Geometry and the Kato square root problem

Lashi Bandara

Centre for Mathematics and its Applications
Australian National University

7 June 2013

Geometric Analysis Seminar
University of Wollongong
Outline

- Brief overview of the Kato square root problem on $\mathbb{R}^n$.
- A motivating application to hyperbolic PDE.
- Recent progress on the Kato square root problem on smooth manifolds by McIntosh and B.
- Recent progress on subelliptic Kato square root problems on Lie groups by ter Elst, McIntosh, and B.
- Kato square root problem on smooth manifolds with non-smooth metrics, connection to geometric flows and PDEs.
The Kato square root problem

Let $A \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ and $a \in L^\infty(\mathbb{R}^n)$. Suppose that there exists $\kappa_1, \kappa_2 > 0$ such that for all $u \in W^{1,2}(\mathbb{R}^n)$,

$$\text{Re} \ a(x) \geq \kappa_1 \quad \text{and} \quad \text{Re} \langle A \nabla u, \nabla u \rangle \geq \kappa_2 \|u\|^2.$$

The Kato square root problem on $\mathbb{R}^n$ is the statement that

$$\mathcal{D}(\sqrt{-a \text{ div } A \nabla}) = W^{1,2}(\mathbb{R}^n)$$

$$\|\sqrt{-a \text{ div } A \nabla} u\| \simeq \|\nabla u\|. \quad (K1)$$

This was answered in the positive in 2002 by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh and Phillipe Tchamitchian in [AHLMcT].
If further $A = A^*$, (K1) is a trivial consequence of the Lax-Milgram Theorem.

Solution to (K1) implies that $\mathcal{D}(\sqrt{-\text{div} A \nabla}) = \mathcal{D}(\sqrt{-\text{div} A^* \nabla})$. We can ask a more abstract question for accretive operators $L$ on a Hilbert space $\mathcal{H}$. There, the question is whether $\mathcal{D}(\sqrt{L^*}) = \mathcal{D}(\sqrt{L})$. In general, this is not true by a counterexample of McIntosh in 1972 in [Mc72].

A second related question is the following. Suppose that $J_t$ is a family of closed, densely-defined, Hermitian forms on $\mathcal{H}$ with domain $\mathcal{W}$ and $L(t)$ the associated self-adjoint operators to $J_t$ with domain $\mathcal{W}$. If $t \mapsto J_t$ extends to holomorphic family (for small $z$), then is $\partial_t \sqrt{L(t)} : \mathcal{V} \to \mathcal{H}$ a bounded operator?

Counterexample to this second question by McIntosh in 1982 in [Mc82].
Motivations from PDE

For \( k = 1, 2 \), let \( L_k = -\text{div} A_k \nabla \) where \( A_k \in L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^n)) \) non-negative self-adjoint and \( L_k \) uniformly elliptic.

As aforementioned, \( D(\sqrt{L_k}) = W^{1,2}(\mathbb{R}^n) \) and \( \|\sqrt{L_k}u\| \simeq \|\nabla u\| \) for \( u \in W^{1,2}(\mathbb{R}^n) \).

Let \( u_k \) be solutions to the wave equation with respect to \( L_k \) with the same initial data. That is,

\[
\partial_t^2 u_k + L_k u_k = 0
\]
\[
\partial_t u_k \big|_{t=0} = g \in L^2(\mathbb{R}^n)
\]
\[
u_k(0) = f \in W^{1,2}(\mathbb{R}^n).
\]
Suppose there exists a $C > 0$

$$\| \sqrt{L_1} u - \sqrt{L_2} u \| \leq C \| A_1 - A_2 \|_\infty \| \nabla u \|. \quad \text{(P)}$$

Then, whenever $t > 0$, the following estimate holds:

$$\| \mathcal{U}_1(t) - \mathcal{U}_2(t) \| + \| \int_0^t \nabla (\mathcal{U}_1(s) - \mathcal{U}_2(s)) \, ds \|$$

$$\leq C t \| A_1 - A_2 \|_\infty (\| \nabla f \| + \| g \|).$$

See [Aus].
The estimate $(P)$ is related to the second question of Kato.

By solving the Kato square root problem $(K1)$ for complex coefficients $A$, we are able to automatically obtain $(P)$ from $(K1)$. 
Kato square root problem on manifolds

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with metric $g$, Levi-Civita connection $\nabla$, and volume measure $\mu$.

Write $\text{div} = -\nabla^*$ in $L^2$ and let $S = (I, \nabla)$.

Assume $a \in L^\infty(\mathcal{M})$ and $A = (A_{ij}) \in L^\infty(\mathcal{M}, \mathcal{L}(L^2(\mathcal{M}) \oplus L^2(T^*\mathcal{M})))$.

Consider the following second order differential operator

$L_A : \mathcal{D}(L_A) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M})$ defined by

$L_A u = aS^*ASu = -a \text{div}(A_{11}\nabla u) - a \text{div}(A_{10}u) + aA_{01}\nabla u + aA_{00}u.$
The main theorem on manifolds

**Theorem (B.-Mc, 2012)**

Let \( \mathcal{M} \) be a smooth, complete Riemannian manifold with \( |\text{Ric}| \leq C \) and \( \text{inj}(\mathcal{M}) \geq \kappa > 0 \). Suppose the following ellipticity condition holds: there exists \( \kappa_1, \kappa_2 > 0 \) such that

\[
\text{Re} \langle av, v \rangle \geq \kappa_1 \| v \|^2 \\
\text{Re} \langle ASu, Su \rangle \geq \kappa_2 \| u \|^2_{W^{1,2}}
\]

for \( v \in L^2(\mathcal{M}) \) and \( u \in W^{1,2}(\mathcal{M}) \). Then, \( \mathcal{D}(\sqrt{L_A}) = \mathcal{D}(\nabla) = W^{1,2}(\mathcal{M}) \) and \( \| \sqrt{L_A} u \| \simeq \| \nabla u \| + \| u \| = \| u \|_{W^{1,2}} \) for all \( u \in W^{1,2}(\mathcal{M}) \).
Since we allow the coefficients $a$ and $A$ to be complex, we obtain the following stability result as a consequence:

**Theorem (B.-Mc, 2012)**

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold with $|\text{Ric}| \leq C$ and $\text{inj}(\mathcal{M}) \geq \kappa > 0$. Suppose that there exist $\kappa_1, \kappa_2 > 0$ such that

$\text{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2$ and $\text{Re} \langle ASu, Su \rangle \geq \kappa_2 \|u\|^2_{W^{1,2}}$ for $v \in L^2(\mathcal{M})$ and $u \in W^{1,2}(\mathcal{M})$. Then for every $\eta_i < \kappa_i$, whenever $\|\tilde{a}\|_{\infty} \leq \eta_1$, $\|\tilde{A}\|_{\infty} \leq \eta_2$, the estimate

$$\|\sqrt{L_A} u - \sqrt{L_{A+\tilde{A}}} u\| \lesssim (\|\tilde{a}\|_{\infty} + \|\tilde{A}\|_{\infty}) \|u\|_{W^{1,2}}$$

holds for all $u \in W^{1,2}(\mathcal{M})$. The implicit constant depends in particular on $A, a$ and $\eta_i$.  

Let $\Omega(\mathcal{M})$ denote the algebra of differential forms over $\mathcal{M}$ under the exterior product $\wedge$.

Let $d$ be the exterior derivative as an operator on $L^2(\Omega(\mathcal{M}))$ and $d^*$ its adjoint, both of which are *nilpotent* operators.

The Hodge-Dirac operator is then the self-adjoint operator $D = d + d^*$. The Hodge-Laplacian is then $D^2 = d d^* + d^* d$.

For an invertible $A \in L^\infty(\mathcal{L}(\Omega(\mathcal{M})))$, we consider perturbing $D$ to obtain the operator $D_A = d + A^{-1} d^* A$. 
Curvature endomorphism for forms

Let \( \{ \theta^i \} \) be an orthonormal frame at \( x \) for \( \Omega^1(M) = T^*M \).

Denote the components of the curvature tensor in this frame by \( R_{ijkl} \).

The curvature endomorphism is then the operator

\[
R \omega = R_{ijkl} \theta^i \wedge (\theta^j \wedge (\theta^k \wedge (\theta^l \wedge \omega)))
\]

for \( \omega \in \Omega_x(M) \).

This can be seen as an extension of Ricci curvature for forms, since

\[
g(R \omega, \eta) = \text{Ric}(\omega^b, \eta^b)
\]

whenever \( \omega, \eta \in \Omega^1_x(M) \) and where

\( b : T^*M \to TM \) is the flat isomorphism through the metric \( g \).

The Weitzenböck formula then asserts that \( D^2 = \text{tr}_{12} \nabla^2 + R \).
Theorem (B., 2012)

Let $\mathcal{M}$ be a smooth, complete Riemannian manifold and let $\beta \in \mathbb{C} \setminus \{0\}$. Suppose there exist $\eta, \kappa > 0$ such that $|\text{Ric}| \leq \eta$ and $\text{inj}(\mathcal{M}) \geq \kappa$. Furthermore, suppose there is a $\zeta \in \mathbb{R}$ satisfying $g(\text{R} u, u) \geq \zeta |u|^2$, for $u \in \Omega_x(\mathcal{M})$ and $A \in L^\infty(\mathcal{L}(\Omega(\mathcal{M})))$ and $\kappa_1 > 0$ satisfying

$$\text{Re} \langle Au, u \rangle \geq \kappa_1 \|u\|^2.$$

Then, $\mathcal{D}(\sqrt{D_A^2 + |\beta|^2}) = \mathcal{D}(D_A) = \mathcal{D}(d) \cap \mathcal{D}(d^* A)$ and

$$\|\sqrt{D_A^2 + |\beta|^2}u\| \simeq \|D_A u\| + \|u\|.$$
Lie groups

Let $G$ be a Lie group of dimension $n$ with Lie algebra $\mathfrak{g}$ and equipped with the left-invariant Haar measure $\mu$.

We say that a linearly independent $\alpha = \{a_1, \ldots, a_k\} \subset \mathfrak{g}$ is an algebraic basis if we can recover a basis for $\mathfrak{g}$ through multi-commutation.

Let $A_i$ denote the right-translation of $a_i$ and $A^i = A_i^*$. Let $\text{span}\{A_1, \ldots, A_k\} = \mathcal{A} \subset TG$ be the bundle obtained through the right-translation of $\alpha$ and $\mathcal{A}^* = \{A^1, \ldots, A^k\}$ the dual of $\mathcal{A}$. 

Lashi Bandara (ANU) 
Geometry and the Kato problem 
7 June 2013 14 / 28
Subelliptic distance

Theorem of Carathéodory-Chow tells us that for any two points \( x, y \in \mathcal{G} \), we can find an absolutely continuous curve \( \gamma : [0, 1] \to \mathcal{G} \) such that

\[
\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t) A_i(\gamma(t)) \in \mathcal{A}.
\]

The length of such a curve then is given by

\[
\ell(\gamma) = \int_0^1 \left( \sum_i |\dot{\gamma}^i(t)|^2 \right)^{1/2} \, dt
\]

Define distance \( d(x, y) \) as the infimum over the length of all such curves.

The measure \( \mu \) is Borel-regular with respect to \( d \).
Subelliptic operators

For $f \in C^\infty(G)$, define

$$\nabla f = A_i f A^i.$$ 

This defines an *sub-connection* on $C^\infty(M)$.

Each vector field $A_i$ is a skew-adjoint differential operator. We consider it as a unbounded operator on $L^2(G)$ with domain $\mathcal{D}(A_i)$.

By also considering $\nabla$ as a closed, densely-defined operator on $L^2(M)$, we obtain the first-order Sobolev space $W^{1,2}(G)' = \mathcal{D}(\nabla) = \bigcap_{i=1}^k \mathcal{D}(A_i)$.

We write the divergence as $\text{div} = -\nabla^*$. Then, the subelliptic Laplacian associated to $\mathcal{A}$ is

$$\Delta = -\text{div} \nabla = -\sum_{i=1}^k A_i^2.$$
Nilpotent Lie groups

The Lie group \( G \) is \textit{nilpotent} if the inductively defined sequence \( g_1 = [g, g], \ g_2 = [g_1, g], \ldots \) is eventually zero.

**Theorem (B.-E.-Mc., 2012)**

Let \((G, d, \mu)\) be a connected, nilpotent Lie group with \( \alpha \) an algebraic basis, \( d \) the associated sub-elliptic distance, and \( \mu \) the left Haar measure. Suppose that \( \alpha, A \in L^\infty \) and that there exist \( \kappa_1, \kappa_2 > 0 \) satisfying

\[
\Re \langle \alpha v, v \rangle \geq \kappa_1 \|v\|^2, \quad \text{and} \quad \Re \langle A \nabla u, \nabla u \rangle \geq \kappa_2 \|\nabla u\|^2.
\]

for every \( v \in L^2(G) \) and \( u \in W^{1,2}(G)' \). Then,

\( \mathcal{D}(\sqrt{-\alpha \ \text{div} \ A \nabla}) = W^{1,2}(G)' \) and \( \|\sqrt{-\alpha \ \text{div} \ A \nabla u}\| \simeq \|\nabla u\| \) for \( u \in W^{1,2}(G)' \).
General Lie groups

Let $S = (I, \nabla)$ as in the manifold case.

**Theorem (B.-E.-Mc., 2012)**

Let $(G, d, \mu)$ be a connected Lie group, $a$ an algebraic basis, $d$ the associated sub-elliptic distance, and $\mu$ the left Haar measure. Let $a, A \in L^\infty$ such that

$$\text{Re} \langle av, v \rangle \geq \kappa_1 \|v\|^2, \quad \text{and} \quad \text{Re} \langle ASu, Su \rangle \geq \kappa_2 \|u\|_{W^{1,2}'}$$

for every $v \in L^2(G)$ and $u \in W^{1,2}(G)'$. Then, $\mathcal{D}(\sqrt{aS^*AS}) = W^{1,2}(G)'$ with

$$\|\sqrt{aS^*AS}u\| \simeq \|u\|_{W^{1,2}'} = \|u\| + \|\nabla u\|.$$
Operator theory

We adapt the framework due to Axelsson (Rosén), Keith, McIntosh in [AKMc].

Let $\mathcal{H}$ be a Hilbert space and $\Gamma : \mathcal{H} \to \mathcal{H}$ a closed, densely-defined, nilpotent operator.

Suppose that $B_1, B_2 \in \mathcal{L}(\mathcal{H})$ such that there exist $\kappa_1, \kappa_2 > 0$ satisfying

$$\text{Re} \langle B_1 u, u \rangle \geq \kappa_1 \| u \|^2 \quad \text{and} \quad \text{Re} \langle B_2 v, v \rangle \geq \kappa_2 \| v \|^2$$

for $u \in \mathcal{R}(\Gamma^*)$ and $v \in \mathcal{R}(\Gamma)$.

Furthermore, suppose the operators $B_1, B_2$ satisfy $B_1 B_2 \mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$ and $B_2 B_1 \mathcal{R}(\Gamma^*) \subset \mathcal{N}(\Gamma^*)$.

The primary operator we consider is $\Pi_B = \Gamma + B_1 \Gamma^* B_2$. 
If the quadratic estimates

$$\int_0^\infty \| t\Pi_B (1 + t^2 \Pi_B^2)^{-1} u \|^2 \sim \| u \|$$  \hspace{1cm} (Q)$$

hold for every \( u \in \mathcal{R}(\Pi_B) \), then, \( \mathcal{H} \) decomposes into the spectral subspaces of \( \Pi_B \) as \( \mathcal{H} = \mathcal{N}(\Pi_B) \oplus E_+ \oplus E_- \) and

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) = \mathcal{D}(\Gamma) \cap \mathcal{D}(\Gamma^* B_2)$$

$$\| \sqrt{\Pi_B^2} u \| \sim \| \Pi_B u \| \sim \| \Gamma u \| + \| \Gamma^* B_2 u \|.$$  

The Kato problems are then obtained by letting

\( \mathcal{H} = L^2(\mathcal{M}) \oplus (L^2(\mathcal{M}) \oplus L^2(\mathcal{T^* M})) \) and letting

$$\Gamma = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}, \quad \Gamma^* = \begin{pmatrix} 0 & S^* \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$
Geometry and harmonic analysis

Harmonic analytic methods are used to prove quadratic estimates (Q).

The idea is to reduce the quadratic estimate (Q) to a Carleson measure estimate. This is achieved via a local $T(b)$ argument.

Geometry enters the picture precisely in the harmonic analysis. We need to perform harmonic analysis on vector fields, not just functions.

One can show that this is not artificial - the Kato problem on functions immediately provides a solution to the dual problem on vector fields.
Elements of the proofs

Similar in structure to the proof of [AKMc] which is inspired from the proof in [AHLMcT].

- A dyadic decomposition of the space
- A notion of averaging (in an integral sense)
- Poincaré inequality - on both functions and vector fields
- Control of $\nabla^2$ in terms of $\Delta$. 
The case of non-smooth metrics on manifolds

We let $\mathcal{M}$ be a smooth, complete manifold as before but now let $g$ be a $C^0$ metric. Let $\mu_g$ denote the volume measure with respect to $g$.

Let $h \in C^0(T^{(2,0)}\mathcal{M})$. Then, define

$$\|h\|_{op,g} = \sup_{x \in \mathcal{M}} \sup_{|u|_g = |v|_g = 1} |h_x(u,v)|.$$  

If $\tilde{g}$ is another $C^0$ metric satisfying $\|g - \tilde{g}\|_{op,g} \leq \delta < 1$, then $L^2(\mathcal{M}, g) = L^2(\mathcal{M}, \tilde{g})$ and $W^{1,2}(\mathcal{M}, g) = W^{1,2}(\mathcal{M}, \tilde{g})$ with comparable norms.
The operator $\Gamma_g$ does not change under the change of metric. However,

$$\Gamma^*_g = C^{-1} \Gamma^*_\tilde{g} C$$

where $C$ is the bounded, invertible, multiplication operator on $L^2(M) \oplus L^2(M) \oplus L^2(T^*M)$.

Thus,

$$\Pi_{B,g} = \Gamma_g + B_1 \Gamma^*_g B_2 = \Gamma_{\tilde{g}} + B_1 C^{-1} \Gamma^*_\tilde{g} C B_2.$$ 

This allows us to reduce the study of $\Pi_{B,g}$ for a $C^0$ metric $g$ to the study of $\Pi_{\tilde{B},\tilde{g}} = \Gamma_{\tilde{g}} + \tilde{B}_1 \Gamma^*_\tilde{g} \tilde{B}_2$ where $\tilde{B}_1 = B_1 C^{-1}$ and $\tilde{B}_2 = C B_2$, but now with a smooth metric $\tilde{g}$. 

$\Pi_B$ under a change of metric
Connection to geometric flows

Given a $C^0$ metric $g$ on a smooth compact manifold, we are able to always find $C^\infty$ metric $\tilde{g}$.

The metric $\tilde{g}$ has $\text{inj}(\mathcal{M}, \tilde{g}) > \kappa$ and $|\text{Ric}(\tilde{g})|_{\tilde{g}} \leq \eta$ so we obtain a corresponding Kato square root estimate in this setting.

The non-compact situation poses issues.

Smooth the metric via mean curvature flow for, say, a $C^2$ imbedding?

Smooth the metric via Ricci flow in the general case? Regularity of the initial metric?
Application to PDE

In the case we are able to find a suitable $C^\infty$ metric near the $C^0$ one, then we have Lipschitz estimates.

Possible application to hyperbolic PDE?

“Stability” of geometries with Ricci bounds and injectivity radius bounds?
References I


References II


