.

Let us first make the following note about these operators.

Proposition 6.4.1. *The operators* d_β *and* δ_β *are nilpotent. Moreover,*

$$\mathrm{D}_\beta^2 = \begin{pmatrix} \mathrm{D}^2 + |\beta|^2 & 0 \\ 0 & \mathrm{D}^2 + |\beta|^2 \end{pmatrix}.$$

Proof. We note that

$$\mathrm{d}_\beta^2 = \begin{pmatrix} \mathrm{d}^2 & 0 \\ \beta\,\mathrm{d} - \mathrm{d}\,\beta & \mathrm{d}^2 \end{pmatrix} = 0.$$

Similarly, $\delta_\beta^2 = 0$. Next, note that

$$\mathrm{D}_\beta = \begin{pmatrix} \mathrm{D} & \overline{\beta} \\ \beta & \mathrm{D} \end{pmatrix}$$

and therefore,

$$\mathrm{D}_\beta^2 = \begin{pmatrix} \mathrm{D}^2 + |\beta|^2 & \overline{\beta}\,\mathrm{D} - D\overline{\beta} \\ \beta\,\mathrm{D} - \mathrm{D}\,\beta & |\beta|^2 + \mathrm{D}^2 \end{pmatrix} = \begin{pmatrix} \mathrm{D}^2 + |\beta|^2 & 0 \\ 0 & \mathrm{D}^2 + |\beta|^2 \end{pmatrix}.$$

□

Let $A \in \mathrm{L}^\infty(\mathcal{L}(\boldsymbol{\Omega}(\mathcal{M})))$ and suppose that A^{-1} exists. Then, define $\mathrm{D}_A = \mathrm{d} + A^{-1}\delta A$. On letting

$$A' = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

define the perturbed inhomogeneous Hodge-Dirac operator $\mathrm{D}_{A,\beta} : \boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M}) \to \boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M})$ by $\mathrm{D}_{A,\beta} = \mathrm{d}_\beta + A'^{-1}\delta_\beta A'$.

The following proposition is a simple calculation similar to that of the proof of Proposition 6.4.1.

Proposition 6.4.2. *The operator*

$$\mathrm{D}_{A,\beta} = \begin{pmatrix} \mathrm{D}_A & \overline{\beta} \\ \beta & -\mathrm{D}_A \end{pmatrix}, \quad \text{and} \quad \mathrm{D}_{A,\beta}^2 = \begin{pmatrix} \mathrm{D}_A^2 + |\beta|^2 & 0 \\ 0 & \mathrm{D}_A^2 + |\beta|^2 \end{pmatrix}.$$

In the following main theorem of this section, we obtain the following solution to the Kato square root problem for the operator D_A.

Theorem 6.4.3. *Let $(\mathcal{M}, \mathrm{g})$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\mathrm{Ric}| \leq \eta$ and $\mathrm{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there exists $\zeta \in \mathbb{R}$ satisfying $\mathrm{g}(\mathrm{R}\,\omega, \omega) \geq \zeta\,|\omega|^2$ for $\omega \in \boldsymbol{\Omega}_x(\mathcal{M})$ and all $x \in \mathcal{M}$, and that $A \in \mathrm{L}^\infty(\mathcal{L}(\boldsymbol{\Omega}(\mathcal{M})))$ such that there exists $\kappa_1 > 0$ satisfying*

$$\mathrm{Re}\,\langle Au, u\rangle \geq \kappa_1 \|u\|^2$$

for all $u \in \mathrm{L}^2(\boldsymbol{\Omega}(\mathcal{M}))$. Then, for $\beta \in \mathbb{C} \setminus \{0\}$, $\mathcal{D}(\sqrt{\mathrm{D}_A^2 + |\beta|^2}) = \mathcal{D}(\mathrm{D}_A) = \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\delta A)$ and $\|\sqrt{\mathrm{D}_A^2 + |\beta|^2} u\| \simeq \|\mathrm{D}_A u\| + \|u\|$.

Proof. On putting $\mathcal{V} = \boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M})$, $\Gamma = \mathrm{d}_\beta$, and $B_2 = A'$, we show that (H1)-(H6), (H7-I) and (H8-I) are satisfied and invoke Theorem 3.2.4 and Theorem 1.8.2.

That (H1) is true is immediate from Proposition 6.4.1. By the ellipticity assumption in the hypothesis, we have that A^{-1} exists and hence that A'^{-1} exists. Furthermore, by an invocation of the spectral mapping theorem for the bounded operator A, we obtain (H2) with $\kappa_2 = 1/\kappa_1$. We assert (H3) holds simply because $B_1 = B_2^{-1}$ and by the nilpotency of d_β and δ_β. The hypotheses (H4) and (H5) are satisfied by assumption.

To prove (H6), fix η a bounded Lipschitz map. It is an easy fact that $\eta\mathcal{D}(\mathrm{d}) \subset \mathcal{D}(\mathrm{d})$ and it is also easy to see that $\mathcal{D}(\mathrm{d}_\beta) = \mathcal{D}(\mathrm{d}) \oplus \mathcal{D}(\mathrm{d})$. Therefore, $\eta\mathcal{D}(\mathrm{d}_\beta) \subset \mathcal{D}(\mathrm{d}_\beta)$. Next, note that at almost every x, and for $u = (u_1, u_2) \in \mathrm{C}_\mathrm{c}^\infty(\mathcal{V})$,

$$\begin{aligned}\mathrm{d}_\beta(\eta u) &= \begin{pmatrix} \mathrm{d}(\eta u_1) \\ \beta\eta u_1 - \mathrm{d}(\eta u_2)\end{pmatrix} = \begin{pmatrix} \mathrm{d}\,\eta \wedge u_1 + \eta\,\mathrm{d}\,u_1 \\ \beta\eta u_1 - \mathrm{d}\,\eta\wedge u_2 - \eta\,\mathrm{d}\,u_2\end{pmatrix} \\ &= \begin{pmatrix} \mathrm{d}\,\eta\wedge u_1 \\ -\,\mathrm{d}\,\eta \wedge u_2\end{pmatrix} + \begin{pmatrix} \eta\,\mathrm{d}\,u_1 \\ \beta\eta u_1 - \eta\,\mathrm{d}\,u_2\end{pmatrix} = \begin{pmatrix} \mathrm{d}\,\eta\wedge u_1 \\ -\,\mathrm{d}\,\eta\wedge u_2\end{pmatrix} - \eta\,\mathrm{d}_\beta\, u.\end{aligned}$$

Thus,

$$[\mathrm{d}_\beta, \eta]\,u = \left[\begin{pmatrix} \mathrm{d} & 0 \\ \beta & \mathrm{d}\end{pmatrix}, \eta\mathrm{I}\right] u = \begin{pmatrix} \mathrm{d}\,\eta\wedge u_1 \\ -\,\mathrm{d}\,\eta\wedge u_2\end{pmatrix}$$

which is a pointwise multiplication operator, and

$$\left|[\mathrm{d}_\beta, \eta]\right|^2 = \left|\mathrm{d}\,\eta\wedge u_1\right|^2 + \left|\mathrm{d}\,\eta\wedge u_2\right|^2 \leq \left|\mathrm{d}\,\eta\right|^2 \left|u\right|^2.$$

But we have that $|\mathrm{d}\,\eta| \lesssim \mathrm{Lip}\,\eta$ for almost all x and hence, by a density argument, (H6) is proved.

Let us now prove (H7-I). First, note that the cube integral appears as follows:

$$\int_Q \mathrm{d}_\beta\, u = \begin{pmatrix} \int_Q \mathrm{d}\,u_1 \\ \int_Q \beta u_1 - \int_Q \mathrm{d}\,u_2\end{pmatrix},$$

for $u \in \mathcal{D}(\mathrm{d}_\beta)$ with $\mathrm{spt}\,u \subset Q$. Therefore,

$$\left|\int_Q \mathrm{d}_\beta\, u\right| \lesssim \left|\int_Q \mathrm{d}\,u_1\right| + \left|\int_Q \mathrm{d}\,u_2\right| + \left|\int_Q \beta u_1\right|.$$

By the same argument as found in the proof of (H7-I) in Theorem 6.3.3, we have that

$$\left|\int_Q \mathrm{d}\, u_1\right| \lesssim \int_Q |u_1|, \quad \text{and} \quad \left|\int_Q \mathrm{d}\, u_2\right| \lesssim \int_Q |u_2|.$$

The passage from this inequality to

$$\left|\int_Q \mathrm{d}_\beta\, u\right| \lesssim \mu(Q)^{\frac{1}{2}} \|u\|$$

is then immediate. Similarly, we can reason that

$$\left|\int_Q \delta_\beta v\right| \lesssim \mu(Q)^{\frac{1}{2}} \|v\|$$

for $v \in \mathcal{D}(\delta_\beta)$ with $\operatorname{spt} v \subset Q$.

We now prove (H8-I). Let us first set $\Xi(u_1, u_2) = (\nabla u_1, \nabla u_2)$ and note that by our assumptions, we can apply Proposition 5.1.3 to deduce (H8-I)-1.

Next, fix $u = (u_1, u_2) \in \mathrm{C}_c^\infty(\mathcal{V})$ and note that

$$\begin{aligned}
\|\mathrm{D}_\beta\, u\|^2 &= \|\mathrm{D}\, u_1 + \overline{\beta} u_2\|^2 + \|\beta u_1 - \mathrm{D}\, u_2\|^2 \\
&= \|\mathrm{D}\, u_1\|^2 + |\beta|^2 \|u_2\|^2 + \beta \langle \mathrm{D}\, u_1, u_2\rangle + \overline{\beta} \langle u_2, \mathrm{D}\, u_1\rangle \\
&\qquad + \|\mathrm{D}\, u_2\|^2 + |\beta|^2 \|u_1\|^2 - \overline{\beta} \langle \mathrm{D}\, u_2, u_1\rangle - \beta \langle u_1, \mathrm{D}\, u_2\rangle .
\end{aligned}$$

But since D is a self-adjoint operator, $\|\mathrm{D}_\beta\, u\|^2 = \|\mathrm{D}\, u_1\|^2 + \|\mathrm{D}\, u_2\|^2 + |\beta|^2 \|u\|^2$. Let as recall from the proof of Theorem 6.3.3 that $\mathrm{D}^2 u_i = \Delta u_i + \mathrm{R}\, u_i$. Then, as before, we can deduce that $\|\nabla u_i\|^2 = \|\mathrm{D}\, u_i\|^2 - \zeta \|u_i\|^2$. As in the proof of Theorem 6.3.3, we choose $C > \max\left\{1, (1-\zeta)\, |\beta|^{-2}\right\}$ to obtain $\|\nabla u\|^2 + \|u\|^2 \leq C\|\mathrm{D}_\beta\, u\|^2$. This proof of (H8-I)-2 is then completed by the density of $\mathrm{C}_c^\infty(\boldsymbol{\Omega}(\mathcal{M}))$ in $\mathrm{W}^{1,2}(\boldsymbol{\Omega}(\mathcal{M}))$.

By the invocation of Theorem 1.8.2 we obtain equality of the domains. The estimate then follows on setting $v = (u, 0)$ for $u \in \mathcal{D}(\mathrm{D}_A)$ and computing the estimate given by Theorem 1.8.2. $\square$

7

Further research

In this concluding chapter, we take the time to describe some open problems and interesting questions which we believe will further advance the understanding of the geometric Kato square root problem.

7.1 Counterexample for the density of smooth compactly supported functions in second-order Sobolev spaces

The recent work [2] by Ambrosio, Gigli and Savare may provide us with a method to construct a counterexample to the density of smooth compactly supported functions in second order Sobolev spaces. These authors study heat flow and Laplacians on measure metric spaces $(\mathcal{X}, d, \mu)$ that satisfy the following growth condition:

$$\int_{\mathcal{X}} \mathrm{e}^{-V(x)^2}\, d\mu < \infty,$$

where $V : \mathcal{X} \to \mathbb{R}$ is a Borel function.

We are interested in specialising to the situation of $\mathcal{X} = \mathcal{M}$, a Riemannian manifold, with volume measure μ and $V(x) = d(x_0, x)$, where $x_0 \in \mathcal{M}$ is fixed. It is plausible that this growth condition allows the measure μ to be at most inverse Gaussian, by which we mean

$$0 < \mu(B(x,tr)) \leq Ct^{\kappa}\mathrm{e}^{(tr)^{\alpha}}\mu(B(x,r)) < \infty$$

for $t > 1$ and $r > 0$ and $\alpha < 2$. The situation $\alpha = 1$ is the exponentially locally doubling situation which is satisfied by hyperbolic manifolds.

Of particular interest is their Laplacian $\Delta_{d,\mu}$, which by construction satisfies the regularity condition $\mathcal{D}(\Delta_{d,m}) = \mathcal{D}(|\nabla^- \mathrm{Ch}_*|)$. Here, ∇^- is the negative slope and Ch_* is the (modified) Cheeger functional. We expect that $\Delta_{d,\mu} = \Delta$, where the latter object is the Bochner Laplacian, and $\mathcal{D}(|\nabla^- \mathrm{Ch}_*|) = \mathrm{W}^{2,2}(\mathcal{M})$. This would imply that $\mathrm{W}^{2,2}_0(\mathcal{M}) \subset \mathrm{W}^{2,2}(\mathcal{M}) = \mathcal{D}(\Delta)$. Thus, to show $\mathrm{W}^{2,2}_0(\mathcal{M}) \subsetneq \mathrm{W}^{2,2}(\mathcal{M})$ would reduce the problem to demonstrating that $\mathrm{W}^{2,2}_0(\mathcal{M}) \subsetneq \mathcal{D}(\Delta)$, which may be an easier problem via the Bochner-Lichnerowicz-Weitzenböck identity. We conjecture the following.

Conjecture 1. *There exists a smooth, complete Riemannian manifold* $(\mathcal{M}, \mathrm{g})$ *such that* $\mathrm{Ric} \geq \eta\mathrm{g}$ *is false for all* $\eta \in \mathbb{R}$ *and* $\mathrm{W}^{2,2}_0(\mathcal{M}) \subsetneq \mathrm{W}^{2,2}(\mathcal{M})$.

7.2 Poincaré inequalities beyond functions

The prototypical Poincaré inequalities we consider on manifolds $\mathcal{M}$ are of the form

$$\|f - f_B\|_{\mathrm{L}^p(B)} \leq C \,\mathrm{rad}(B)\|\nabla f\|_{\mathrm{L}^p(B)},$$

for $f \in \mathrm{C}^{\infty}(\mathcal{M})$, where B is a ball, $\mathrm{rad}(B)$ is the radius of B and C is a constant (which typically depends on p and n). Without complicating issues too much by considering general p, let us just state the problem for $p = 2$. Also, we are interested in such Poincaré inequalities for small balls, i.e., $\mathrm{rad}(B) \leq 1$. Such estimates are valid under the bound $\mathrm{Ric} \geq \eta\mathrm{g}$, where $\eta \in \mathbb{R}$ and g is the metric on $\mathcal{M}$.

The main question is then to ask what are the "correct" analogues of such inequalities on vector bundles. Such a question is primarily motivated by attempting to weaken the geometric assumptions necessary to prove Kato square type results. In particular, it would be desirable to dispense the injectivity radius assumption and only assume a Ricci bound from below. The hope is that a weaker Poincaré inequality more natural to the vector bundle setting would provide us a method with which to do this.

In [9], Badr and Bernicot prove that

$$\|f - e^{-\operatorname{rad}^2(B)\Delta} f\|_{\mathrm{L}^2(B)} \leq C \operatorname{rad}(B) \|\nabla f\|_{\mathrm{L}^2(B)}.$$

In fact, they prove this in L^p. They actually prove this "weaker" version by using the classical Poincaré inequality, presumably because they are interested in demonstrating Calderon-Zygmund decompositions under this weaker estimate, and because they are not necessarily concerned with a study of such estimates for its own sake.

This inequality is suggestive. Certainly, given a vector bundle $\mathcal{V}$ with connection ∇ and metric h, we can certainly define the heat operator $e^{-t\Delta}$ where now, $\Delta = \nabla^* \nabla$ (with ∇ considered as an operator on L^2). Thus, this may be the right way to proceed in proving Poincaré inequalities on vector bundles. A proof of this weaker Poincaré inequality for even just functions avoiding the use of a classical Poincaré inequality could be instructive.

7.3 Geometric counterexample to the Kato square root problem

Recall that Kato asked two questions: the first was about the comparability of a square root and its adjoint, and the second was about the holomorphic dependency of the square root of an operator defined via a holomorphic form.

McIntosh obtained counterexamples to each of these questions; for the first in [42] and for the second in [43]. Both these counterexamples are in an abstract setting and it is desirable to be able to obtain a counterexample to the Kato square root problem on manifolds under natural geometric conditions.

Perhaps a direction which this can go is the following. It is easy to see that the Davies-Gaffney estimates are always satisfied for the operator Π_B if we begin with Γ constructed out of ∇ and B_2 constructed out of $A \in \mathrm{L}^\infty$. These decay estimates are exponential. Thus, there is hope that if the growth of the underlying manifold is faster than the decay provided by the Davies-Gaffney estimates, the Kato square root problem may fail.

If the coefficients A are real symmetric, then Lax-Milgram implies a solution to the

Kato square root problem. Thus, the lack of symmetry of A may play an important role in the construction of a geometric counterexample. Also, for a complete manifold, we always have that $\mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{M})$ is dense in $\mathrm{W}^{1,2}(\mathcal{M})$. Thus, if the Kato square root estimates are valid, then we would also have that $\mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{M})$ is dense in $\mathcal{D}(\sqrt{\mathrm{L}_A})$. Thus, a way to produce a counterexample may be to show that $\mathrm{C}_{\mathrm{c}}^{\infty}(\mathcal{M})$ is not dense in the $\mathcal{D}(\sqrt{\mathrm{L}_A})$ for a manifold that lacks $\mathrm{Ric} \geq \eta \mathrm{g}$ and when A is non-symmetric. In fact, we make the following conjecture.

Conjecture 2. *There exists* $(\mathcal{M}, \mathrm{g})$*, a complete, smooth Riemannian manifold such that no* $\eta \in \mathbb{R}$ *exists satisfying* $\mathrm{Ric} \geq \eta \mathrm{g}$ *and for which the Kato square root problem on manifolds has a negative answer.*

7.4 Stability of inhomogeneous Hodge-Dirac operators

In the paper [8], the authors show that the Hodge-Dirac operator is stable under small perturbations of the metric on a compact manifold. This is obtained as a direct generalisation of the *holomorphic dependency* of the functional calculus, which they list in §6 of [8]. Let us describe this result. Let $\mathrm{D}_{\mathrm{g}} = \mathrm{d} + \delta_{\mathrm{g}}$ with respect to the metric g, the stability result stated as Theorem 7.3 in [8] is as follows.

Theorem 7.4.1 (Theorem 7.3 in [8])**.** *Let* $\mathcal{M}$ *be a compact Riemannian manifold with metric* g *and let* $\mathrm{g} + \mathrm{h}$ *be a measurable perturbation of* g *with* $\|\mathrm{g}\|_\infty < 1/4$*, and let* $0 < \mu < \pi/2$ *be given by* $\mu = \arccos(1/4)$*. Then we have*

$$\|f(\mathrm{D}_{\mathrm{g}+\mathrm{h}} - f(\mathrm{D}_{\mathrm{g}})\| \lesssim \|f\|_\infty \|\mathrm{h}\|_\infty$$

for every bounded $f : \mathrm{S}_\mu^o \cup \{0\} \to \mathbb{C}$ *holomorphic on* S_μ^o*. The constant above depends only on* $\mathcal{M}$*.*

This is obtained by results which demonstrate the holomorphic dependency of the functional calculus. In §5.4 we prove similar results. However, the main consequence used to prove Theorem 7.3 in [8] is their Theorem 6.5. The generalisation of this is Theorem 1.8.5. The proof of Theorem 6.5 in [8] has a slight oversight, and the extra condition (ii) and (iii) were included in Theorem 1.8.5 to make this proof work. This oversight is repaired in the paper [37] of Hytönen, McIntosh and Portal, but this technology seems to be particular to the setting of $\mathbb{R}^n$. It is these extra conditions imposed in Theorem 1.8.5 which prevent us from proving a statement similar to Theorem 7.3 in [8]. Thus, our work falls short of accessing a stability result for the inhomogeneous Dirac operators we consider. We expect, however, that further study should render such a result possible.

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Notation

$(\mathcal{X}, d)$ Metric space $\mathcal{X}$ with metric d.

$(\mathcal{X}, d, \mu)$ Space $\mathcal{X}$ with metric d and Borel-regular measure μ.

Γ^* Adjoint of the fundamental AKM operator Γ.

Γ_B^* Perturbation of the operator Γ^*.

T^* Adjoint of the operator T.

$\mathcal{A}_t$ Dyadic averaging operator.

$\mathcal{L}(\mathcal{V})$ The vector bundle of bounded maps $T_x : \mathcal{V}_x \to \mathcal{V}_x$.

$\mathcal{L}(\mathscr{H})$ Bounded operators on $\mathscr{H}$.

$\mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ Bounded operators which map $\mathscr{H}_1$ to $\mathscr{H}_2$.

Δ_{B} Bochner Laplacian.

$\mathscr{B}$ The Borel σ-algebra of a topological space.

ρ The GBG radius.

$\mathcal{A}$ Sub-bundle associated with algebraic basis $\mathfrak{a}$

$\operatorname{card} S$ Cardinality of a set S.

$\mathcal{C}$ Space of Carleson measures.

$\mathcal{C}_{t_0}$ Space of local Carleson measures.

R_Q Carleson box over cube Q.

$\mathcal{E}_\tau$ Dyadic cutoff interior portion.

$\mathrm{C}(\nu)$ Carleson function of upper half space Borel measure ν.

$\mathcal{G}$ Lie group.

χ^+ Projection to the right side of the imaginary axis in $\mathbb{C}$.

χ^- Projection to the left side of the imaginary axis in $\mathbb{C}$.

Γ^i_{jk} Connection coefficients or Christoffel symbols.

$\mathrm{C}^k_{\mathrm{c}}(\mathcal{M})$ The compactly supported C^k functions over $\mathcal{M}$.

$\mathrm{C}^k(\mathcal{M})$ The C^k functions over $\mathcal{M}$.

$\mathrm{C}^k(\mathcal{V})$ The space of k differentiable sections on a vector bundle $\mathcal{V}$.

$\mathrm{C}^k(\mathcal{V})$ The space of compactly supported k differentiable sections on a vector bundle $\mathcal{V}$.

$\overline{T}$ Closure of a closeable operator T.

$\mathcal{M}_{\mathrm{st}}$ The spacetime manifold $\mathcal{M} \times \mathbb{S}^1$ or $\mathcal{M} \times \mathbb{R}$.

$\rhd$ Co-Laplacian on L^2.

$\rhd_c$ Connection co-Laplacian.

$[X, Y]$ Commutator of X and Y or Lie derivative when X and Y are vector fields.

$\Gamma(x)$ Cone over point x.

∇_c Restriction of connection ∇ to $\mathrm{C}^\infty_{\mathrm{c}}$.

$\mathrm{T}^*\mathcal{M}$ Cotangent bundle of $\mathcal{M}$.

$\mathrm{T}^*_x\mathcal{M}$ Cotangent space of $\mathcal{M}$ at x.

$\mathrm{c}_L(r)$ Speed of propagation of operator L in a ball of radius r.

$\mathrm{c}_L(x)$ Speed of propagation of L at x.

$\mathcal{V}$ Vector bundle.

$\mathcal{V}_x$ Fibre of the vector bundle $\mathcal{V}$ over x.

δ_{ij} The direct delta function on the indices i and j.

D Hodge-Dirac operator in L^2.

D^2 Hodge-Laplace operator.

D_∞ Smooth Hodge-Dirac operator.

D_{β} The inhomogeneous Hodge-Dirac operator on $\mathbf{\Omega}(\mathcal{M}) \oplus \mathbf{\Omega}(\mathcal{M})$.

D_A The perturbed Hodge-Dirac operator $\mathrm{d} + A^{-1}\delta A$.

D_c Hodge-Dirac operator on compactly supported smooth forms.

$\sqcup\mathscr{C}$ The disjoint union over the collection $\mathscr{C}$.

div Divergence, the adjoint of ∇_c.

$\mathcal{D}(T)$ Domain of the operator T.

$\dot{\gamma}(t)$ The derivative with respect to t of γ.

$\mathscr{Q}$ Dyadic cubes of all levels.

$\mathscr{Q}^k$ Level k dyadic cubes.

$\mathscr{Q}_t$ Level t dyadic cubes.

e Identity element of a Lie group

exp Exponential map of Lie group

$\exp_x$ Exponential map at x.

d Exterior derivative.

d_∞ Exterior derivative as an operator on smooth forms.

d_{β} The inhomogeneous exterior derivative on $\mathbf{\Omega}(\mathcal{M}) \oplus \mathbf{\Omega}(\mathcal{M})$.

d_c Exterior derivative on compactly supported smooth forms.

$\mathfrak{a}$ Algebraic basis of a Lie algebra.

$\mathfrak{g}$ The Lie algebra of a Lie group.

$\mathfrak{g}_k$ Inductively defined ideals $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_{k+1}]$.

$⨍_S f$ Average of a function on the set S (possibly through GBG coordinates).

$\flat$ The canonical isomorphism from the cotangent space to the tangent space through the metric.

$\mathbf{\Omega}(\mathcal{M})$ Exterior algebra of $\mathcal{M}$.

$\mathbf{\Omega}^p(\mathcal{M})$ Bundle of p forms.

Γ The fundamental closed, densely-defined, nilpotent operator in the AKM framework.

$\Gamma_{t_1}(x)$ Local truncated cone $\Gamma_{t_1}(x)$ at point x.

$\mathscr{G}(T)$ Graph of the operator T.

$\mathscr{H}$ Hilbert space.

$\mathrm{Hol}^\infty(\mathrm{S}^o_\mu)$ Bounded functions $f : \mathrm{S}^o_\mu \cup 0 \to \mathbb{C}$ holomorphic on S^o_μ.

$\mathrm{L}^{\mathrm{H}}_A$ Perturbed homogeneous sub-Laplacian with coefficients A.

I Identity operator or Euclidean inner product.

$\mathrm{L}^{\mathrm{I}}_A$ Inhomogeneous perturbed sub-Laplacian.

$\mathbb{Z}$ The integers.

$\mathbb{Z}_+$ The non-negative integers.

χ_S Indicator function of S.

$\mathrm{inj}(\mathcal{M})$ Injectivity radius of the manifold $\mathcal{M}$.

$\mathrm{inj}(\mathcal{M}, x)$ Injectivity radius at x.

$\langle \cdot, \cdot \rangle$ Inner product on a Hilbert space.

$\langle \cdot, \cdot \rangle$ Inner product.

$\langle x \rangle$ $1 + |x|$.

δ_∞ Interior derivative as an operator on smooth forms.

δ_β The inhomogeneous interior derivative on $\boldsymbol{\Omega}(\mathcal{M}) \oplus \boldsymbol{\Omega}(\mathcal{M})$.

δ_c Interior derivative on compactly supported smooth forms.

J Scale below GBG radius, $C_1 \delta^J \leq \rho/5$.

Δ_c Connection Laplacian.

$\mathscr{L}$ The Lebesgue measure.

$\ell(\gamma)$ Length of a curve γ

$\ell(\gamma)$ Length of the curve γ.

$\ell(Q^k_\alpha)$ Length of cube Q^k_α.

$\mathbf{Lip}\, f$ Lipschitz constant of f.

$\mathrm{Lip}\, f$ Lipschitz pointwise constant.

$\mathrm{L}^2_{\mathscr{Q}_t}(\mathcal{X})$ Functions $f : \mathcal{X} \to \mathbb{C}^N$ that are in $\mathrm{L}^2(Q)$ for each $Q \in \mathscr{Q}_t$.

$\mathrm{L}^1_{\mathrm{loc}}(\mathcal{V})$ The locally integrable sections of $\mathcal{V}$.

$\mathrm{L}^p(\mathcal{V})$ The L^p space over the bundle $\mathcal{V}$.

$\mathrm{L}^p(\mathcal{X})$ The L^p space over the bundle $\mathcal{X} \times \mathbb{C}$.

$\mathcal{M}f$ Maximal function (uncentred) of f.

$\max S$ maximum of S.

$\mathcal{M}^* f(x)$ Non-tangential maximal function of f at point x.

$\mathcal{M}^\beta f(x)$ Exponential maximal function of f at x with growth $\beta > 0$.

$\mathcal{M}^\infty_{\mathrm{loc}} f(x)$ Local non-tangential maximal function.

$\mathscr{M}$ The (largest) measure σ-algebra.

d Metric on the metric space $\mathcal{X}$.

$d(E, F)$ Distance between sets E and F induced by d.

h Metric on the vector bundle $\mathcal{V}$.

$|\cdot|_x$ Norm on a vector bundle on the fibre at x.

$\mathbb{N}$ The natural numbers or positive integers $\{1, 2, \ldots\}$.

$\nabla^{\mathcal{M}}$ Levi-Cevita connection on $(\mathcal{M}, \mathrm{g})$ or extension of such a connection to a product manifold.

$\mathcal{N}$ Space of non-tangential functions.

$\mathcal{N}_{t_1}$ Local Nirenberg function space.

$\|\cdot\|$ The L^2 norm.

$\|\cdot\|_p$ The L^p norm.

$\|\cdot\|_{\mathcal{C}, t_0}$ Local Carleson measure norm.

$\|\cdot\|_{\mathcal{C}}$ Carleson norm.

$\|\cdot\|_{\mathcal{N}}$ Non-tangential function space norm.

$\|\cdot\|_{\mathrm{W}^{k,p}}$ The $k,\ p$ Sobolev norm.

$\|u\|_T$ Operator norm of T.

$\mathcal{N}(T)$ Null space of the operator T.

$\omega \wedge \eta$ Exterior product between the forms ω and η.

$\widehat{Q}$ Parent cube of Q at top level.

Π The operator $\Gamma + \Gamma^*$.

Π_B The operator $\Gamma + \Gamma^*_B$.

$\gamma_t(x)$ Principal part operator at x.

$\mathbf{P}_A$ Projection operator onto the subspace A.

$\mathscr{P}(\mathcal{T})$ Powerset of $\mathcal{T}$.

$\Psi(\mathrm{S}^{\mathrm{o}}_{\mu})$ Rapidly decaying holomorphic functions on the open μ-sector.

Q^k_α Christ cube with index α and of level k.

$\mathbb{R}^+$ The positive real numbers $(0, \infty)$.

$\mathbb{R}^n$ Euclidean space of n dimensions.

$\mathbb{R}_+$ The non-negative real numbers $[0, \infty)$.

$\mathcal{R}(T)$ Range of the operator T.

$\mathcal{R}(\cdot\,,\cdot)$ The curvature operator (3-tensor) associated to a connection.

R_T Resolvent of T.

Ric Ricci curvature.

Rm Riemann curvature tensor.

$\mathrm{R_s}$ Scalar curvature.

$\rho(T)$ Resolvent set of T.

r_g Right multiplication (by g) operator.

$\mathrm{t_S}$ Dyadic scale.

$\mathrm{K}(X, Y)$ Sectional curvature spanned by the vector fields X and Y.

$\mathrm{S}^{\mathrm{o}}_{\omega}$ Open bi-sector of angle ω.

$\mathbf{\Gamma}(\mathcal{V})$ The measurable sections of the vector bundle $\mathcal{V}$.

S_{ω} Closed bi-sector of angle ω.

sgn The signum operator $\chi^{+} - \chi^{-}$.

$\mathrm{W}^{k,p}_0$ The $k,\ p$ Sobolev space as the completion of compactly supported sections of $\mathcal{V}$.

$\mathrm{W}^{k,p}_0$ The $k,\ p$ Sobolev space as the completion of compactly supported smooth functions on $\mathcal{M}$.

$\mathrm{W}^{k,p}_0(\mathcal{G})'$ The k-th order sub-Sobolev space in L^p defined as the completion of compactly supported sections.

$\mathrm{W}^{k,p}_0(\mathcal{V})'$ The k-th order sub-Sobolev space in L^p defined as the completion of compactly supported sections.

$\mathrm{W}^{k,p}(\mathcal{G})'$ The k-th order sub-Sobolev space in L^p.

$\mathrm{W}^{k,p}(\mathcal{M})$ The $k,\ p$ Sobolev space over functions on $\mathcal{M}$.

$\mathrm{W}^{k,p}(\mathcal{V})$ The $k,\ p$ Sobolev space on $\mathcal{V}$.

$\mathrm{W}^{k,p}(\mathcal{V})'$ The k-th order sub-Sobolev space in L^p.

$\mathcal{X}$ Metric space.

$\mathcal{X}_+$ Upper half space $\mathcal{X} \times \mathbb{R}^+$.

$\mathcal{X}_{+,t_0}$ Truncated upper half space.

$\sigma(T)$ Spectrum of T.

$\mathcal{S}_B(\zeta)(\psi)$ Operator $\psi(t\Pi_{B(\zeta)})$.

$\sigma_L(x,v)$ Principal symbol of operator L, at x in direction v.

$\mathrm{T}\mathcal{M}$ Tangent bundle of $\mathcal{M}$.

$\mathrm{T}_x\mathcal{M}$ Tangent space to $\mathcal{M}$ at x.

$\mathcal{T}^{(p,q)}\mathcal{M}$ The bundle of tensors of contravarient rank p and covarient rank q.

$\mathrm{T}(O)$ Tent over open set O.

Θ_t The bounded operator $t\Gamma^*(\mathrm{I} + t^2\Pi^2)^{-1}$.

Θ_t^B The bounded operator $t\Gamma_B^*(\mathrm{I} + t^2\Pi_B^2)^{-1}$.

$\tilde{\mathcal{E}}_\tau$ Dyadic cutoff boundary portion.

$\operatorname{tr} T$ $\operatorname{tr}_{12} T$.

$\operatorname{tr}_{ij} T$ Trace of the tensor T of the indices i and j with respect to the metric.

A_i Differential operators defined by left regular representation as the infinitesimal generator.

$B(x, r)$ Open ball of centre x and radius r.

$d\,\mathrm{r}_g$ Differential of the map r_g.

D_i right-translated vector field of a_i.

$f_{Q,\varepsilon}^w$ Test function for T(b) argument.

P_t The bounded operator $(\mathrm{I} + t^2\Pi^2)^{-1}$.

P_t^B The bounded operator $(\mathrm{I} + t^2\Pi_B^2)^{-1}$.

Q_t The bounded operator $t\Pi(\mathrm{I} + t^2\Pi^2)^{-1}$.

Q_t^B The bounded operator $t\Pi_B(\mathrm{I} + t^2\Pi_B^2)^{-1}$.

R_t The bounded operator $(\mathrm{I} + t\Pi)^{-1}$.

R_t^B The bounded operator $(\mathrm{I} + t\Pi_B)^{-1}$.

S_k^p Set of $u \in \mathrm{C}^\infty(\mathcal{V}) \cap \mathrm{L}^p(\mathcal{V})$ such that $\nabla^i u \in \mathrm{C}^\infty(\mathcal{V}) \cap \mathrm{L}^p(\mathcal{T}^{(i,0)}\mathcal{M} \otimes \mathcal{V})$.

u_Q Cube average of u on Q.

(Exp) Exponential measure.

(Pol) Polynomial measure.

(Qest) The main quadratic estimate.

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