Square roots of perturbed sub-elliptic operators on Lie groups

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(Joint work with Tom ter Elst, Auckland and Alan McIntosh, ANU)

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August 13, 2012

POSTI/Mprime Seminar
University of Calgary
Let $\mathcal{G}$ be a Lie group of dimension $n$ and $\mathfrak{g}$ is Lie algebra.
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We let $d\mu$ denote the left invariant *Haar* measure.
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Algebraic basis and vectorfields

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The vectorfields \( \{A_i\} \) are linearly independent and *global*. 
Distance

Theorem of Carathéodory-Chow tells us that for any two points $x, y \in G$, we can find a curve $\gamma : [0, 1] \to G$ such that

$$\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t) A_i(\gamma(t)) \in \text{span} \{A_i(\gamma(t))\}.$$
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The length of such a curve then is given by

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Define distance \( d(x, y) \) as the infimum over the length of all such curves.

The measure \( d\mu \) is Borel-regular with respect to \( d \) and we consider \((\mathcal{G}, d, d\mu)\) as a measure metric space.
Define an associated \textit{sub-Laplacian} by:

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\Delta = - \sum_i A_i^2.
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This is a densely-defined, self-adjoint operator on \( L^2(\mathcal{G}) \).
We say that a Lie group is *nilpotent* if

\[ g_1 = [g, g], \ g_2 = [g, g_1], \ g_3 = [g_1, g_2], \ldots \]

is eventually 0. That is, there is a \( k \) such that \( g_k = 0 \).
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On such spaces, we consider the uniformly elliptic second order operator

\[ D_H = -b \sum_{i,j} A_i b_{ij} A_j \]

where \( b, b_{ij} \in L^\infty(G) \).
The main theorem for nilpotent Lie groups

**Theorem (B.-E.-Mc)**

Let $G$ be a connected nilpotent and suppose there exist $\kappa_1, \kappa_2 > 0$ such that

$$\text{Re } b(x) \geq \kappa_1 \quad \text{and} \quad \text{Re } \int_G \sum_{i,j} b_{ij} A_i u \overline{A_j u} \geq \kappa_2 \sum_i \|A_i u\|^2$$

for almost all $x \in G$ and $u \in H^1(G)$. Then,

(i) $\mathcal{D}(\sqrt{D_H}) = \bigcap_{i=1}^m \mathcal{D}(A_i) = H^1(G)$, and

(ii) $\|\sqrt{D_H u}\| \simeq \sum_{i=1}^m \|A_i u\|$ for all $u \in H^1(G)$. 

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Stability

Theorem (B.-E.-Mc)

Let $0 < \eta_i < \kappa_i$ and suppose that $\tilde{b}, \tilde{b}_{ij} \in L^\infty(\mathcal{G})$ such that $\|\tilde{b}\|_\infty \leq \eta_1$ and $\|(\tilde{b}_{ij})\|_\infty \leq \eta_2$. Then,

$$\|\sqrt{D_H}u - \sqrt{\tilde{D}_H}u\| \lesssim (\|\tilde{b}\|_\infty + \|(\tilde{b}_{ij})\|_\infty) \sum_{i=1}^k \|A_i u\|,$$

for $u \in H^1(\mathcal{G})$ and where

$$\tilde{D}_H = (b + \tilde{b}) \sum_{i,j=1}^k A_i (b_{ij} + \tilde{b}_{ij}) A_j.$$
Operator theory

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(H1) The operator $\Gamma : D(\Gamma) \subset \mathcal{H} \to \mathcal{H}$ is closed, densely-defined and nilpotent ($\Gamma^2 = 0$).
Operator theory

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\hspace{1cm} \textbf{(H1)} The operator $\Gamma : D(\Gamma) \subset \mathcal{H} \to \mathcal{H}$ is closed, densely-defined and \textit{nilpotent} ($\Gamma^2 = 0$).

\hspace{1cm} \textbf{(H2)} The operators $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ satisfy

\begin{align*}
\text{Re} \langle B_1 u, u \rangle & \geq \kappa_1 \| u \| \quad \text{whenever } u \in \mathcal{R}(\Gamma^*) \\
\text{Re} \langle B_2 u, u \rangle & \geq \kappa_2 \| u \| \quad \text{whenever } u \in \mathcal{R}(\Gamma)
\end{align*}

where $\kappa_1, \kappa_2 > 0$ are constants.
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(H3) The operators $B_1, B_2$ satisfy $B_1 B_2(\mathcal{R}(\Gamma)) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1(\mathcal{R}(\Gamma^*)) \subset \mathcal{N}(\Gamma^*)$.
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Let $\Gamma_B^* = B_1 \Gamma^* B_2$, $\Pi_B = \Gamma + \Gamma_B^*$, and $\Pi = \Gamma + \Gamma^*$. 
Harmonic analysis and Kato square root type estimates

**Theorem (Kato square root type estimate)**

Suppose that \((\Gamma, B_1, B_2)\) satisfy (H1)-(H3) and

\[
\int_0^\infty \| t\Pi_B (1 + t^2 \Pi_B^2)^{-1} u \|^2 \frac{dt}{t} \simeq \| u \|^2
\]

for all \(u \in \overline{\mathcal{R}(\Pi_B)} \subset \mathcal{H} \). Then,

(i) There is a spectral decomposition \(\mathcal{H} = \mathcal{N}(\Pi_B) \oplus E^+_B \oplus E^-_B\), where \(E^\pm_B\) are spectral subspaces and the sum is in general non-orthogonal, and

(ii) \(\mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma_B^*) = \mathcal{D}(\Pi_B) = \mathcal{D}(\sqrt{\Pi_B^2})\) with

\[
\| \Gamma u \| + \| \Gamma_B u \| \simeq \| \Pi_B u \| \simeq \| \sqrt{\Pi_B^2} u \| \text{ for all } u \in \mathcal{D}(\Pi_B).
\]
Homogeneous conditions

(H4) Let \( X \) be a complete, connected metric space and \( \mu \) a Borel-regular measure on \( X \) that is doubling. Then set \( H = L^2(X, C^N; d\mu) \).

(H5) The operators \( B_i \) are matrix-valued pointwise multiplication operators such that the function \( x \mapsto B_i(x) \) are \( L^\infty(X, L(C^N)) \).

(H6) For every bounded Lipschitz function \( \xi: X \rightarrow C \), multiplication by \( \xi \) preserves \( D(\Gamma) \) and \( M_\xi = [\Gamma, \xi I] \) is a multiplication operator. Furthermore, there exists a constant \( m > 0 \) such that \( |M_\xi(x)| \leq m |\text{Lip} \xi(x)| \) for almost all \( x \in X \).

(H7) For each open ball \( B \), we have \( \hat{\Diamond} B \Gamma u \; d\mu = 0 \) and \( \hat{\Diamond} B \Gamma^* v \; d\mu = 0 \) for all \( u \in D(\Gamma) \) with \( \text{spt} u \subset B \) and for all \( v \in D(\Gamma^*) \) with \( \text{spt} v \subset B \).
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(H7) For each open ball $B$, we have

$$\int_B \Gamma u \ d\mu = 0 \quad \text{and} \quad \int_B \Gamma^* v \ d\mu = 0$$

for all $u \in \mathcal{D}(\Gamma)$ with $\text{spt} \ u \subset B$ and for all $v \in \mathcal{D}(\Gamma^*)$ with $\text{spt} \ v \subset B$. 
(H8) -1 (Poincaré hypothesis)
There exists $C' > 0$, $c > 0$ and an operator
$\Xi : \mathcal{D}(\Xi) \subset L^2(\mathcal{X}, \mathbb{C}^N) \rightarrow L^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and

$$\int_B |u - u_B|^2 \, d\mu \leq C'r^2 \int_B |\Xi u|^2 \, d\mu$$

for all balls $B = B(x, r)$ and $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$. 

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(\H8) -2 (Coercivity hypothesis)
There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,
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This is slightly different from (H8) in [Bandara].
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This is slightly different from (H8) in [Bandara].
Theorem (B.)

Let $\mathcal{X}, (\Gamma, B_1, B_2)$ satisfy (H1)-(H8). Then, $\Pi_B$ satisfies the quadratic estimate

$$\int_0^\infty \| t \Pi_B (1 + t^2 \Pi_B^2)^{-1} u \|^2 \frac{dt}{t} \simeq \| u \|^2$$

for all $u \in \overline{R(\Pi_B)} \subset L^2(\mathcal{X}, \mathbb{C}^N)$. 

Define the bundle $W = \text{span} \{ A_i \} \subset T^*G$ and complexify it.

Equip $W$ with the inner product $h(A_i, A_j) = \delta_{ij}$.

Equip $G$ with the sub-connection $\nabla f = A_k f A_k$ where $A_k = A_k^* \in W^*$.

Equip $W$ with the sub-connection $\tilde{\nabla}(u_i A_i) = (\nabla u_i) \otimes A_i$.

We have that $W \sim = C^k$ and $L^2(G) \oplus L^2(W) \sim = L^2(C^{k+1})$. 

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Geometric setup

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where $A^k = A_k^* \in \mathcal{W}^*$. 
Geometric setup

Define the bundle $\mathcal{W} = \text{span} \{ A_i \} \subset TG$ and complexify it.

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We have that $\mathcal{W} \cong \mathbb{C}^k$ and $L^2(\mathcal{G}) \oplus L^2(\mathcal{W}) \cong L^2(\mathbb{C}^{k+1})$. 
Define: $\Gamma : \mathcal{D}(\Gamma) \subset L^2(\mathcal{G}) \oplus L^2(\mathcal{W}^*) \to L^2(\mathcal{G}) \oplus L^2(\mathcal{W}^*)$ by

$$\Gamma = \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}.$$
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Then,

$$\Gamma^* = \begin{pmatrix} 0 & -\text{div} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi = \begin{pmatrix} 0 & -\text{div} \\ \nabla & 0 \end{pmatrix},$$

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Let $B = (b_{ij})$. Then, define

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$
Proof of the homogeneous problem

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(H2) By accretivity assumptions.
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(H2) By accretivity assumptions.

(H3) By construction.
Proof (cont.)

(H4) The measure $d\mu$ is Borel-regular and the nilpotency of $G$ implies that it is doubling.

(H5) By assumption.

(H6) It is an easy fact that for all bounded Lipschitz $\xi : G \to \mathbb{C}$,
\[ |M_{\xi}(x)| = |[\Gamma, \xi(x)]I| = |\nabla \xi(x)| \leq k \text{Lip} \xi(x) \]
for almost all $x \in G$.

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for all almost all $x \in G$.

(H7) By the left invariance of the measure $d\mu$. 
Proof (cont.)

The nilpotency of $G$ implies the following Poincaré inequality:

$$\hat{B} |f - f_B|^2 d\mu \lesssim r^2 \hat{B} |\nabla f|^2 d\mu$$

for all balls $B$, and $f \in C^\infty(B)$. See [SC, (P.1), p118].

Define $\Xi u = (\nabla u_1, \tilde{\nabla} u_2)$.

The crucial fact needed here is the regularity result [ERS, Lemma 4.2] which gives

$$\|A_i A_j f\| \lesssim \|\Delta f\|$$

for $f \in H^2(G) = D(\Delta)$. 
(H8) The nilpotency of $G$ implies the following Poincaré inequality

$$\int_B |f - f_B|^2 \, d\mu \lesssim r^2 \int_B |\nabla f|^2 \, d\mu$$

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Inhomogeneous problem

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For general Lie groups, we need to consider operators with lower order terms.

Let $b, b_{ij}, c_k, d_k, e \in L^\infty(G)$. Define the following uniformly elliptic second order operator

$$D_I = -b \sum_{ij=1}^{m} A_i b_{ij} A_j u - b \sum_{i=1}^{m} A_i c_i u - b \sum_{i=1}^{m} d_i A_i u - be u.$$
Theorem (B.-E.-Mc)

Let $G$ be a connected Lie group and suppose there exists $\kappa_1, \kappa_2 > 0$ such that

\[
\text{Re} b(x) \geq \kappa_1,
\]

\[
\text{Re} \int_G \left( eu + \sum_{i=1}^{m} d_i A_i u \right) \, \overline{u} + \sum_{i=1}^{m} \left( c_i u + \sum_{j=1}^{m} b_{ij} A_j u \right) \overline{A_i u} \, d\mu \geq \kappa_2 \left( \|u\|^2 + \sum_{i=1}^{m} \|A_i u\|^2 \right)
\]

for almost all $x \in G$ and $u \in H^1(G)$. Then,

(i) $D(\sqrt{D_I}) = \bigcap_{i=1}^{m} D(A_i) = H^1(G)$, and

(ii) $\|\sqrt{D_I} u\| \simeq \|u\| + \sum_{i=1}^{m} \|A_i u\|$ for all $u \in H^1(G)$. 
Spaces of exponential growth

$(\mathcal{X}, d, \mu)$ an exponentially locally doubling measure metric space. That is: there exist $\kappa, \lambda \geq 0$ and constant $C \geq 1$ such that

$$0 < \mu(B(x, tr)) \leq Ct^\kappa e^{\lambda tr} \mu(B(x, r))$$

for all $x \in \mathcal{X}$, $r > 0$ and $t \geq 1$. 
Changes to (H7) and (H8)

The following (H7) from [Morris]:

(H7) There exist $c > 0$ such that for all open balls $B \subset X$ with $r \leq 1$,

$$\left| \int_B \Gamma u \, d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|u\| \quad \text{and} \quad \left| \int_B \Gamma^* v \, d\mu \right| \leq c\mu(B)^{\frac{1}{2}} \|v\|$$

for all $u \in \mathcal{D}(\Gamma)$, $v \in \mathcal{D}(\Gamma^*)$ with $\text{spt } u$, $\text{spt } v \subset B$. 
We introduce the following \textit{local} (H8):

\textbf{(H8) -1} \quad \text{(Local Poincaré hypothesis)}
There exists $C' > 0$, $c > 0$ and an operator
$\Xi : \mathcal{D}(\Xi) \subset L^2(\mathcal{X}, \mathbb{C}^N) \to L^2(\mathcal{X}, \mathbb{C}^M)$ such that $\mathcal{D}(\Pi) \cap \mathcal{R}(\Pi) \subset \mathcal{D}(\Xi)$ and
\[
\int_B |u - u_B|^2 \, d\mu \leq C' r^2 \int_B (|\Xi u|^2 + |u|^2) \, d\mu
\]
for all balls $B = B(x, r)$ and for $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$.

\textbf{-2} \quad \text{(Coercivity hypothesis)}
There exists $\tilde{C} > 0$ such that for all $u \in \mathcal{D}(\Pi) \cap \mathcal{R}(\Pi)$,
\[
\|\Xi u\| + \|u\| \leq \tilde{C} \|\Pi u\|.
\]
Theorem (Morris)

Let \( \mathcal{X}, (\Gamma, B_1, B_2) \) satisfy (H1)-(H8). Then, \( \Pi_B \) satisfies the quadratic estimate

\[
\int_0^\infty \| t\Pi_B (1 + t^2 \Pi_B^2)^{-1} u \|^2 \frac{dt}{t} \lesssim \| u \|^2
\]

for all \( u \in \overline{\mathcal{R}(\Pi_B)} \subset L^2(\mathcal{X}, \mathbb{C}^N) \).
Setup

Set $\mathcal{X} = \mathcal{G}$ and $\mathcal{H} = L^2(\mathcal{G}) \oplus L^2(\mathcal{G}) \oplus L^2(\mathcal{W}) \cong L^2(\mathbb{C}^{k+2})$. 

Let $S = (I, \nabla)$, $S^* = [I - \text{div}]$. 

Let $\Gamma = \begin{pmatrix} 0 & S \end{pmatrix}^T$, $\Gamma^* = \begin{pmatrix} S^* & 0 \end{pmatrix}$, and $\Pi^* = \begin{pmatrix} S^* & S \end{pmatrix}^T$. 

Let $\tilde{B}_{00} = e$, $\tilde{B}_{10} = (c_1, \ldots, c_m)_\text{tr}$, $\tilde{B}_{01} = (d_1, \ldots, d_m)_\text{tr}$, $\tilde{B}_{11} = (b_{ij})$, and $B = (\tilde{B}_{ij})$. 

Then, we can write $B_1 = \begin{pmatrix} b_{00} & 0 \\ 0 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. 

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$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \Pi^* = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}.$$
Setup

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\]

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Then, we can write

\[
B_1 = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.
\]
Proof

The proofs of (H1)-(H6) are similar to the homogeneous situation.

(H7) The proof is the same as the homogeneous situation except the lower order term introduces the term $\mu(B)^{1/2} \|u\|$ on the right.
Proof

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(H7) The proof is the same as the homogeneous situation except the lower order term introduces the term \( \mu(B)^{1/2} \|u\| \) on the right.

(H8) The existence of a local Poincaré inequality is guaranteed by [ER2, Proposition 2.4]:

\[
\int_B |f - f_B|^2 \, d\mu \lesssim r^2 \int_B (|\nabla f|^2 + |f|^2) \, d\mu
\]

for all balls \( B = B(x, r) \) and where \( f \in C^\infty(B) \).
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Define $\Xi u = (\nabla u_1, \nabla u_2, \tilde{\nabla} u_3)$.
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The crucial fact needed here is the regularity result in [ER, Theorem 7.2],

\[
\|A_i A_j u\|^2 \lesssim \|\Delta u\|^2 + \|u\|^2
\]

for $u \in H^2(G) = \mathcal{D}(\Delta)$. 

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References I


