2016.04.27

p-derivations and Witt vectors, following Bourbaki (Cartier), Joyal, Buium.

p : prime number
R : ring (comm.)

A Frobenius lift is a ring homom. ψ making the following diagram comm.

$$\begin{tikzcd}
R \arrow[r, "\psi"] \arrow[d] & R \arrow[d] \\
R/pR \arrow[r, "F"] & R/pR \\
r \arrow[r, mapsto] & r^p
\end{tikzcd}$$

Ex $R = \mathbb{Z}[x]$, $\psi(x) = x^p + pf(x)$, any f.

Rings with Frobenius lifts naturally form a category, but not a good one! The reason is that to be a Frobenius lift,

$$\forall x \in R, \exists y \in R : \psi(x) = x^p + py,$$

tacitly involves an existensial quatifier. This is responsible for the category of rings with Frob. lifts not to have equalizers. The solution is to make the choice of y part of the structure.

So introduce a map $\delta : R \rightarrow R$ s.t.

$$\varphi(x) = x^p + p\,\delta(x).$$

For φ to be a ring homom., we now express in terms of δ as

$$\varphi(x+y) = \varphi(x) + \varphi(y) \rightsquigarrow$$

$$(x+y)^p + p\,\delta(x+y) = x^p + p\,\delta(x) + y^p + p\,\delta(y) \rightsquigarrow$$

$$(1) \quad \delta(x+y) = \delta(x) + \delta(y) - \sum_{0<i<p} \frac{1}{p}\binom{p}{i} x^{p-i} y^{i}$$

$$\varphi(xy) = \varphi(x)\,\varphi(y) \rightsquigarrow$$

$$(xy)^p + p\,\delta(xy) = (x^p + p\,\delta(x))(y^p + p\,\delta(y)) \rightsquigarrow$$

$$(2) \quad \delta(xy) = \delta(x)\,y^p + x^p\,\delta(y) + p\,\delta(x)\,\delta(y)$$

$$\varphi(1) = 1 \rightsquigarrow$$

$$1^p + p\,\delta(1) = 1 \rightsquigarrow$$

$$(3) \quad \delta(1) = 0$$

$$\varphi(0) = 0 \rightsquigarrow$$

$$(4) \quad \delta(0) = 0.$$

<u>Def</u> A p-derivation on a ring R is a map $\delta : R \to R$ satisfying (1)-(4).

The map

$$\{\delta\text{-str. on } R\} \longrightarrow \{\text{Frob. lift. on } R\}$$

$$\delta(x) \longmapsto \varphi(x) = x^p + p\,\delta(x)$$

is a bijection if R is p-torsion free.

<u>Ex</u> If p is nilpotent in R and if R admits a p-derivation, then R is a zero ring.

<u>Ex</u> The operator δ on K_0 defined by the symmetric function

$$\frac{1}{p}\left(x_1^p + \cdots + x_n^p - (x_1 + \cdots + x_n)^p \right)$$

is a p-derivation. The associated Frobenius lift is the p'th Adams operator. //

A morphism of δ-rings

$$(R, \delta) \xrightarrow{f} (R', \delta')$$

Is a ring homom. $f : R \to R'$ s.t. $f \circ \delta = \delta' \circ f$.

Witt vectors: Several descent situations

$$\begin{tikzcd}
\delta\text{-rings} \arrow[d] & & C\text{-mod} \arrow[d] \\
\text{rings} \arrow[u, bend left, "\Lambda_p \odot -"] \arrow[u, bend right, "W"'] & & \text{ab} \arrow[u, bend left, "C \otimes_{\mathbb{Z}} -"] \arrow[u, bend right, "\mathrm{Hom}_{\mathbb{Z}}(C, -)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Diff rings} \arrow[d] \\
\text{rings} \arrow[u, bend left, "D \odot -"] \arrow[u, bend right, "W^{\mathrm{diff}}"']
\end{tikzcd}$$

$D \odot A$ = "diff. ring gen. by A"

$$= \mathbb{Z}[d^{\circ n}(x) \mid n \geq 0,\ x \in A] \Big/ \left(\begin{matrix}\text{Leibniz rules} \\ \text{for } d^{\circ n}\end{matrix} \right)$$

$$d^{\circ n}(x+y) = d^{\circ n}(x) + d^{\circ n}(y)$$

$$d^{\circ n}(1) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

$$d^{\circ n}(xy) = \sum_{i+j=n} \binom{n}{i} d^{\circ i}(x)\, d^{\circ j}(y).$$

$$A \xrightarrow{\eta} D \odot A \quad \text{ring homom.}$$

$$a \longmapsto d^{\circ 0}(a)$$

$$D \odot A \xrightarrow{d} D \odot A$$

$$d^{\circ n}(a) \longmapsto d^{\circ (n+1)}(a) .$$

$$\mathrm{Hom}_{\text{ring}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\text{diff-ring}}(D \odot A, (B, d_B))$$

$$A \xrightarrow{\varphi} B \longmapsto D \odot A \xrightarrow{\tilde{\varphi}} B$$

$$\tilde{\varphi}(d^{\circ n}(a)) = d_B^{\circ n}(\varphi(a)) .$$

$$W^{\text{diff}}(A) = \left\{ \sum_{n \geq 0} a_n \cdot \text{``}\frac{t^n}{n!}\text{''} \;\middle|\; a_n \in A \right\}$$

$$d = \frac{d}{dt}$$

$$\mathrm{Hom}_{\text{ring}}(R, A) \xrightarrow{\sim} \mathrm{Hom}_{\text{diff-ring}}((R, d_R), W^{\text{diff}}(A))$$

$$R \xrightarrow{\varphi} A \longmapsto R \xrightarrow{\tilde{\varphi}} W^{\text{diff}}(A)$$

$$\tilde{\varphi}(f) = \varphi(\text{Taylor exp. of } f)$$

$$= \sum_{n \geq 0} \varphi(d_R^{\circ n}(f)) \cdot \text{``}\frac{t^n}{n!}\text{''}$$

Alternative point of view: Mult.

$$(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots)$$

$$= (\underbrace{a_0 b_0, a_1 b_0 + a_0 b_1, a_2 b_0 + 2 a_1 b_1 + a_0 b_2}_{\text{Leibniz rule for } d^{\circ n}}, \ldots)$$

The mult. on $A \times A \times \cdots$ is just a syntactic re-expression of the Leibniz rules for $d^{\circ n}$. So let us do the same for p-derivations.

Witt vectors

$$\Lambda_p \odot A = \mathbb{Z}[\delta^{\circ n}(a) \mid n \geq 0, a \in A] / \binom{\text{Leib.}}{\text{rules}}$$

$$\delta^{\circ 0}(x+y) = \delta^{\circ 0}(x) + \delta^{\circ 0}(y)$$

$$\delta^{\circ 0}(xy) = \delta^{\circ 0}(x)\,\delta^{\circ 0}(y)$$

$$\delta^{\circ 1}(x+y) = \delta^{\circ 1}(x) + \delta^{\circ 1}(y) - \sum \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

$$\delta^{\circ 1}(xy) = \delta^{\circ 1}(x)\,y^p + x^p\,\delta^{\circ 1}(y) + p\,\delta^{\circ 1}(x)\,\delta^{\circ 1}(y)$$

$$\delta^{\circ 2}(x+y) = \delta(\delta(x+y))$$

$$= \delta\left(\delta(x) + \delta(y) + \sum \frac{1}{p}\binom{p}{i} x^i y^{p-i}\right)$$

$$= \cdots$$

So makes it clear how, in principle, to write out the Leibniz rules for $\delta^{\circ n}$, but, in practice, one should never do so.

Define right-adjoint as the set

$$W(A) = A \times A \times \cdots$$

with ring operations given by syntactically re-expressing Leibniz rules for $\delta^{\circ n}$:

$$(a_0, a_1, \dots) + (b_0, b_1, \dots)$$

$$= (a_0+b_0, a_1+b_1 - \Sigma \tfrac{1}{p}\tbinom{p}{i} a_0^i b_0^{p-i}, \dots)$$

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots)$$

$$= (a_0 b_0, a_0 b_1^p + a_1 b_0^p + p a_1 b_1, \dots)$$

$$0 = (0, 0, \dots)$$

$$1 = (1, 0, 0, \dots)$$

$$\delta(a_0, a_1, \dots) = (a_1, a_2, \dots)$$

$$\tilde{\varphi}(r) = (\varphi(r), \varphi(\delta(r)), \varphi(\delta^{\circ 2}(r)), \dots)$$

N.B. The coordinates $(a_0, a_1, a_2, \ldots)$ are not the usual Witt coord.

Have constructed adjunctions

$$\begin{tikzcd} \delta\text{-rings} \arrow[d] \\ \text{rings} \arrow[u, bend left=60, "\Lambda_p \odot -"] \arrow[u, bend right=60, "W"'] \end{tikzcd}$$

so the forgetful functor creates both limits and colimits in δ-rings. Moreover,

$\mathrm{fgt} \circ W$ is a comonad

$$W(A) \xrightarrow{\Delta} W(W(A))$$

$$W(A) \xrightarrow{\varepsilon} A \;;$$

$\mathrm{fgt} \circ (\Lambda_p \odot -)$ is a monad

$$\Lambda_p \odot (\Lambda_p \odot A) \longrightarrow \Lambda_p \odot A$$

$$A \xrightarrow{\eta} \Lambda_p \odot A$$

Canonical lifts:

(R, δ_R) : δ-ring

Serre-Tate

$X = \mathrm{Spec}(R)$

$$\begin{tikzcd}
X(A) \arrow[d, hook] \arrow[r, phantom, "="] & \mathrm{Hom}(R, A) \arrow[d, "\sim"] \\
X(W(A)) \arrow[r, phantom, "\supset"] & \mathrm{Hom}_\delta((R, \delta_R), W(A))
\end{tikzcd}$$

So any object over A lifts can. to one over $W(A)$, if the moduli space has a δ-structure.

Ghost components:

δ^{on} — computation is hard

ψ^{on} — computation is easy

So define a ψ-ring to be a pair (R, ψ) of a ring R and a ring homom. $\psi: R \to R$. In other words a ψ-ring "is" a ring R with an action of the additive monoid $\mathbb{N}$. Forgetful functor

$$\delta\text{-rings} \longrightarrow \varphi\text{-rings}$$

$$(R, \delta) \longmapsto (R, \varphi)$$

$$\varphi(x) = x^p + p\,\delta(x)$$

and adjunctions

$$\begin{tikzcd}
\delta\text{-rings} \arrow[dr] \arrow[dd] & \\
& \varphi\text{-rings} \arrow[ul, bend right, "\mathrm{Hom}_\varphi(\Lambda_p, -) = W_\varphi(-)"'] \arrow[ul, bend left, "\Lambda_p \odot_\varphi -"] \arrow[dl] \\
\text{rings} \arrow[uu, bend left=50, "\Lambda_p \odot -"] \arrow[uu, bend right, "W"'] \arrow[ur, bend left] \arrow[ur, bend right, "A \mapsto A^{\mathbb{N}}"'] &
\end{tikzcd}$$

The ghost map is the counit

$$\begin{tikzcd}
W(A) \arrow[r, "w"] \arrow[d, equal] & A^{\mathbb{N}} \\
(\mathrm{Fgt} \circ W_\varphi)(A^{\mathbb{N}}) \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

of the adjunction $(\mathrm{Fgt}, W_\varphi)$ at the φ-ring $A^{\mathbb{N}}$. It is a φ-ring homom.

Prop (1) If A is p-tors. free, then w is injective.

(2) If A is a $\mathbb{Z}[1/p]$-alg., then w is bijective.

Pf (1) To be a δ-str. is a property of a φ-str.

(2) A δ-str. determines and is determined by a φ-str. //

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Last time: $W(A) = A \times A \times \cdots$

$$\begin{tikzcd} \delta\text{-rings} \arrow[d, "!"'] \\ \text{rings} \arrow[u, bend right, "W"'] \end{tikzcd}$$

" δ - components"

(Buium, Joyal)

Not Witt components

First consider diff. rings

$$\begin{tikzcd} \text{diff. rings} \arrow[d] \\ \text{rings} \arrow[u, bend left, "D\odot-"] \arrow[u, bend right, "W^{\text{diff}}"'] \end{tikzcd} \qquad \begin{tikzcd} C\text{-mod} \arrow[d] \\ \text{ab} \arrow[u, bend left, "C\otimes_{\mathbb{Z}}-"] \arrow[u, bend right, "\mathrm{Hom}_{\mathbb{Z}}(C,-)"'] \end{tikzcd}$$

D = free diff. ring on one gen.

C = free C-mod. on one gen.

Def $D = D \odot \mathbb{Z}[e]$
$= \mathbb{Z}[e, d, d^{\circ 2}, \ldots]$.

D = "ring of diff. operators"

If $\xi \in D$ and (R, d_R) is a diff. ring, then we get a map $\xi_R : R \to R$ by substituting d_R for d. Can prove

$D = \{$nat. unary oper. on diff. rings$\}$;

will not actually use this.

Note that D represents W^{diff},

$$W^{\mathrm{diff}}(A) = \mathrm{Hom}_{\mathrm{rings}}(\mathbb{Z}[e], W^{\mathrm{diff}}(A))$$

$$= \mathrm{Hom}_{\mathrm{diff}}(D, W^{\mathrm{diff}}(A))$$

$$= \mathrm{Hom}_{\mathrm{rings}}(D, A)$$

$$\sum_{n\geq 0} a_n \text{“}\frac{t^n}{n!}\text{”} \longmapsto (d^{\circ n} \longmapsto a_n).$$

δ-rings: Same procedure

$$\Lambda_p = \text{free } \delta\text{-ring on one gen. } e$$

$$= \Lambda_p \odot \mathbb{Z}[e]$$

$$= \mathbb{Z}[e, \delta, \delta^{\circ 2}, \ldots]$$

$$= \text{“arithmetic diff. oper.”}$$

$$= \text{“nat. unary oper. on } \delta\text{-rings.”}$$

$$W(A) = \mathrm{Hom}_{\mathrm{rings}}(\Lambda_p, A)$$

$$(a_0, a_1, \ldots) \longmapsto (\delta^{\circ n} \longmapsto a_n).$$

These are the δ-components. But any free gen. set of Λ_p will give "new" components of Witt vectors. For example, we could replace p by $-p$ everywhere to get a new system of components.

ψ-rings and ghost components

Ψ_p = free ψ-ring on one gen. e

$= \mathbb{Z}[e, \psi, \psi^{\circ 2}, \dots]$;

ghost map repr. by ring homom.

$$\begin{tikzcd}
\Psi_p \arrow[r] \arrow[d, equal] & \Lambda_p \arrow[d, equal] \\
\mathbb{Z}[e, \psi, \psi^{\circ 2}, \dots] \arrow[r] & \mathbb{Z}[e, \delta, \delta^{\circ 2}, \dots]
\end{tikzcd}$$

$$\begin{aligned}
e &\longmapsto e \\
\psi &\longmapsto e^p + p\delta \\
\psi^{\circ 2} &\longmapsto (e^p + p\delta)\circ(e^p + p\delta) \\
&\vdots
\end{aligned}$$

Illustrate expansion

$$(e^p + p\delta)\circ(e^p + p\delta)$$

$$= e^p \circ (e^p + p\delta) + p\delta \circ (e^p + p\delta)$$

$$= (e^p + p\delta)^p + p\Big(\delta(e^p) + \delta(p\delta) - \sum_{0<i<p} \tfrac{1}{p}\tbinom{p}{i}(e^p)^i (p\delta)^{p-i},$$

$$= (e^p + p\delta)^p - \sum_{0<i<p} \binom{p}{i} (e^p)^i (p\delta)^{p-i} + p\delta(e^p) + p\delta(p\delta)$$

$$= e^{p^2} + \cancel{(p\delta)^p} + \sum_{0<i\leq p} \binom{p}{i} e^{pi} (p\delta)^{p-i}$$

$$+ p\delta^p + p^2\delta^{\circ 2} - \cancel{(p\delta)^p} = (e^p + p\delta)^p + p\delta^p + p^2\delta^{\circ 2}$$

So in terms of δ-components, the ghost coordinates start

$$w_0 = a_0$$

$$w_1 = a_0^p + pa_1$$

$$w_2 = (a_0^p + pa_1)^p + pa_1^p + p^2 a_2$$

$w_3 = ??$ – should not be hard...

The ring homom. above induces an isomorphism

$$\Psi_p[\tfrac{1}{p}] \xrightarrow{\sim} \Lambda_p[\tfrac{1}{p}],$$

and since all these rings are p-torsion free, we have

$$\Psi_p \hookrightarrow \Lambda_p \hookrightarrow \Psi_p[1/p].$$

Define Witt components:

Def For $n \geq 0$, $\theta_n \in \Lambda_p[1/p]$ recursively by the formula

$$\psi^{\circ n} = \theta_0^{p^n} + p\,\theta_1^{p^{n-1}} + \cdots + p^n \theta_n$$

Thm (Witt) In fact, $\theta_n \in \Lambda_p$, and

$$\Lambda_p = \mathbb{Z}[\theta_0, \theta_1, \theta_2, \ldots]. \quad /$$

The Witt components are

$$W(A) \xrightarrow{\sim} A^{\mathbb{N}}$$

$$a \longmapsto (n \longmapsto a(\theta_n)).$$

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Thm The subrings $\mathbb{Z}[\theta_0, \theta_1, \dots, \theta_n]$ and $\mathbb{Z}[e, \delta, \dots, \delta^{\circ n}]$ of $\Lambda_p[1/p]$ are equal and are free on the indicated families of element.

Proof First, assume

(*) $\delta(\theta_n) = \theta_{n+1} +$ pol. in $\theta_0, \dots, \theta_n$

and show that

$$\delta^{\circ(n+1)} = \theta_{n+1} + \text{pol. in } \theta_0, \dots, \theta_n$$

and $\mathbb{Z}[\theta_0, \dots, \theta_{n+1}] = \mathbb{Z}[e, \delta, \dots, \delta^{\circ(n+1)}]$.
Inductively,

$$\begin{aligned}
\delta(\delta^{\circ n}) &= \delta(\theta_n + f(\theta_0, \dots, \theta_{n-1})) \\
&= \delta(\theta_n + g(e, \delta, \dots, \delta^{\circ(n-1)})) \\
&= \delta(\theta_n) + h(e, \delta, \dots, \delta^{\circ n}) \\
&= \theta_{n+1} + j(\theta_0, \theta_1, \dots, \theta_n),
\end{aligned}$$

and hence, also

$$\begin{aligned}
\mathbb{Z}[\theta_0, \dots, \theta_{n+1}] &= \mathbb{Z}[\theta_0, \dots, \theta_n][\delta^{\circ(n+1)}] \\
&= \mathbb{Z}[\delta^{\circ 0}, \dots, \delta^{\circ n}, \delta^{\circ(n+1)}].
\end{aligned}$$

It remains to prove (*). Recall:

Lemma If A is a comm. ring, and if $x, y \in A$ are congruent mod pA, then for every pos. integer m,

$$x^{p^m} \equiv y^{p^m} \mod p^{m+1}A.$$

Pf Binomial formula.

Now, expand $\psi^{\circ(n+1)}$ in two ways:

① $\psi^{\circ(n+1)} = \sum_{i=0}^{n+1} p^i \theta_i^{p^{n+1-i}}$

② $\psi^{\circ(n+1)} = \psi\left(\sum_{i=0}^{n} p^i \theta_i^{p^{n-i}}\right)$

$$= p^n \psi(\theta_n) + \sum_{i=0}^{n-1} p^i \psi(\theta_i)^{p^{n-i}}$$

$$= p^n \psi(\theta_n) + \sum_{i=0}^{n-1} p^i \left(\theta_i^p + p\,\delta(\theta_i)\right)^{p^{n-i}}$$

which, by induction, is equal to

$$p^n \psi(\theta_n) + \sum_{i=0}^{n-1} p^i \left(\theta_i^p + p\,\theta_{i+1} + p f_i(\theta_0, \dots, \theta_i)\right)^{p^{n-i}}$$

$$\equiv p^n \psi(\theta_n) + \sum_{i=0}^{n-1} p^i \theta_i^{p^{n+1-i}} \mod p^{n+1}\mathbb{Z}[\theta_0, \dots, \theta_n]$$

Comparing ① and ②, we find

$$p^n \psi(\theta_n) \equiv p^n \theta_n^p + p^{n+1}\theta_{n+1} \mod p^{n+1}\mathbb{Z}[\theta_0, \dots, \theta_n].$$

and hence,

$$\begin{array}{l} \psi(\theta_n) \equiv \theta_n^p + p\,\theta_{n+1} \mod p\,\mathbb{Z}[\theta_0,\dots,\theta_n] \\ \quad \parallel \\ \theta_n^p + p\,\delta(\theta_n) \end{array}$$

So we conclude

$$\delta(\theta_n) \equiv \theta_{n+1} \mod \mathbb{Z}[\theta_0,\dots,\theta_n]$$

as desired. //

Classically, Witt vectors are developed using Witt components, forcing one to prove congruences as above (due to Kummer). Using the δ-components instead, this becomes (mostly) unnecessary, and the Witt components are then more properly viewed as a calculational device.

Witt vectors of finite length:

$$\begin{tikzcd} \Lambda_{p,n} \arrow[d, hook] & = & \mathbb{Z}[e,\delta,\dots,\delta^{\circ n}] \\ \Lambda_p & = & \mathbb{Z}[e,\delta,\dots,\delta^{\circ n},\dots] \end{tikzcd}$$

Since the Leibniz rules for $\delta^{\circ m}$ are in terms of $\delta^{\circ 0}, \ldots, \delta^{\circ m}$, this subring is a sub-biring. It represents

$$\begin{tikzcd} W_n(R) \arrow[r, phantom, "="] & \mathrm{Hom}(\Lambda_{p,n}, R) \\ W(R) \arrow[u, two heads] \arrow[r, phantom, "="] & \mathrm{Hom}(\Lambda_p, R) \arrow[u, two heads] \end{tikzcd}.$$

It is not a sub-δ-ring, but δ induces a map (not ring homom.)

$$W_{n+1}(R) \xrightarrow{\delta} W_n(R);$$

and $\psi(x) = x^p + p\delta(x)$ induces a ring homomorphism

$$W_{n+1}(R) \xrightarrow{\psi} W_n(R).$$

Also, the inclusion

$$\mathbb{Z}[e, \psi, \ldots, \psi^{\circ n}] \hookrightarrow \mathbb{Z}[e, \delta, \ldots, \delta^{\circ n}]$$

defines the truncated ghost map

$$W_n(R) \xrightarrow{w} R^{[0,n]};$$

it is a ring homomorphism.

<u>Rmk</u> Beware that $W_0(R) = R$.

Plethories (a general theory of unary operators on rings):

$$\begin{tikzcd} S\text{-rings} \arrow[d, "\mathrm{fgt}"] \\ \text{rings} \arrow[u, bend left=60, "\Lambda_p \odot -"] \arrow[u, bend right=60, "\mathrm{Hom}_{\text{ring}}(\Lambda_p, -)"'] \end{tikzcd}$$

Compare to additive situation: For a ring homom. $k \to C$,

$$\begin{tikzcd} {}_C\mathrm{Mod} \arrow[d, "\mathrm{fgt}"] \\ {}_k\mathrm{Mod} \arrow[u, bend left=60, "C \otimes_k -"] \arrow[u, bend right=60, "{}_k\mathrm{Mod}(C, -)"'] \end{tikzcd}$$

Here C is an algebra; similarly, Λ_p is a plethory. Literature:

Tall-Wraith (biring triples)

Bergmann-Hausknecht

Borger-Wieland (k-plethory)

A plethory is a system of unary operators with Leibniz rules and closed under $+, \times, \circ$.

Recall $\otimes$ - Hom adjunction: For a k-k'-bimod. C, adj.

$$ {}_k\mathrm{Mod}(C \otimes_{k'} M, N) $$
$$ \xrightarrow{\sim} {}_{k'}\mathrm{Mod}(M, {}_k\mathrm{Mod}(C, N)). $$

Would like similarly to have

$$ \mathrm{Alg}_k(Q \odot_{k'} R, S) $$
$$ \xrightarrow{\sim} \mathrm{Alg}_{k'}(R, \mathrm{Alg}_k(Q, S)). $$

But $\mathrm{Alg}_k(Q, -)$ takes values in sets — need to have structure on Q to lift this set-valued functor to a k'-algebra-valued one.

<u>Def</u> A k-k'-biring structure on a k-algebra Q is, equivalently,

(1) a lift

$$\begin{tikzcd}
 & \mathrm{Alg}_{k'} \arrow[d, "\mathrm{fgt}"] \\
\mathrm{Alg}_k \arrow[ur, dashed] \arrow[r, "\mathrm{Alg}_k(Q,-)"'] & \mathrm{Set}
\end{tikzcd}$$

;

(2) a comm. k'-algebra structure on the k-scheme $\operatorname{Spec}(Q)$;

(3) a pair of k-alg. morphisms

$$\Delta^{+} : Q \longrightarrow Q \otimes_k Q$$

$$\Delta^{\times} : Q \longrightarrow Q \otimes_k Q$$

satisfying cocomm., coassoc., codistr., counitality, coinverse, together with a ring homom.

$$k' \xrightarrow{\beta} \operatorname{Alg}_k(Q, k). \qquad //$$

Joyal : Every cocontinuous functor

$$\operatorname{Alg}_{k'} \longrightarrow \operatorname{Alg}_k$$

is, up to unique natural isom., of the form

$$R \longmapsto Q \odot_{k'} R$$

for some k-k'-biring Q.

Construct $Q \odot_{k'} R$ in two steps :

① $Q \tilde{\odot}_{k'} R = k[f \odot x \mid f \in Q,\ x \in R] / \sim$

$$(f+g) \odot x \sim (f \odot x) + (g \odot x)$$

$$(fg) \odot x \sim (f \odot x)(g \odot x)$$

$$c \odot x \sim c \quad \text{for all } c \in k.$$

② For every $x, y \in R$, the relations in ① makes the map

$$Q \otimes_k Q \xrightarrow{\langle -,\, x \otimes y \rangle} Q \tilde{\odot}_{k'} R$$

$$f \otimes g \longmapsto (f \odot x)(g \odot y)$$

a ring homom. Now define

$$Q \odot_{k'} R = Q \tilde{\odot}_{k'} R / \approx$$

$$f \odot (x+y) \approx \langle \Delta^{+}_{Q}(f),\, x \otimes y \rangle$$

$$f \odot (xy) \approx \langle \Delta^{\times}_{Q}(f),\, x \otimes y \rangle$$

$$f \odot c' \approx \beta_Q(c')(f)$$

where $f \in Q$, $x, y \in R$, and $c' \in k'$.

<u>Check</u> $Q \odot_{k'} -$ is a left adjoint of $\mathrm{Alg}_{k}(Q, -)$.

Let us write

$$W_Q(-) = \mathrm{Alg}_k(Q, -).$$

We have

$$W_{Q \odot_{k'} R}(S) = \mathrm{Alg}_k(Q \odot_{k'} R, S)$$
$$\cong \mathrm{Alg}_k(R, W_Q(S))$$
$$\cong W_R(W_Q(S)).$$

Since this functor takes values in k''-algebras, $Q \odot_{k'} R$ is a k-k''-biring. Also, if $k = k' = k''$, then $\odot_k$ is a monoidal product on k-k-birings with unit $k[e]$, $\Delta^+(e) = e \otimes 1 + 1 \otimes e$, $\Delta^\times(e) = (e \otimes 1) \cdot (1 \otimes e)$.

Def A k-plethory P is a monoid w.r.t. to the monoidal structure $(\odot_k, \text{---})$ on k-k-biring.

2015.05.06

Notation: $W_Q(R) = \mathrm{Alg}_k(Q, R)$, esp. when Q is a k-k'-biring or k-plethory.

Biring examples:

(1) $Q = \Lambda_p = \mathbb{Z}[e, \delta, \delta^{\circ 2}, \dots]$

$W_{\Lambda_p}(R) = W(R)$

$\Lambda_p = \mathbb{Z}$-$\mathbb{Z}$-biring

$\Lambda_p \xrightarrow{\Delta^+, \Delta^\times} \Lambda_p \otimes \Lambda_p$ encode Leibniz rules

For example, since $\Delta^\times(\psi) = \psi \otimes \psi$ with $\psi = e^p + p\delta$, we find

$$\Delta^\times(\delta) = \delta \otimes e^p + e^p \otimes \delta + p\,\delta \otimes \delta.$$

(2) $Q = \Lambda_{p,n} = \mathbb{Z}[e, \delta, \dots, \delta^{\circ n}] < \Lambda_p$ is a sub-$\mathbb{Z}$-$\mathbb{Z}$-biring and $W_{\Lambda_{p,n}}(R) = W_n(R)$.

(3) $Q = D = \mathbb{Z}[e, d, d^{\circ 2}, \dots]$ is a $\mathbb{Z}$-$\mathbb{Z}$-biring with

$W_D(R) = \Gamma_R\{t\}^\wedge$

$$\Delta^+(d^{\circ n}) = d^{\circ n} \otimes 1 + 1 \otimes d^{\circ n}$$

$$\Delta^\times(d^{\circ n}) = \sum_{i+j=n} \binom{n}{i} d^{\circ i} \otimes d^{\circ j}$$

The ring homom. $D \longrightarrow D$ that to ξ assigns $\xi \circ d$ induces

$$W_D(R) \xrightarrow{d/dt} W_R(R) .$$

(4) $Q = D_n = \mathbb{Z}[e, d, \dots, d^{\circ n}] \subset D$,

$$W_{D_n}(R) = \Gamma_R\{t\} / J^{[n+1]} .$$ //

Reminder: For k-k'-biring Q and k'-k''-biring R, define

$$Q \,\tilde{\odot}_{k'}\, R = k'[f \odot x \mid f \in Q, x \in R] / \sim ,$$

where $\sim$ expresses that $f \odot x$ is a ring homom. in f, and

$$Q \odot_{k'} R = Q \,\tilde{\odot}_{k'}\, R / \approx$$

where $\approx$ is gen. by

$$f \odot (x+y) \approx \langle \Delta^+(f), x \otimes y \rangle ,$$

$$f \odot (xy) \approx \langle \Delta^\times(f), x \otimes y \rangle .$$

Writing $\Delta^+(f) = \sum f_i^{(1)} \otimes f_i^{(2)}$, the first of these expresses

$$f \odot (x+y) = \sum f_i^{(1)}(x) f_i^{(2)}(y) ,$$

which is meaningful (independent of how we write $\Delta^+(f)$ as a sum of tensors) in $Q \bar{\odot}_k R$, but not in $k[f \odot x \mid f \in Q, x \in R]$.

A k-plethory is a monoid P in k-k-biring w.r.t. the monoidal structure $(\odot_k, k[e], \ldots)$. It defines a monad $P \odot_k -$ and a comonad $\mathrm{Alg}_k(P, -)$ on Alg_k, and an algebra for $P \odot_k -$, or equivalently, a coalgebra for $\mathrm{Alg}_k(P, -)$ is called a P-*ring*. We have the adjunctions

$$\begin{tikzcd}
\text{P-rings} \arrow[d, "\mathrm{fgt}"] \\
\text{rings} \arrow[u, bend left=60, "P \odot_k -"] \arrow[u, bend right=60, "W_P"']
\end{tikzcd}$$

and a Tanaka-Krein converse: Given adjunctions

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "U"] \\
\text{k-alg} \arrow[u, bend left=60, "F"] \arrow[u, bend right=60, "F'"']
\end{tikzcd}$$

such that $\mathcal{C}$ has all small limits and colimits and such that U reflects isomorphisms, then, by Beck's thm., U is monadic and comonadic. So let

$$P = (U \circ F)(k[e]).$$

Since

$$(U \circ F')(R) = \mathrm{Alg}_k(k[e], (U \circ F')(R))$$

$$= \mathrm{Alg}_k((U \circ F)(k[e]), R)$$

$$= W_P(R),$$

we see that P represents a comonad; that P is a k-plethory; and that $\mathcal{C}$ = P-rings.

2015.05.11

Recall

$$\begin{tikzcd}
\delta\text{-rings} \arrow[d, "\mathrm{fgt}"] \\
\text{rings} \arrow[u, bend left=60, "\Lambda_p \odot -"] \arrow[u, bend right=60, "W"']
\end{tikzcd}$$

$\rightsquigarrow$ Λ_p is a plethory

Alternative (and more pedestrian) approach:

$$\begin{array}{ccc}
\Lambda_p \odot \Lambda_p & \longrightarrow & \Lambda_p \\
\| & & \\
\Lambda_p \odot \mathbb{Z}[e, \delta, \delta^{\circ 2}, \dots] & & \\
\| & & \\
\bigotimes_{n \geq 0} \Lambda_p \odot \mathbb{Z}[\delta^{\circ n}] & & \\
f \odot \delta^{\circ n} & \longmapsto & f \circ \delta^{\circ n}
\end{array}$$

shows that

$$\begin{array}{ccc}
\Lambda_p \odot \Lambda_p & \longrightarrow & \Lambda_p \\
f \odot g & \longmapsto & f(g)
\end{array}$$

is well-defined. Similarly, the birings

$$D = \mathbb{Z}[e, d, d^{\circ 2}, \dots]$$

and

$$\Psi_p = \mathbb{Z}[e, \psi, \psi^{\circ 2}, \dots]$$

are plethories. Also, the biring homomorphism

$$\begin{tikzcd} \Psi_p \arrow[r, "i"] & \Lambda_p \\ \psi \arrow[r, mapsto] & e^p + p\delta \end{tikzcd}$$

is a morphism of plethories, and

$$\Psi_p[1/p] \xrightarrow{i} \Lambda_p[1/p]$$

is an isomorphism.

More generally, if G is a monoid, not necessarily commutative, then

$$\begin{tikzcd} G\text{-rings} \arrow[d, "fgt"] \\ \text{rings} \arrow[u, bend left=60, "(-)^{\otimes G}"] \arrow[u, bend right=60, "(-)^{G}"'] \end{tikzcd}$$

gives the plethory

$$P = \mathbb{Z}[e]^{\otimes G} = \mathbb{Z}[\psi_g \mid g \in G].$$

consisting of "ring-theoretic" words in G.

Enveloping principle: Any object that "knows how" to act on a k-algebra (k-module, ...) should act via a k-plethory, typically formed by taking the closure under composition and pointwise $+$ and $\cdot$ (resp. operations in the language). E.g.

$$\mathbb{Z}[\psi_g \mid g \in G] \odot \mathbb{Z}[\psi_h \mid h \in G]$$

$$\longrightarrow \mathbb{Z}[\psi_g \mid g \in G]$$

$$\psi_g \odot \psi_h \longmapsto \psi_{gh}$$

Ex (1) $\mathfrak{g}$: Lie alg. $/k$ acting on k-algebras R in such a way that

$$\xi(xy) = \xi(x)y + x\xi(y)$$

$$\xi(c) = 0 \qquad (c \in k).$$

$$\begin{tikzcd}
\mathfrak{g}\text{-rings}/k \arrow[d, "\text{fgt}"] \\
\text{rings}/k \arrow[u, bend left=60] \arrow[u, bend right=60]
\end{tikzcd} \qquad P = ?$$

(2) H : cocomm. bialg. / k

$$H \xrightarrow{\Delta} H\otimes H \,, \qquad H \xrightarrow{\varepsilon} k .$$

From

$$\begin{tikzcd}
P \sim & H\text{-rings}/k \arrow[r, shift right, "\mathrm{fgt}"'] \arrow[d, "\mathrm{fgt}"] & H\text{-mod} \arrow[l, shift right, "\mathrm{Sym}"'] \arrow[d, "\mathrm{fgt}"] \\
& \text{rings}/k \arrow[u, bend left] \arrow[u, bend right] \arrow[r, shift right, "\mathrm{fgt}"'] & k\text{-mod} \arrow[l, shift right, "\mathrm{Sym}"'] \arrow[u, bend left] \arrow[u, bend right]
\end{tikzcd}$$

conclude that

$$P = \mathrm{Sym}_k(H).$$

Moreover, Δ^+ is easy to work out, since H acts linearly on an H-module, while $\Delta^\times$ uses the comultiplication Δ. //

Def A k-plethory P is *linear* if it is isomorphic, as a k-plethory, to $\mathrm{Sym}_k(H)$, for some cocomm. k-bialgebra H; it is *weakly linear* if it is generated as a k-algebra by additive operators ($\Delta^+(f) = f\otimes 1 + 1\otimes f$)., Rmk Λ_p is not weakly linear, since

$$S \not\subset \text{Additive operators}$$
$$\parallel$$
$$\bigoplus_{n\geq 0} \mathbb{Z}\cdot\psi^{\circ n}$$

//

Compare:

left k-module	k-algebra
k-k'-bimod.	k-k'-biring
${}_k\mathrm{Mod}({}_kM_{k'}, {}_kN_{k''})$ $\sim$ k'-k''-bimod	$\mathrm{Alg}_k({}_kQ_{k'}, {}_kR_{k''})$ $\sim$ k'-k''-biring
${}_kM_{k'} \otimes_{k'} {}_{k'}N_{k''}$ $\sim$ k-k''-bimod.	${}_kQ_{k'} \odot_{k'} {}_{k'}R_{k''}$ $\sim$ k-k''-biring
representable comonads	k-plethories
$\underset{\text{ring map}}{k \xrightarrow[\sim]{} C} \sim \text{ring}$	P
C-modules	P-rings

Some differences:

k-module $=$ k-$\mathbb{Z}$-bimodule	k-algebra $\neq$ k-$\mathbb{Z}$-biring
all operators in C are add.	operators in P have own Leibniz rules
fin. coproducts = fin. products.	coprod. $= \otimes$ product $= \prod$
base-change: if A, B are k-alg., then so is $A \otimes B$	no base-change: if P_1, P_2 are plethories, then the biring $P_1 \odot P_2$ is typically not a plethory.

2016.05.13

Recall that while $\mathrm{Sym}(H)$, $\mathrm{Sym}(U(\mathfrak{g}))$, $\mathrm{Sym}(k[G])$ are linear, Λ_p is not. However, $\Lambda_p[1/p]$ is linear. The following theorem was proved only recently. May prove this later.

Thm (Magnus Carlson)

(1) If k is a field of characteristic zero, then every k-plethories are linear.

(2) If k is a finite field, then there exists non-weakly linear k-plethories. /

Inverting Frobenius ($\sim$ TF):
Recall that an $\mathbb{F}_p$-algebra R is perfect if $Fr_p : R \to R$, $Fr_p(x) = x^p$, is an isomorphism.

$$\begin{tikzcd}[column sep=large, row sep=huge]
& \text{Perf } \mathbb{F}_p\text{-alg} \arrow[d] & \\
\begin{array}{c}\text{coperfection} \\ R \mapsto \operatorname{colim}_{n, Fr} R\end{array} & \mathbb{F}_p\text{-alg} \arrow[u, bend left=40] \arrow[u, bend right=40] & \begin{array}{c}\text{perfection} \\ R \mapsto R_{pf} = \lim_{n, Fr} R\end{array}
\end{tikzcd}$$

Ex If R is reduced, then

$$R_{pf} = \bigcap_{n \geq 0} R^{p^n}.$$

For instance, $\mathbb{F}_p[x]_{pf} = \mathbb{F}_p$. So the canonical projection $R_{pf} \to R$ is not always injective. //

Ex $\mathcal{O}_{\bar{\mathbb{Q}}_p}^{\flat} := (\mathcal{O}_{\bar{\mathbb{Q}}_p} / p\, \mathcal{O}_{\bar{\mathbb{Q}}_p})_{pf}$ Fontaine,

$$W(\mathcal{O}_{\bar{\mathbb{Q}}_p}^{\flat}) = A_{\inf} . \; //$$

By general theory developed, if

$$P = \mathbb{F}_p[e]^{pf} = \bigcup_{n \geq 0} \mathbb{F}_p[e^{p^{-n}}]$$

$$\Delta^+(e^{p^{-n}}) = e^{p^{-n}} \otimes 1 + 1 \otimes e^{p^{-n}}$$

$$\Delta^\times(e^{p^{-n}}) = (e^{p^{-n}} \otimes 1) . (1 \otimes e^{p^{-n}})$$

$$e^{p^{-m}} \circ e^{p^{-n}} = e^{p^{-(m+n)}}$$

then

$$P\text{-rings} = \text{Perfect } \mathbb{F}_p\text{-algebras}.$$

The fact that being perfect is property is reflected by

$$P \odot P \xrightarrow[\sim]{\circ} P .$$

Def A δ-ring R is perfect if the ring homom. $\varphi : R \to R$ is an isomorphism.

Ex $\mathbb{Z}$ and $\mathbb{Z}_p$ are perfect, but $\mathbb{Z}[x]$ and Λ_p are not. //

Since $\varphi = e^p + p\delta$ is a map of δ-rings, we can define adjoints

$$\begin{tikzcd}
 & \text{Perf } \delta\text{-rings} \arrow[d] & \\
\begin{array}{c}\text{coperfection} \\ R \mapsto R^{pf} = \operatorname{colim}_{n,\varphi} R\end{array} & \delta\text{-rings} \arrow[u, bend left] \arrow[u, bend right] & \begin{array}{c}\text{perfection} \\ R \mapsto R_{pf} = \lim_{n,\varphi} R\end{array}
\end{tikzcd}$$

So the plethory giving rise to perfect δ-rings is

$$P = \Lambda_p^{pf} = \operatorname{colim}_{n,\varphi} \Lambda_p$$

Write $\Lambda_p\langle \varphi^{\pm 1}\rangle$ for this plethory. The corresponding Witt vectors are

$$W_{\Lambda_p\langle\varphi^{\pm 1}\rangle}(R) = \operatorname{Hom}(\Lambda_p\langle\varphi^{\pm 1}\rangle, R)$$

$$= \operatorname{Hom}(\operatorname{colim}_{n,\varphi} \Lambda_p, R)$$

$$= \lim_{n,\varphi} \operatorname{Hom}(\Lambda_p, R)$$

$$= \lim_{n,\psi} W(R) = W(R)_{pf} .$$

Ex $A_{inf} = W_{\Lambda_p\langle\psi^{\circ-1}\rangle}(\mathcal{O}_{\overline{\mathbb{Q}}_p}/p\mathcal{O}_{\overline{\mathbb{Q}}_p})$.

Recall that ψ induces

$$W_n(R) \xrightarrow{\psi} W_{n-1}(R).$$

So may also consider

$$W'(R) = \lim_{n,\psi} W_n(R).$$

Is this the Witt vectors corresp. to a plethory P? The functor W' is represented by $\operatorname{colim}_{n,\psi} \Lambda_{p,n}$.

$$\begin{tikzcd}
\mathbb{Z}[e] \arrow[r, "\psi"] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}[e,\delta] \arrow[r, "\psi"] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}[e,\delta,\delta^{\circ 2}] \arrow[r] \arrow[d, phantom, "\shortparallel"] & \cdots \\
\Lambda_{p,0} & \Lambda_{p,1} & \Lambda_{p,2} & \\
\mathbb{Z}[e] \arrow[u, "id"'] \arrow[r, hook] & \psi^{\circ-1}\circ\mathbb{Z}[e,\delta] \arrow[u, "\psi", "\sim"'] \arrow[r, hook] & \psi^{\circ-2}\circ\mathbb{Z}[e,\delta,\delta^{\circ 2}] \arrow[u, "\psi^{\circ 2}", "\sim"'] \arrow[r] & \cdots
\end{tikzcd}$$

Composition? We imagine

$$(\psi^{\circ-m}\circ f)\circ(\psi^{\circ-n}\circ g) = \psi^{\circ-(m+n)}\circ f\circ g$$

and rewrite this

$$\begin{tikzcd}
\Lambda_{p,m} \odot \Lambda_{p,n} \arrow[r, "\circ"] \arrow[d, "\psi \odot \psi"] & \Lambda_{p,m+n} \arrow[d, "\psi^{\circ 2}"] \\
\Lambda_{p,m+1} \odot \Lambda_{p,n+1} \arrow[r, "\circ"] & \Lambda_{p,m+n+2}
\end{tikzcd}$$

which gives

$$\begin{aligned}
&(\operatorname{colim}_{m,\psi} \Lambda_{p,m}) \odot (\operatorname{colim}_{n,\psi} \Lambda_{p,n}) \\
&\xleftarrow{\sim} \operatorname{colim}_{m,n,\psi\odot\psi} \Lambda_{p,m} \odot \Lambda_{p,n} \\
&\xrightarrow{\circ} \operatorname{colim}_{2k,\psi^{\circ 2}} \Lambda_{p,2k} \\
&\xrightarrow{\sim} \operatorname{colim}_{k,\psi} \Lambda_{p,k}
\end{aligned}$$

Write this

$$\Lambda_p' \odot \Lambda_p' \xrightarrow{\circ} \Lambda_p'.$$

So what is a Λ_p'-ring explicitly? Define $a, b \in \Lambda_p'$ by

$$\begin{aligned}
\Lambda_{p,1} &\xrightarrow{in_1} \Lambda_p' \\
e, -\delta &\longmapsto a, b
\end{aligned}$$

Sign gives nicer formulas but is not necessary

Formally,

$$a = \psi^{\circ -1}, \quad b = -\psi^{\circ -1} \circ \delta.$$

Leibniz rules for a and b:

(1) a "is" a ring homom.:

$$\Delta^{+}(a) = a\otimes 1 + 1\otimes a$$

$$\Delta^{\times}(a) = (a\otimes 1).(1\otimes a)$$

(2) b is $-\delta$, but must express Leibniz rules using a and b:

$$\delta(x+y) = \delta(x)+\delta(y) - \frac{1}{p}\sum_{0<i<p}\binom{p}{i} x^{i}y^{p-i}$$

so

$$b(x+y) = b(x)+b(y) + \frac{1}{p}\sum_{0<i<p} a(x)^{i}a(y)^{p-i}$$

Similarly,

$$\delta(xy) = \delta(x)y^{p} + x^{p}\delta(y) + p\,\delta(x)\,\delta(y)$$

and applying $\psi^{\circ-1}$, we get

$$b(xy) = b(x)a(y)^{p} + a(x)^{p}b(y) + p\,b(x)\,b(y).$$

Also $b(1) = 0$.

We understand the relation

$$a^p = e + pb$$

as a is a lift of 'the inverse Frobenius' and b is a "witness" to this lifting.

So any $\hat{\Lambda}_p$-ring has two operators a and b, where a is a ring homomorphism; b satisfies

$$b(x+y) = b(x) + b(y) + \frac{1}{p} \sum_{0<i<p} \binom{p}{i} a(x)^i a(x)^{p-i}$$

$$b(xy) = b(x) a(y)^p + a(x)^p b(y) + p\, b(x) b(y)$$

$$b(1) = 0 \,;$$

and where $a^p = e + pb$, i.e.

$$a(x)^p = x^p + p\, b(x) \quad , \forall x.$$

Exercise: Show that the converse is true.

Summary :

$$\begin{tikzcd} \Lambda_p\langle\psi^{\circ-1}\rangle & \Lambda_p \arrow[l] \\ \Lambda'_p \arrow[u] \arrow[ur, phantom, "\text{push out}"] & \mathbb{Z}[e] \arrow[l] \arrow[u] \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} W(R)_{pf} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull back}"] & W(R) \arrow[d] \\ W'(R) \arrow[r] & R \end{tikzcd}$$

Ghost components :

$$\begin{tikzcd} R^{\mathbb{Z}} \arrow[r] \arrow[d] & R^{\mathbb{N}} \arrow[d] \\ R^{-\mathbb{N}} \arrow[r] & R^{\{0\}} \end{tikzcd}$$

Compare

$$\begin{tikzcd} \mathbb{A}^1 \setminus \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1 \arrow[d, hook] \\ \mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$

Would like analog of $\mathbb{P}^1$ in our situation.

Neckless components:
If R is p-torsion free, then

$$W(R) \xhookrightarrow{w} R^{\mathbb{N}}.$$

What is the image? Answer:
It is the largest sub-φ-ring on which φ is a Frob. lift.
This is not explicit!

Thm If R is p-torsion free and has a δ-structure, then

$$\langle x_0, x_1, \ldots \rangle \in R^{\mathbb{N}}$$

is in the image of the ghost map if and only if

$$x_{n+1} \equiv \varphi(x_n) \mod p^{n+1}R$$

for all $n \in \mathbb{N}$. //

Ex $$W(\mathbb{Z}) = \left\{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}^{\mathbb{N}} \;\middle|\; x_{n+1} \equiv x_n \mod p^{n+1}\mathbb{Z} \right\}$$

so consists of certain p-adic Cauchy sequences.

2016.05.18

Correction: The square

$$\begin{tikzcd}
\Lambda_p \arrow[r] & \Lambda_p\langle \psi^{\circ -1}\rangle \\
\mathbb{Z}[e] \arrow[u] \arrow[r] & \Lambda_p' \arrow[u]
\end{tikzcd}$$

is not cocartesian, and

$$\begin{tikzcd}
W(R) \arrow[d] & \lim_{n,\psi} W(R) \arrow[l] \arrow[d] \\
R & \lim_{n,\psi} W_n(R) \arrow[l]
\end{tikzcd}$$

is not cartesian. For example,

$$\begin{tikzcd}
W(\mathbb{F}_p) \arrow[d] & \lim_{n,\psi} W(\mathbb{F}_p) \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
\mathbb{F}_p & \lim_{n,\psi} W_n(\mathbb{F}_p) \arrow[l]
\end{tikzcd}$$

is obviously not cartesian. //

Necklace components:

<u>Thm</u> If R is a p-tors. free δ-ring, then

$$W(R) \xrightarrow{w} R^{\mathbb{N}}$$

has image $\{\langle x_0, x_1, \ldots\rangle \mid x_n \equiv \psi(x_{n-1}) \bmod p^n R\}$.

Reformulate statement:

$$\begin{tikzcd}[column sep=small] W_n(R) = \mathrm{Hom}_{\text{ring}}(\Lambda_{p,n}, R) \ni & a \arrow[d, leftrightarrow] \\ = \mathrm{Hom}_{\Lambda_p\text{-ring}}(\Lambda_p \odot \Lambda_{p,n}, R) \ni & a' \end{tikzcd}$$

The ghost coordinates are

$$x_n = a(\psi^{\circ n}) = a'(e \odot \psi^{\circ n}),$$

so

$$\psi(x_{n-1}) = \psi(a(\psi^{\circ(n-1)}))$$

$$= a'(\psi \circ \psi^{\circ(n-1)}).$$

Hence,

$$x_n = \psi(x_{n-1}) + p^n(?)$$

or

$$a'(e \odot \psi^{\circ n} - \psi \circ \psi^{\circ(n-1)}) = p^n(?).$$

Natural to consider $c_0 = e \odot e$ and

$$c_n = \frac{1}{p^n}(e \odot \psi^{\circ n} - \psi \circ \psi^{\circ(n-1)})$$

$$\in \Lambda_p \odot \Lambda_{p,n}[1/p]. \qquad (n \geq 1)$$

Then (?) will be $a'(c_n)$. So theorem will follow from:

Thm The elements

$$c_0, c_1, \dots, c_n \in \Lambda_p \odot \Lambda_{p,n}[1/p]$$

lie in $\Lambda_p \odot \Lambda_{p,n}$ and generate it freely as a Λ_p-ring.

Pf Calculate:

$$e \odot \psi^{\circ n} = e \odot \sum_{0 \le i \le n} p^i \theta_i^{p^{n-i}}$$

$$\psi \odot \psi^{\circ(n-1)} = \psi \odot \sum_{0 \le i < n} p^i \theta_i^{p^{n-1-i}}$$

$$= \sum_{0 \le i < n} p^i (\psi \odot \theta_i)^{p^{n-1-i}}$$

$$= \sum_{0 \le i < n} p^i (e^p \odot \theta_i + p\delta \odot \theta_i)^{p^{n-1-i}}$$

$$= p^{n-1}(e^p \odot \theta_{n-1} + p\delta \odot \theta_{n-1})$$

$$+ \sum_{0 \le i < n-1} p^i (e^p \odot \theta_i)^{p^{n-1-i}} + p^n f$$

$$(f \in \Lambda_{p,1} \odot \mathbb{Z}[\theta_0, \dots, \theta_{n-2}])$$

$$= p^n \delta \odot \theta_{n-1} + \sum_{0 \le i < n} p^i (e \odot \theta_i)^{p^{n-i}} + p^n f$$

so we find

$$p^{-n}(e \odot \psi^{\circ n} - \psi \odot \psi^{\circ(n-1)})$$

$$= e \odot \theta_n - \delta \odot \theta_{n-1} - f,$$

and hence,

$$c_n = e \odot \theta_n - \delta \odot \theta_{n-1}$$

modulo $\Lambda_{p,1} \odot \mathbb{Z}[\theta_0, \dots, \theta_{n-2}]$. Now, by induction on $n \geq 0$, the case $n = 0$ being trivial, we find

$$\Lambda_p \odot \Lambda_{p,n} = \Lambda_p \odot \mathbb{Z}[\theta_0, \dots, \theta_n]$$

$$= \Lambda_p \odot (\Lambda_{p,n-1} \otimes \mathbb{Z}[\theta_n])$$

$$= (\Lambda_p \odot \Lambda_{p,n-1}) \otimes (\Lambda_p \odot \mathbb{Z}[\theta_n])$$

$$= (\Lambda_p \odot \mathbb{Z}[c_0, \dots, c_{n-1}]) \otimes (\Lambda_p \odot \mathbb{Z}[\theta_n])$$

which by the formula above becomes

$$= (\Lambda_p \odot \mathbb{Z}[c_0, \dots, c_{n-1}]) \otimes (\Lambda_p \odot \mathbb{Z}[c_n])$$

$$= \Lambda_p \odot \mathbb{Z}[c_0, \dots, c_n]).$$

This proves the induction step. //

Rmk (1) The composition map

$$\Lambda_p \odot \Lambda_p \xrightarrow{\circ} \Lambda_p$$

maps c_0 to e and c_n with $n \geq 1$ to zero. Accordingly,

$$\delta(\theta_n) = \theta_{n+1} + \bar{f}$$

with $\bar{f} \in \Lambda_{p,1} \cdot \mathbb{Z}[\theta_0, \dots, \theta_n]$. Note that we use the θ_n to prove this; there should be some simplification, not making use of the θ_n.

The c_i give a new bijection

$$W_n(R) \xrightarrow{\sim} R^{[0,n]},$$

$$a \longmapsto [a'(c_0), \dots, a'(c_n)]$$

call the $a'(c_i)$ the necklace components of a. To understand + and × in necklace components, recall that

$$e \odot \psi^{\circ i} = \psi \odot \psi^{\circ(i-1)} + p^i c_i,$$

and hence,

$$a'(e \odot \psi^{\circ i}) = \psi(a'(e \odot \psi^{\circ(i-1)})) + p^i a'(c_i).$$

So writing $\langle x_0, \ldots, x_n \rangle$ for the ghost comp. of a and $[b_0, \ldots, b_n]$ for the necklace comp., we have

$$x_i = \psi(x_{i-1}) + p^i b_i,$$

from which we find

$$[b_0, b_1, \ldots] + [b_0', b_1', \ldots]$$

$$= [b_0 + b_0', b_1 + b_1', \ldots]$$

$$[b_0, b_1, \ldots] \times [b_0', b_1', \ldots]$$

$$= [b_0 b_0', \psi(b_0) b_1' + b_1 \psi(b_0') + p b_1 b_1', \ldots]$$

$$[b_0, b_1, \ldots] \xmapsto{\psi} [\psi(b_0) + p b_1, p b_2, \ldots].$$

To work these formulas out, use formula for ghost coordinates in terms of necklace coordinates:

$$x_0 = b_0$$

$$x_1 = \psi(b_0) + p b_1$$

$$x_2 = \psi^{\circ 2}(b_0) + p \psi(b_1) + p^2 b_2$$

$$\vdots$$

2016.05.20

Recall that

$$W_n(R) = \mathrm{Hom}_{\mathrm{ring}}(\Lambda_{p,n}, R)$$

and that we have the different components

$$\begin{tikzcd}
\mathbb{Z}[e, \delta, \ldots, \delta^{\circ n}] \arrow[dr, "\sim"] & & \\
\mathbb{Z}[c, \psi, \ldots, \psi^{\circ n}] \arrow[r, hook] & \Lambda_{p,n} \arrow[r, "a'"] & R \\
\mathbb{Z}[\theta_0, \theta_1, \ldots, \theta_n] \arrow[ur, "\sim"'] & &
\end{tikzcd}$$

inducing, respectively,

$$\begin{tikzcd}
 & R^{[0,n]} & \delta\text{-components} \\
W_n(R) \arrow[ur, bend left, "\sim"] \arrow[r] \arrow[dr, bend right, "\sim"'] & R^{[0,n]} & \text{ghost components} \\
 & R^{[0,n]} & \text{Witt components}
\end{tikzcd}$$

If (R, δ) is a Λ_p-ring, then also

$$W_n(R) = \mathrm{Hom}_{\Lambda_p\text{-ring}}(\Lambda_p \odot \Lambda_{p,n}, R)$$

and

$$\Lambda_p \odot \mathbb{Z}[c_0, \ldots, c_n] \xrightarrow{\sim} \Lambda_p \odot \Lambda_{p,n} \xrightarrow{a'} R,$$

where

$$c_i = \begin{cases} e \odot e & (i=0) \\ p^{-i}(e \odot \varphi^{\circ i} - \varphi \circ \varphi^{\circ (i-1)}) & (0<i\leq n), \end{cases}$$

gives the necklace components

$$W_n(R) \xrightarrow{\sim} R^{[0,n]}.$$

<u>Rmk</u> Here R may have p-torsion. //

Let $\langle x_0, \ldots, x_n\rangle$ and $[b_0, \ldots, b_n]$ be the ghost comp. and necklace comp. of $a \in W_n(R)$, respectively. Recursively,

$$x_0 = b_0$$

$$x_i = \varphi(x_{i-1}) + p^i b_i \qquad (0<i\leq n).$$

Can work out Witt vector arithmetic in necklace components, e.g.

$$[b_0, b_1, \ldots] \cdot [b_0', b_1', \ldots]$$

$$= [b_0 b_0', \ldots, \sum_{\substack{0\leq s,t\leq i \\ s=i \text{ or } t=i}} p^{s+t-i} \varphi^{\circ(i-s)}(b_s)\varphi^{\circ(i-t)}(b_t'), \ldots]$$

<u>Cor</u> (Dwork's lemma) If (R, ϕ) is a Λ_p-ring, then a vector $\langle x_0, \dots, x_n \rangle$ is in the image of the ghost map

$$W_n(R) \xrightarrow{w} R^{[0,n]}$$

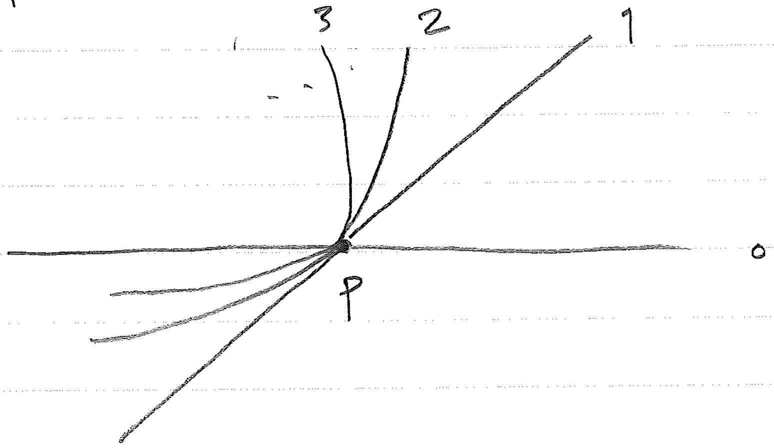
if and only if

$$x_i \equiv \phi(x_{i-1}) \mod p^i R,$$

for all $1 \leq i \leq n$. //

If, in addition, R is p-torsion free, then the ghost map is injective. Hence, in this case Dwork's lemma identifies $W_n(R)$ as a subring of $R^{[0,n]}$.

<u>Ex</u> $\operatorname{Spec} W_n(\mathbb{Z})$:



Similarly,

$$W(\mathbb{Z}_p) = \{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}_p^{\mathbb{N}} \mid x_i \equiv x_{i-1} \bmod p^i \mathbb{Z}_p \},$$

so have ring homom.

$$W(\mathbb{Z}_p) \xrightarrow{w_\infty} \mathbb{Z}_p$$

$$\langle x_0, x_1, \ldots \rangle \longmapsto x_\infty := \lim_{i \to \infty} x_i.$$

It is clearly surjective, and its kernel is

$$I = \{ \langle x_0, x_1, \ldots \rangle \in \mathbb{Z}_p^{\mathbb{N}} \mid x_i \in p^{i+1} \mathbb{Z}_p \}.$$

We claim that I is also the kernel of the surjective ring homomorphism

$$W(\mathbb{Z}_p) \longrightarrow W(\mathbb{F}_p)$$

induced by the unique ring homom. from $\mathbb{Z}_p$ to $\mathbb{F}_p$. First show that the latter map takes I to zero. Let $(a_0, a_1, \ldots)$ be the Witt comp. corresponding to $\langle x_0, x_1, \ldots \rangle$ in I. We must show that $a_i \in p\mathbb{Z}_p$, for all $i \geq 0$. This is clear for $i = 0$, since $a_0 = x_0 \in p\mathbb{Z}_p$, and for $i \geq 1$, it follows inductively from

$$x_i = a_0^{p^i} + p a_1^{p^{i-1}} + \cdots + p^i a_i .$$

Running this argument backwards, we find that if all Witt comp. $a_0, a_1, \ldots$ are divisible by p, then the i'th ghost comp. x_i is divisible by p^{i+1}. So w_p annihilates the kernel of $W(\mathbb{Z}_p) \to W(\mathbb{F}_p)$. We therefore conclude, finally, that

$$\mathbb{Z}_p \xrightarrow{\sim} W(\mathbb{F}_p) . \quad //$$

Remark that $\Lambda_p \odot \Lambda_{p,n}$ represents

$$\Lambda_p\text{-rings} \xrightarrow{W_n} \text{Rings} = \mathbb{Z}[e]\text{-rings};$$

so $\Lambda_p \odot \Lambda_{p,n}$ is a Λ_p-$\mathbb{Z}[e]$-biring. Also, while $\Lambda_{p,n}$ is not generated by additive elements, $\Lambda_p \odot \Lambda_{p,n}$ is generated by the necklace elements, which are additive. I.e., $\Lambda_{p,n}$ is not (weakly) linear but becomes linear rel. to Λ_p after applying the "base-change" $\Lambda_p \odot -$.

More generally, ...

Question: Is it true that for every plethory P, the P-ring $P \odot P$ is generated by additive elements?

If X is a scheme, then its n'th jet space $J^n X$ is defined by

$$(J^n X)(R) = X(R[t]/(t^{n+1}));$$

it is again a scheme. Similarly, define the n'th arithmetic jet sp.:

$$(J_p^n X)(R) = X(W_n(R));$$

it, too, is a scheme. We note

$$J_p^n \operatorname{Spec}(A) = \operatorname{Spec}(\Lambda_{p,n} \odot A)$$

and also

$$\begin{tikzcd} \mathrm{Sch} \arrow[r, shift left, "J_p^n"] & \mathrm{Sch}. \arrow[l, shift left, "W_n"] \end{tikzcd}$$

Since J_p^n has a ~~left~~ adjoint, it preserves limits; in particular, it takes group schemes to group schemes.

Verschibung and Teichmüller :

Define Teichmüller representative of $a \in R$ to be the Witt vector

$$[a) = (a, 0, 0, \dots)_\delta \in W(R).$$

Universal example :

$$a \in \mathbb{Z}[a]$$

with Λ_p-structure $\psi(a) = a^p$, i.e.

$$\delta^{\circ n}(a) = \begin{cases} a & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

The resulting ring homom.

$$\mathbb{Z}[a] \longrightarrow W(\mathbb{Z}[a])$$

takes a to $[a]$; more generally,

$$f \longmapsto (f, \delta(f), \delta^{\circ 2}(f), \dots)_\delta.$$

This ring homom. defines a map of sets

$$R \xrightarrow{[-]} W(R).$$

In ghost components,

$$[a] = \langle a, a^p, a^{p^2}, \ldots \rangle,$$

which shows that

$$[ab] = [a][b].$$

Indeed, this formula is satisfied in the universal case $R = \mathbb{Z}[a, b]$, in which case the ghost map is injective; and hence, it is satisfied in general.

Define Verschiebung operator

$$W(R) \xrightarrow{V_p} W(R)$$

to be the set map given in Witt components by the shift

$$(a_0, a_1, \ldots) \longmapsto (0, a_0, a_1, \ldots).$$

In ghost components, this becomes

$$\langle x_0, x_1, \ldots \rangle \longmapsto \langle 0, px_0, px_1, \ldots \rangle,$$

which shows that V_p is additive, but not multiplicative.

Rmk The definition of V_p implicitly uses the choice of generator p of the maximal ideal $p\mathbb{Z} \subset \mathbb{Z}$. We could as well have used the generator $-p$ to get V_{-p}. //

Similarly, by checking in ghost components, we find that

$$\tau(V_p(a)) = p \cdot a ,$$

$$a = \sum_{i \geq 0} V_p^i [a_i]$$

where a_i is the i'th Witt comp. of a.

2016.05.27

Frobenius lifts in general:

So far: Lifting of p'th power Frob. on R/pR.

Here p plays two roles:

- exponent
- generator of ideal pR

Wish to consider generalization to $\mathfrak{p} \subset \mathcal{O}_K \longrightarrow R$, $[K:\mathbb{Q}] < \infty$. So wish to generalize both occurences of p above.

Context:

A : ring ("base")

$\mathfrak{p} \subset A$: ideal, invertible as A-mod.

$k = A/\mathfrak{p}$

$F \in k[e]$ k-algebra like:

$\Delta^{+}(F) = F\otimes 1 + 1\otimes F$, $\Delta^{\times} = (F\otimes 1)\cdot(1\otimes F)$, ...

so corepresents k-alg. homom.

Def A pre-$\Lambda_{A,\mathfrak{p},F}$-ring is a pair (R, ψ) of an A-algebra R and an A-alg. homom. $\psi : R \to R$ s.t.

$$\begin{tikzcd}
R \arrow[r, "\psi"] \arrow[d] & R \arrow[d] \\
R/\mathfrak{p}R \arrow[r, "F"] & R/\mathfrak{p}R
\end{tikzcd}$$

commutes.

Ex $[K : \mathbb{Q}] < \infty$, $A = \mathcal{O}_K$, $\mathfrak{p} \subset A$ maximal ideal, $k = A/\mathfrak{p}$, $F = e^q$, where $q = (\# k)^r$ for some $r \geq 0$.

Ex As in previous example, but with K a local field or global field ($A = \mathbb{F}_p[t]$, $\mathbb{Z}_p$, $\mathbb{F}_p[[t]]$, ...)

Ex Can consider A, $\mathfrak{p}$ any ring and ideal and $F = e$.

Ex $A = \mathbb{Z}$, $\mathfrak{p} = p_1 p_2 \mathbb{Z}$, p_i prime, $k = \mathbb{F}_{p_1} \times \mathbb{F}_{p_2}$, $F \in \mathbb{F}_{p_1}[e] \times \mathbb{F}_{p_2}[e]$,

$$F = (e^{p_1^{r_1}}, e^{p_2^{r_2}}).$$

Rmk Other variants possible: Could ask for endom. ψ s.t. $\psi^{\circ n}$ lifts F.

Want: A-plethory $\Lambda_{A,\mathfrak{p},F}$ s.t. for $\mathfrak{p}$-torsion A-algebras,

pre-$\Lambda_{A,\mathfrak{p},F}$-structure $\rightsquigarrow$

$\Lambda_{A,\mathfrak{p},F}$-structure

Note We'll abbreviate

$$\Lambda = \Lambda_{A,\mathfrak{p},F}.$$

Ex For $[K:\mathbb{Q}] < \infty$, suppose $\mathfrak{p} \subset \mathcal{O}_K$ principal. Choosing a uniformizer $\pi \in \mathfrak{p}$, the Frobenius lift cond. becomes

$$\psi(x) = x^q + \pi \delta(x),$$

where, as before, δ satisfies

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{0<i<q} \frac{1}{\pi}\binom{q}{i} x^i y^{q-i}$$

$$\delta(xy) = x^q \delta(y) + \delta(x) y^q + \pi \delta(x)\delta(y)$$

$$\delta(1) = 0.$$

Note that $\frac{1}{\pi}\binom{q}{i} \in \mathcal{O}_K$. Define

$$\Lambda = \Lambda_{A,\mathfrak{p},F} = \mathcal{O}_K[e, \delta, \delta^{\circ 2}, \dots]$$

and

$$W(R) = W_{A,\beta,F}(R) = R^{\mathbb{N}}$$

with ring structure defined as earlier. Two issues:

(1) How does this depend on π?

(2) What if β is not principal?

One solution is to ask for operators

$$\delta_\eta(x) = \eta \cdot (\psi(x) - x^q)$$

for every $\eta \in \beta^{-1}$. However, this will necessitate to understand rel.

$$\delta_{\eta_1+\eta_2} = \dots$$

$$\delta_{a\eta} = \dots$$

$$\delta_{\eta_1} \circ \delta_{\eta_2} = \dots$$

which is obviously a huge mess.

It is clearly enough to consider δ_η for η ranging over a family of generators of β^{-1}. In the case $A = O_K$, $[K:\mathbb{Q}] < \infty$, always have $p^{-1} = (1, y)$, so only have two operater δ_1 and δ_y. Still need to show that the resulting theory is independent on choice of y. So not the way to go.

Better approach:

$$\begin{tikzcd}
\left(\begin{matrix}p\text{-tors. free}\\ \text{pre-}\Lambda\text{-rings}\end{matrix}\right) \arrow[r, hook] \arrow[d] & \Lambda\text{-rings} \arrow[d] \\
\begin{matrix}p\text{-tors. free}\\ A\text{-algebras}\end{matrix} \arrow[u, bend right, "W_\Lambda^{tf}"'] \arrow[r, hook] & A\text{-alg.} \arrow[u, bend left, "\Lambda\odot -"] \arrow[u, bend right, "W_\Lambda"']
\end{tikzcd}$$

It is better to produce the comonad $\mathrm{fgt} \circ W_\Lambda$ than to produce the monad $\mathrm{fgt} \circ (\Lambda \odot -)$. Reason: the latter does not preserve p-tors. freeness. So first produce

$$W_\Lambda^{tf} = W_\Lambda^{p\text{-tors-free}}$$

and use Kan extension to get W_Λ.

Def An A-mod. M is $\mathfrak{p}$-torsion-free, if the A-module

$$\mathrm{Tor}_1^A(M, A/\mathfrak{p})$$

is trivial. //

Equivalently, an A-module is $\mathfrak{p}$-torsion-free if the map

$$\mathfrak{p} \otimes_A M \to M$$

induced by the canonical incl. is injective.

Def Fix $(A, \mathfrak{p}, F)$ as above and for A-algebras R, define a sequence of sub-A-algebras

$$W^{(0)}(R) \supset \cdots \supset W^{(m)}(R) \supset \cdots,$$

recursively, by

$$W^{(0)}(R) = R^{\mathbb{N}}$$

$$W^{(m+1)}(R) = \{ a \in W^{(m)}(R) \mid$$

$$\varphi(a) \equiv F(a) \bmod \mathfrak{p} W^{(m)}(R) \}.$$

Prop (1) $W^{(m)}(R) \subset R^{\mathbb{N}}$ is a sub-ψ-ring.

(2) $W^{(\infty)}(R) = \lim_m W^{(m)}(R)$ has the Frobenius lift property.

(3) $W^{(\infty)}(R)$ is the cofree pre-Λ-ring on R.

Pf (1) Proof by induction on $m \geq 0$, the case $m = 0$ being trivial. By definition,

$$\begin{tikzcd}
 & & W^{(m)}(R) \arrow[dr, two heads] & \\
W^{(m+1)}(R) \arrow[r] & W^{(m)}(R) \arrow[ur, "\psi"] \arrow[dr, two heads] & & W^{(m)}(R)/pW^{(m)}(R) \\
 & & W^{(m)}(R)/pW^{(m)}(R) \arrow[ur, "F"] &
\end{tikzcd}$$

is an equalizer, so $W^{(m+1)}(R)$ is a sub-A-alg. of $W^{(m)}(R)$. Must prove that it is stable under ψ. So let $a \in W^{(m+1)}(R)$, i.e.

$$\psi(a) \equiv F(a) \mod pW^{(m)}(R).$$

So

$$\psi^{\circ 2}(a) \equiv \underset{\substack{\shortparallel \\ F(\psi(a))}}{\psi(F(a))} \mod \psi(pW^{(m)}(R))$$

and

$$\psi(\mathfrak{p} W^{(m)}(R)) = \mathfrak{p}\, \psi(W^{(m)}(R))$$
$$\subset \mathfrak{p} W^{(m)}(R)$$

by inductive hypothesis. So

$$\psi(\psi(a)) \equiv F(\psi(a)) \bmod \mathfrak{p} W^{(m)}(R)$$

which shows that $\psi(a) \in W^{(m+1)}(R)$ as desired.

(2) Since $\mathfrak{p}$ is finitely generated and projective (if this does not follow from $\mathfrak{p}$ being invertible, then we will assume it),

$$\lim \mathfrak{p} W^{(m)}(R)$$
$$= \lim \operatorname{im}(\mathfrak{p} \otimes W^{(m)}(R) \to R^{\mathbb{N}})$$
$$\overset{?}{=} \operatorname{im}(\lim \mathfrak{p} \otimes W^{(m)}(R) \to R^{\mathbb{N}})$$
$$\overset{?}{=} \operatorname{im}(\mathfrak{p} \otimes \lim W^{(m)}(R) \to R^{\mathbb{N}})$$

which is what we need. Will justify ?? next time and prove (3).

2016.06.01

Continue with proof from last time. Rewrite equalizer diagram defining $W^{(m)}(R)$ as

$$W^{(m+1)}(R) \longrightarrow W^{(m)}(R) \underset{F}{\overset{\gamma}{\rightrightarrows}} k \otimes_A W^{(m)}(R).$$

Taking limits over m, we get the equalizer diagram

$$W^{(\infty)}(R) \longrightarrow W^{(\infty)}(R) \underset{F}{\overset{\gamma}{\rightrightarrows}} \lim_m k \otimes_A W^{(m)}(R).$$

What we wish to prove, to get the statement (2), is that

$$W^{(\infty)}(R) \longrightarrow W^{(\infty)}(R) \underset{F}{\overset{\gamma}{\rightrightarrows}} k \otimes_A W^{(\infty)}(R)$$

is an equalizer, so we need to show that the canonical map

$$k \otimes_A \lim_m W^{(m)}(R) \longrightarrow \lim_m k \otimes_A W^{(m)}(R)$$

is a monomorphism. The sequence

$$0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow k \longrightarrow 0$$

is exact. We will now assume that R is $\mathfrak{p}$-torsion free. Then so is $W^{(m)}(R)$, so we get a diagram with exact rows

$$\begin{tikzcd}
 & \mathfrak{p}\otimes_A \lim W^{(m)} \arrow[r] \arrow[d, "\beta"] & \lim W^{(m)} \arrow[r] \arrow[d, equal] & k\otimes_A \lim W^{(m)} \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \lim \mathfrak{p}\otimes_A W^{(m)} \arrow[r] & \lim W^{(m)} \arrow[r] & \lim k\otimes_A W^{(m)} \arrow[r] & 0
\end{tikzcd}$$

where we abbreviate $W^{(m)} = W^{(m)}(R)$. So we may instead show that the map β is surjective. Claim:

(i) $\mathfrak{p}$ is finitely generated as an A-module.

(ii) $\mathfrak{p}$ is projective as an A-mod.

Granting this, $\mathfrak{p}$ is dualizible as an A-module, and hence, the functor $\mathfrak{p}\otimes_A -$ preserves limits. This shows that β is an isom. So actually α is an isomorphism.

It remains to justify the claims (i)–(ii). First, by EGA I.5.4.1, an A-module M is invertible if and only if M is locally free of rk. 1. Now, by EGA IV.2.5.3 and Bourbaki, Comm. Alg., Chap. I, §3, no. 6, Prop. 11, the properties of being finitely generated and finitely presented are local properties for the fpqc topology. Since we know that, locally for the Zariski topology, $\mathfrak{p}$ is free of rk. 1 we conclude that $\mathfrak{p}$ is finitely presented. So (i) holds. Similarly, being flat is a local property, so $\mathfrak{p}$ is flat. But flat and finitely presented is projective by Bourbaki, ibid, §2, p. 64. So (ii) holds.

Finally, we also used that $W^{(m)}(R)$ is $\mathfrak{p}$-torsion free if R is. Again, we may prove this locally on A, so we can assume that $\mathfrak{p} = (\pi)$ is principal. But

$$\begin{tikzcd}
W^{(m)}(R) \arrow[r, hook] \arrow[d, "\pi"] & R^{\mathbb{N}} \arrow[d, "\pi"] \\
W^{(m)}(R) \arrow[r, hook] & R^{\mathbb{N}}
\end{tikzcd}$$

shows that $W^{(m)}(R)$ is $\mathfrak{p}$-tors. free. This completes the proof of (2), but we had to assume R to be $\mathfrak{p}$-torsion free.

The remaining statement (3) was that $W^{(n)}(R)$ is the cofree pre-Λ-ring on R, if R is p-torsion free. So let S be a pre-Λ-ring and let $\gamma: S \to R$ be an A-algebra map. We wish to show that γ admits a unique lifting to a map of pre-Λ-rings, or equivalently, a map of ψ-rings

$$\begin{tikzcd} & W^{(\infty)}(R) \arrow[d] \\ S \arrow[ur, dashed, "\exists ! \tilde{\gamma}"] \arrow[r, "\gamma"] & R \end{tikzcd}$$

Uniqueness is clear, since

$$\tilde{\gamma}(s) = \langle \gamma(s), \gamma(\psi(s)), \gamma(\psi^{\circ 2}(s)), \dots \rangle,$$

and to prove existence, it suffices to show that for all $m \geq 0$,

$$\tilde{\gamma}(S) \subset W^{(m)}(R) \subset R^{\mathbb{N}}.$$

We proceed by induction, the case $m=0$ being trivial. So we assume $\tilde{\gamma}(S) \subset W^{(m-1)}(R)$ and show that $\tilde{\gamma}(S) \subset W^{(m)}(R)$. We calculate

$$\psi(\tilde{\gamma}(s)) - F(\tilde{\gamma}(s))$$

$$= \tilde{\gamma}(\psi(s)) - \tilde{\gamma}(F(s))$$

$$= \tilde{\gamma}(\psi(s) - F(s))$$

$$\in \tilde{\gamma}(p S)$$

$$\subset p\, \tilde{\gamma}(S)$$

$$\subset p\, W^{(m-1)}(R) \qquad \text{(by induction)}$$

which proves the induction step, by the definition of $W^{(m)}(R)$ (as an equalizer). //

2016.06.03

We proceed to show that the functor $W^{(\infty)}$ from $\mathfrak{p}$-torsion free A-algebras to pre-Λ-rings is representable. We use the representing object to extend $W^{(\infty)}$ to all A-algebras.

We first define $A[\mathfrak{p}^{-1}]$. If $\mathfrak{p} = (\pi)$, then we have $A[\pi^{-1}]$, which we wish to generalize. The A-module

$$\mathfrak{p}^{-1} = \mathrm{Hom}_A(\mathfrak{p}, A)$$

is invertible, and the canonical inclusion $\mathfrak{p} \hookrightarrow A$ induces a map of A-modules $A \longrightarrow \mathfrak{p}^{-1}$. This map again is injective, since $\mathfrak{p}$ is locally principal. We define $A[\mathfrak{p}^{-1}]$ to be the pushout

$$\begin{tikzcd}
\mathrm{Sym}_A(A) \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[d] \\
\mathrm{Sym}_A(\mathfrak{p}^{-1}) \arrow[r] & A[\mathfrak{p}^{-1}]
\end{tikzcd}$$

The morphism $A \longrightarrow A[\mathfrak{p}^{-1})$ is injective and flat, since being so are local properties and these are true locally.

NB $A[\mathfrak{p}^{-1}] \neq S^{-1}A$ with $S = \mathfrak{p}$, for e.g. $0 \in \mathfrak{p}$ is not inverted in $A[\mathfrak{p}^{-1}]$. //

We first show that each $W^{(n)}$ is representable.

Def The family of A-algebras

$$(\Lambda^{(m)} \mid m \in \mathbb{N})$$

are defined recursively by:

$$\Lambda^{(0)} = \Psi = A[c, \psi, \psi^{\circ 2}, \ldots]$$

$\Lambda^{(m)}$ = sub-$\Lambda^{(m-1)}$-algebra of $A[\mathfrak{p}^{-1}] \otimes_A \Psi$ generated by

$$(f \circ \delta_\eta \mid f \in \Lambda^{(m-1)},\ \eta \in \mathfrak{p}^{-1}),$$

where $\delta_\eta = \eta(\psi - F)$. //

Prop The canonical inclusion

$$\Lambda^{(m)} \hookrightarrow \Psi$$

becomes an isomorphism after cobase-change along $A \hookrightarrow A[\mathfrak{p}^{-1}]$.

~~Pf~~ Proof by induction on $m \in \mathbb{N}$, the case $m=0$ being trivial. To prove the induction step, consider

$$\begin{tikzcd}
A[p^{-1}] \otimes_A \Lambda^{(m)} \arrow[rr] \arrow[dr] & & A[p^{-1}] \otimes_A \Psi \\
& A[p^{-1}] \otimes_A \Lambda^{(m-1)} \arrow[ur, "\text{isom. by induction}"'] &
\end{tikzcd}$$

The left-hand slanted map is an isomorphism, by the definition of $\Lambda^{(m)}$, since

$$A[p^{-1}] \otimes_A A \longrightarrow A[p^{-1}] \otimes_A A[p^{-1}]$$

is an isomorphism. //

<u>Prop</u> The A-algebra $\Lambda^{(m)}$ is p-torsion free and represents the endofunctor $W^{(m)}$ on the category of p-torsion free A-algebras.

<u>Pf</u> Wish to show

$$\begin{tikzcd}
\mathrm{Hom}_{A\text{-alg}}(\Lambda^{(0)}, R) \arrow[r] & R^{\mathbb{N}} \\
\mathrm{Hom}_{A\text{-alg}}(\Lambda^{(m)}, R) \arrow[u] \arrow[r, dashed, "\exists!", "\sim"'] & W^{(m)}(R) \arrow[u]
\end{tikzcd}$$

and proceed by induction on $m \in \mathbb{N}$, the case $m=0$ being trivial. To prove induction step, given

$$\Lambda^{(m-1)} \xrightarrow{a} R,$$

consider

$$\begin{tikzcd}
 & \Lambda^{(m-1)} \arrow[r, "a"] \arrow[dl, hook] \arrow[d, hook] & R \arrow[d, hook, "\text{injective since } R \text{ p-tors. free}"] \\
\Lambda^{(m)} \arrow[dr, hook] & A[p^{-1}] \otimes_A \Lambda^{(m-1)} \arrow[r] \arrow[d, "\sim"] & A[p^{-1}] \otimes_A R \\
 & A[p^{-1}] \otimes_A \Psi \arrow[ur, "a'"'] &
\end{tikzcd}$$

and calculate:

$$\{ \Lambda^{(m)} \xrightarrow{a} R \}$$

$$= \{ \Lambda^{(m-1)} \xrightarrow{a} R \mid \forall \eta \in p^{-1}, \forall f \in \Lambda^{(m-1)}, \; a'(f \circ \delta_\eta) \in R \}$$

inductive hypothesis

$$= \{ a \in W^{(m-1)}(R) \mid \forall \eta, \delta_\eta(a)(\Lambda^{(m-1)}) \subset R \}$$

$$= \{ a \in W^{(m-1)}(R) \mid \forall \eta, \delta_\eta(a') \in W^{(m-1)}(R) \}$$

$$= \{ a \in W^{(m-1)}(R) \mid \forall \eta, \eta(\psi(a') - F(a')) \in W^{(m-1)}(R) \}$$

$$= \{ a \in W^{(m-1)}(R) \mid \psi(a') - F(a') \in pW^{(m-1)}(R) \}$$

$$= W^{(m)}(R). \qquad //$$

Cor $\Lambda^{(\infty)} = \bigcup \Lambda^{(m)}$ represents $W^{(\infty)}$ on p-torsion free A-algebras. //

Rmk The canonical map $A \to A[p^{-1}]$ identifies $\mathrm{Spec}(A[p^{-1}])$ with the open subscheme of $\mathrm{Spec}(A)$ on which $j: p \to A$ is invertible. //

Recall:

$$\begin{tikzcd}
\left(\begin{array}{c} p\text{-tors. free} \\ \text{pre-}\Lambda\text{-ring} \end{array}\right) \arrow[r, hook, "?"] \arrow[d, shift right, "\mathrm{fgt}"'] & \Lambda\text{-rings} \arrow[d, shift right, "\mathrm{fgt}"'] \\
\left(\begin{array}{c} p\text{-tors. free} \\ A\text{-alg.} \end{array}\right) \arrow[r, hook, "i"] \arrow[u, shift right, bend right, "W^{(\infty)}"'] & A\text{-algebras} \arrow[u, shift right, bend right, "W_\Lambda"']
\end{tikzcd}$$

Here $W^{(\infty)}(R)$ is the maximal sub-ψ-ring of $R^{\mathbb{N}}$ on which ψ has the Frobenius lift property. We have now proved that the composite functor $\mathrm{fgt} \circ W^{(\infty)}$ is represented by $\Lambda^{(\infty)}$. Addition and multiplication of A-algebras are represented by

$$\Lambda^{(\infty)} \xrightarrow{\Delta^+, \Delta^\times} (\Lambda^{(\infty)} \otimes \Lambda^{(\infty)}) / \mathfrak{p}\text{-torsion}$$

$$\parallel$$

coprod. in $\mathfrak{p}$-tors. free A-alg.

If ($\mathfrak{p}$-tors. free) $\otimes$ ($\mathfrak{p}$-tors. free) again is $\mathfrak{p}$-tors. free, then we obtain that $\Lambda^{(\infty)}$ is an A-A-biring, and hence, represents the functor W_Λ that we wish to define. But this is not true in the generality that we work in here. The following result gets us around this difficulty.

Prop If $\mathfrak{p} = (\pi)$, then

$$\Lambda^{(m)} = A[e, \delta, \dots, \delta^{\circ m}, \psi \circ \delta^{\circ m}, \psi^{\circ 2} \circ \delta^{\circ m}, \dots],$$

where $\delta = \frac{1}{\pi}(\psi - F)$.

Pf Induction, $m = 0$ O.K.

$\Lambda^{(m)} =$ sub-$\Lambda^{(m-1)}$-alg. gen. by

$(f \circ \delta_y \mid f \in \Lambda^{(m-1)}, y \in \mathfrak{p}^{-1})$

$=$ sub-$\Lambda^{(m-1)}$-alg. gen. by

$(f \circ \delta \mid f \in \Lambda^{(m-1)})$.

Use Leibniz rules to prove equality. So enough to let f range over a family of generators of the A-alg. $\Lambda^{(m-1)}$. Hence,

$$\Lambda^{(m)} = \text{sub-}\Lambda^{(m-1)}\text{-alg. gen. by}$$

$$(e\circ\delta, \delta\circ\delta, \ldots, \delta^{\circ(m-1)}\circ\delta, \psi\circ\delta^{\circ(m-1)}\circ\delta, \ldots),$$

where we use the inductive hypothesis. Using it again, we get

$$\Lambda^{(m)} = \text{sub-}A\text{-algebra gen. by the family consisting of}$$

$$e, \delta, \ldots, \delta^{\circ(m-1)}, \psi\circ\delta^{\circ(m-1)}, \psi^{\circ 2}\circ\delta^{\circ(m-1)}, \ldots$$

and

$$\delta, \ldots, \delta^{\circ(m-1)}, \delta^{\circ m}, \psi\circ\delta^{\circ m}, \psi^{\circ 2}\circ\delta^{\circ m}, \ldots$$

We claim that the subfamily of

$$e, \delta, \ldots, \delta^{\circ m}, \psi\circ\delta^{\circ m}, \psi^{\circ 2}\circ\delta^{\circ m}, \ldots$$

generates $\Lambda^{(m)}$ as an A-algebra. This follows from the calculation

$$\psi^{\circ i} \circ \delta^{\circ(m-1)} = \psi^{\circ(i-1)} \circ (F + \pi\delta) \circ \delta^{\circ(m-1)}$$

$$= \psi^{\circ(i-1)} \circ F \circ \delta^{\circ(m-1)} + \psi^{\circ(i-1)} \circ \pi\delta \circ \delta^{\circ(m-1)}$$

$$= F \circ \psi^{\circ(i-1)} \circ \delta^{\circ(m-1)} + \pi\,\psi^{\circ(i-1)} \circ \delta^{\circ m}.$$

Indeed, $F \in A[c]$ and $\pi \in A$, so claim follows by induction on i. Finally, the generators $\delta^{\circ i}$ and $\psi^{\circ j} \circ \delta^{\circ m}$ are differential operators of order i and $j+m$, respectively, and therefore, are algebraically independent. //

Cor $\Lambda^{(m)}$ is locally free as an A-algebra, and hence, as an A-module. //

In particular, for all m, n, the A-module

$$(\Lambda^{(m)})^{\otimes n}$$

is locally free. Hence, the A-mod.

$$(\Lambda^{(\infty)})^{\otimes n} = \operatorname*{colim}_{m} (\Lambda^{(m)})^{\otimes n}$$

is flat, and hence, p-torsion free.

It follows that $\Lambda^{(m)}$ and $\Lambda^{(\infty)}$ are A-A-birings; the co-ring axioms hold because $(\Lambda^{(m)})^{\otimes 3}$ and $(\Lambda^{(\infty)})^{\otimes 3}$.

2016.06.08

Recall the context we are considering:

A : ring.

$\mathfrak{p} \subset A$: ideal, invertible as an A-module.

$F \in A/\mathfrak{p}[c]$: $A/\mathfrak{p}$-algebra-like.

Have constructed left-hand vertical functor in the diagram

$$\begin{tikzcd}
\left\{\begin{array}{l}\mathfrak{p}\text{-tors.-free} \\ A\text{-algebras}\end{array}\right\} \arrow[r, hook, "\iota"] \arrow[d, "\mathrm{fgt}\circ W^{(\infty)}"] & \mathrm{Alg}_A \arrow[d, dashed, "\text{``}\mathrm{fgt}\circ W\text{''}"] \\
\left\{\begin{array}{l}\mathfrak{p}\text{-tors.-free} \\ A\text{-algebras}\end{array}\right\} \arrow[r, hook, "\iota"] & \mathrm{Alg}_A
\end{tikzcd}$$

and we may define the right-hand functor by left Kan extension,

$$(\mathrm{fgt}\circ W)(R) = \operatorname*{colim}_{\tilde{R} \to R} (\mathrm{fgt}\circ W^{(\infty)})(\tilde{R}).$$

It is not clear that this extension exists in the universe of discourse; we know that it does, because it is represented by the (p-torsion-free) A-algebra representing $\mathrm{fgt} \circ W^{(\infty)}$. We note that, since i is fully faithful, we have

$$\text{"}\mathrm{fgt} \circ W\text{"} \circ i = i \circ \mathrm{fgt} \circ W^{(\infty)}.$$

Question: Why is $\Lambda := i(\Lambda^{(\infty)})$ a plethory?

In other words, why is $\mathrm{fgt} \circ W$ a comonad?

Answer: Because $\mathrm{fgt} \circ W^{(\infty)}$ is.

In detail, by Yoneda,

$$\{\mathrm{fgt} \circ W^{(\infty)} \to \mathrm{fgt} \circ W^{(\infty)} \circ \mathrm{fgt} \circ W^{(\infty)}\}$$

$$= (\mathrm{fgt} \circ W^{(\infty)} \circ \mathrm{fgt} \circ W^{(\infty)})(\Lambda^{(\infty)})$$

$$= (\text{"}\mathrm{fgt} \circ W\text{"} \circ \text{"}\mathrm{fgt} \circ W\text{"})(\Lambda)$$

$$= \{\text{"}\mathrm{fgt} \circ W\text{"} \to \text{"}\mathrm{fgt} \circ W\text{"} \circ \text{"}\mathrm{fgt} \circ W\text{"}\}$$

which gives the comonadal (co-) product on "$\mathrm{fgt} \circ W$", and the counit is defined similarly.

It satisfies the comonad axioms because $\mathrm{fgt} \circ W^{(\infty)}$ does. This defines a canonical comonad structure, or equivalently, a canonical plethory structure on

$$\Lambda = \Lambda_{A,\mathfrak{p},F} .$$

Def A $\Lambda_{A,\mathfrak{p},F}$ - structure on an A-algebra R is an action of the plethory $\Lambda_{A,\mathfrak{p},F}$. //

We may define the category of $\Lambda_{A,\mathfrak{p},F}$ - rings, equivalently to be the category of coalgebras for "$\mathrm{fgt} \circ W$" with the comonad structure define above. If R is an A-algebra, then the A-algebra "$\mathrm{fgt} \circ W$"(R) has a canonical $\Lambda_{A,\mathfrak{p},F}$ - structure; this gives the functors

$$\begin{tikzcd}
\left\{ \begin{array}{l} \mathfrak{p}\text{-tors.-free} \\ \text{pre-}\Lambda\text{-rings} \end{array} \right\} \arrow[d, shift right, "\mathrm{fgt}"'] & \Lambda_{A,\mathfrak{p},F}\text{-rings} \arrow[l, hook', "\iota"'] \arrow[d, shift right, "\mathrm{fgt}"'] \\
\left\{ \begin{array}{l} \mathfrak{p}\text{-tors.-free} \\ A\text{-algebras} \end{array} \right\} \arrow[u, shift right, "W^{(\infty)}"'] & \mathrm{Alg}_A \arrow[l, hook', "\iota"'] \arrow[u, shift right, "W"']
\end{tikzcd}$$

Classical structures in general contexts:

(1) Ghost components : canonical
$\rightsquigarrow$ ψ-operators

(2) δ-components : exist if $\mathfrak{p} = (\pi)$;
$\rightsquigarrow$ δ-operators depend on π

(3) Verschibung filtr. : canonical
$\rightsquigarrow$ $W_n(R)$

(4) Verschibung operater, : exist under
Witt components, further
Θ_n-operaters assump., e.g.
$A/\mathfrak{p}$ = fin. field
$F = e^q$, $q \neq 1$,
$\mathfrak{p} = (\pi)$.

Explain:

(1) By def.

(2) $\Lambda = A[e, \delta, \delta^{\circ 2}, \dots]$, $\delta = \frac{1}{\pi}(\psi - F)$.

$$W(R) \xrightarrow{\sim} R^{\mathbb{N}}$$

$$(\Lambda \xrightarrow{x} R) \mapsto (x(e), x(\delta), x(\delta^{\circ 2}), \dots)$$

NB. The δ-components depend on the choice of π in a complicated way.

(3) Define $(\Psi_n = A[e, \psi, \dots, \psi^{\circ n}])$

$$\begin{tikzcd}
\Lambda_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{pullback}"] & \Lambda \arrow[d, hook] \\
A[\beta^{-1}] \underset{A}{\otimes} \Psi_n \arrow[r, hook] & A[\beta^{-1}] \underset{A}{\otimes} \Psi
\end{tikzcd}$$

Locally, $\beta = (\pi)$, so

$$\Lambda_n = A[e, \delta, \dots, \delta^{\circ n}]$$

Is a retract of Λ; therefore, the A-A-biring structure on Λ defines one on Λ_n:

$$\begin{tikzcd}
\Lambda_n \arrow[r, hook] \arrow[d, dashed, "\exists!\,\Delta^{+,\times} \quad \Longleftarrow"] & \Lambda \arrow[d, "\Delta^{+,\times}"] \\
\Lambda_n \otimes \Lambda_n \arrow[r, hook] & \Lambda \otimes \Lambda
\end{tikzcd}$$

↳ b/c locally a retract

Consider three tensor factors to verify axioms.

$$\begin{tikzcd}
W_n(R) \arrow[r, phantom, ":="] & \mathrm{Hom}_{A\text{-alg}}(\Lambda_n, R) \\
W(R) \arrow[u] \arrow[r, phantom, "="] & \mathrm{Hom}_{A\text{-alg}}(\Lambda, R) \arrow[u] \\
V^n W(R) \arrow[u] \arrow[r, phantom, "="] & \text{kernel} \arrow[u, hook]
\end{tikzcd}$$

Since $\operatorname{colim}_n \Lambda_n \xrightarrow{\sim} \Lambda$,

$$W(R) \xrightarrow{\sim} \lim_n W_n(R).$$

(4) Assume $\mathfrak{p} = (\pi)$, $A/p = k$ a finite field, and $F = c^q$, $q \neq 1$. Note that we need $q > 1$ to get

$$x \equiv y \mod \mathfrak{p} \Rightarrow$$

$$x^{q^n} \equiv y^{q^n} \mod \mathfrak{p}^{n+1}.$$

(For $q = 1$, we have in fact,

$$\theta_2 = \frac{1}{\pi}(\theta_1 + \theta_1 \circ \theta_1)$$

$$= \frac{1}{\pi}\delta + \delta \circ \delta. \quad)$$

We may now define an operator

$$W(R) \xrightarrow{V_\pi} W(R)$$

in π-Witt coordinates by the usual formula

$$(a_0, a_1, \ldots) \longmapsto (0, a_0, a_1, \ldots).$$

In ghost components,

$$\langle x_0, x_1, \ldots \rangle \longmapsto \langle 0, \pi x_0, \pi x_1, \ldots \rangle,$$

so V_π does depend on the choice of π; $(F \circ V_\pi)(x) = \pi x$.

Can remove dependence on π as follows: The map

$$\mathfrak{p}^n \otimes_A W(R) \xrightarrow{V^n} W(R)$$

defined locally by

$$\pi^n \otimes a \longmapsto V_\pi^n(a)$$

does not depend on π and exists even if $\mathfrak{p}$ is not principal. But the assumptions $F = e^q$, $q > 1$, etc. are needed.

For it is given in ghost coord. by

$$\gamma \otimes \langle x_0, x_1, \dots \rangle$$

$$\longmapsto \langle \underbrace{0, \dots, 0}_{n}, \gamma x_0, \gamma x_1, \dots \rangle .$$

10/6 - 16

Witt Vectors

Geometry, $\mathbb{F}_1$-picture

Multiple primes: O_K ring of integers in a number field K

Def: pre-$\Lambda_{(\dots)}$-ring

i) An O_K-algebra R

maximal → ii) $\forall$ primes $\mathfrak{p} \subset O_K$, an O_K-alg endo $\psi_{\mathfrak{p}} : R \to R$

iii) $\psi_{\mathfrak{p}_1} \circ \psi_{\mathfrak{p}_2} = \psi_{\mathfrak{p}_2} \circ \psi_{\mathfrak{p}_1}$

iv) $\psi_{\mathfrak{p}}$ reduces to the q'th power Frob mod pR where $q = \#(O_K/\mathfrak{p})$

Ex: $K = \mathbb{Q}$ $O_K = \mathbb{Z}$ primes (p), p prime

Commuting ring maps $\psi_p : R \to R$ s.t.

$$\psi_p(x) \equiv x^p \mod pR$$

Ex: $K = \mathbb{Q}(i)$, $O_K = \mathbb{Z}[i]$

$p = (1+i)$

$p \equiv 3 \pmod 4$ $\mathfrak{p} = (p)$

$p \equiv 1 \pmod 4$ $p = (a+ib)$ $\mathfrak{p} = (a-ib)$ $a^2+b^2=p$

$\psi_{(1+i)}(x) \equiv x^2 \mod (1+i)R$

$\psi_{(p)}(x) \equiv x^{p^2} \mod pR$

$\psi_{(a+ib)}(x) \equiv x\, x^p \mod (a+bi)R$

Witt Vectors

Also works for function fields

(Fin ext. of) $\mathbb{F}_p(t) = K$

$\mathbb{F}_p[t] = \mathcal{O}_K$

We pass from pre-Λ_K-rings to Λ_K-rings in the usual way.

i) $\underline{p} = (\pi_{\mathfrak{p}})$ all $\mathfrak{p}$

Define $\delta_{\mathfrak{p}}$ operators

ii) Usual case ghost ring $R^{\mathbb{N}} \circlearrowleft \mathbb{N}\ \psi^{\circ n}$

Multiple prime case $R^{\mathrm{Id}(K)} \circlearrowleft \psi_I$

I any ideal $I = \mathfrak{p}_1 \cdots \mathfrak{p}_n$

$\mathrm{Id}(K)$ = monoid of non zero ideals under mult

$$\cong \bigoplus_{\substack{\text{max ideals} \\ \mathfrak{p}}} \mathbb{N}$$

$R^{\mathrm{Id}(K)}$ is the cofree ψ_K-ring on R

Witt Vectors

Ex: $R = O_K[z]$

$\psi_p : z \mapsto z^{q_p}$

$K = \mathbb{Q}$ $\quad R = \mathbb{Z}[z]$ $\quad z \mapsto z^p$

$$\begin{tikzcd}
\Lambda_K\text{-rings} \arrow[d] \\
O_K\text{-alg} \arrow[u, bend left=60, "\Lambda_K \odot -"] \arrow[u, bend right=60, "W_K"']
\end{tikzcd}$$

Most important example: $K = \mathbb{Q}$ $\quad W_K$ = big Witt-vectors

$W_\mathbb{Q}$ ring = λ-ring in K-theory

Thm: $\Lambda_\mathbb{Q} = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \dots]$

Like $\theta_1, \theta_2, \dots$

NB: $\overbrace{\mathrm{Spec}(R^{\mathbb{N}})}^{\text{q. compact}} \neq \coprod_{\mathbb{N}} \mathrm{spec}(R)$, $R \neq (0)$

Only true for finite sets

$\neq$ avoid this!

X scheme $\rightsquigarrow$ $W_{K,n}(X)$

Cover by spec(R)
Use $W_{K,n}(R)$

Want colimit, clever way to be well behaved

Witt Vectors

$X = \mathrm{spec}(\mathbb{F}_p)$

Work with the usual p-typical W

$W_n(\mathbb{F}_p) = \mathbb{Z}/p^{n+1}\mathbb{Z}$

$W(\mathbb{F}_p) = \mathbb{Z}_p$

$\mathrm{Spec}(\mathbb{Z}_p)$

~~Spec(~~ $\operatorname*{colim}_n \mathrm{Spec}(\mathbb{Z}/p^{n+1}\mathbb{Z}) =: \underset{\substack{\uparrow \\ \text{formal}}}{\mathrm{Spf}}(\mathbb{Z}_p)$

$W(\mathrm{Spec}(\mathbb{F}_p)) = \mathrm{Spf}\ \mathbb{Z}_p$

R ring

$$\mathrm{Hom}(\mathrm{Spec}(R), \mathrm{Spec}(\mathbb{Z}_p)) = \mathrm{Hom}(\mathbb{Z}_p, R)$$

$$\mathrm{Hom}(\mathrm{Spec}(R), \mathrm{Spf}(\mathbb{Z}_p))$$

$$= \mathrm{Hom}(\mathrm{Spec}(R), \mathrm{colim}\,\mathrm{Spec}(\mathbb{Z}_{\cancel{p}}/p^{n+1}\mathbb{Z}))$$

$$= \operatorname*{colim}_n \mathrm{Hom}(\mathrm{Spec}\,R, \mathrm{spec}\,\mathbb{Z}/p^{n+1}\mathbb{Z}))$$

$$= \operatorname*{colim}_n \mathrm{Hom}(\mathbb{Z}/p^{n+1}\mathbb{Z}, R)$$

$\mathrm{Spec}(\mathbb{Z}_p)$ ignores top. on $\mathbb{Z}_p$

$\mathrm{Spf}(\mathbb{Z}_p)$ uses the p-adic topology

Witt Vectors

$$\mathbb{Z}_p = \lim_n \mathbb{Z}/p^{n+1}\mathbb{Z}$$

$$\mathrm{Spf}(\mathbb{Z}_p) = \operatorname{colim}_n \mathrm{Spec}(\mathbb{Z}/p^{n+1}\mathbb{Z})$$

Remembering $\mathbb{Z}_p$ is a limit.

Fact: $W_{\mathbb{Q}}(R) = \lim_n \left(W^{(p_1)} \circ \dots \circ W^{(p_n)}(R) \right)$

$p_1, \dots, p_n$ first n primes

$= " \dots \circ W^{(5)} \circ W^{(3)} \circ W^{(2)}(R) "$

Big Witt Vectors

$$W_{\mathbb{Q}}(\mathbb{F}_p) = \prod_{\substack{n \geq 1 \\ p \nmid n}} \mathbb{Z}_p$$

$$\begin{tikzcd}
W_{\mathbb{Q}}(\mathbb{Z}) \arrow[r, hook, "\text{ghost map}"'] & \mathbb{Z} \times \mathbb{Z} \times \dots\dots \arrow[d, equal] \\
 & \mathbb{Z}^{\{1,2,\dots\}}
\end{tikzcd}$$

p'th frob mult. by p

(p-typ case ⇝ shift $\begin{array}{c}(0,1,2,\dots) \\ \updownarrow \\ (p^0, p^1, p^2, \dots)\end{array}$

$$W_{\mathbb{Q}}(\mathbb{Z}) = \{ \langle a_1, a_2, \dots \rangle \in \mathbb{Z}^{\infty} \mid \forall n \ \forall \text{prime } p$$

$$a_{np} \equiv a_n \mod p^{1+\mathrm{ord}_p(n)}$$

Witt Vectors

~~$\frac{1}{p} \in R$~~ $\frac{1}{q} \in R$ for all $q \neq p$ i.e $R = \mathbb{Z}_{(p)}$-alg

$$W_{\mathbb{Q}}(R) = \prod_{\substack{n \geq 1 \\ p \nmid n}} W^{(p)}(R)$$

W_K is a comonad on $\mathcal{O}_K$-alg

⇝ W_K will be a monad on $\mathcal{O}_K$-schemes

$$\begin{tikzcd}
\text{“}\Lambda_K\text{-schemes”} \arrow[d] \\
\mathcal{O}_K\text{-schemes} \arrow[u, bend left, "W_K"] \arrow[u, bend right, "I_K"']
\end{tikzcd}$$

aritmetic jet space

K-alg

L-alg

$G = \mathrm{Gal}(L/K)$

Witt Vectors

$$\begin{tikzcd}
K\text{-mod} \arrow[r, shift left, "L\otimes_K -"] \arrow[d, bend right=60, "L\otimes_K -"'] \arrow[d, bend left=60, "\mathrm{Hom}_K(L,-)"] & L\text{-mod with twisted } G\text{-action} \arrow[l, shift left, "\text{inv.}"] \arrow[d, "\text{fgt}"] \\
L\text{-mod} \arrow[u, "\text{fgt}"] \arrow[r, equal] & L\text{-mod} \arrow[u, bend right=60, "\prod_G -"']
\end{tikzcd}$$

$K \longrightarrow L$ subalgebra

$G = \mathrm{Gal}(L/K)$

- G-action is descend data from L to K

$K = \mathbb{Q}$

$$\begin{tikzcd}
\text{Schemes} \arrow[d, bend right=60, "W"'] \arrow[d, bend left=60, "I"] & & \mathbb{Z}\text{-schemes} \arrow[d, bend right=60, "\text{base forg.}"'] \arrow[d, bend left=60, "\text{Weil restric. of scalars}"] \\
\Lambda_{\mathbb{Q}}\text{-schemes} \arrow[u, "\text{fgt}"] & =: & \mathbb{F}_1\text{-schemes} \arrow[u, "\substack{\mathbb{Z}\otimes_{\mathbb{F}_1} - \\ \text{base change}}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F}_q[x] & \mathbb{Z} \\
\mathbb{F}_q \arrow[u] & \text{“}2.\mathbb{F}_1\text{”} \arrow[u]
\end{tikzcd}$$

$\Lambda_{\mathbb{Q}}$-structure is descend data from $\mathbb{Z}$ to $\mathbb{F}_1$

Like \$

Witt Vectors

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Z}) \arrow[d, mapsto] \\ \boxed{\begin{array}{l} W(\mathrm{Spec}(\mathbb{Z})) \\ \text{as a } \Lambda_{\mathbb{Q}}\text{-scheme} \end{array}} \end{tikzcd}$$

$= $ "spec $\mathbb{Z}$ viewed as an $\mathbb{F}_1$-scheme"

$W(\mathrm{Spec}(\mathbb{Z}))$ viewed as a scheme $=$ "$\mathrm{spec}\,\mathbb{Z} \times_{\mathrm{spec}\,\mathbb{F}_1} \mathrm{Spec}\,\mathbb{Z}$"

"$\Lambda_{\mathbb{Q}} = \mathbb{Z}[\mathrm{Gal}(\mathbb{Z}/\mathbb{F}_1)]$"

rmk notation here means the <u>free</u> pol. alg. on G.

$$= \mathbb{Z}[\psi_g \mid g \in \mathrm{Gal}(\mathbb{Z}/\mathbb{F}_1)]$$

$$\mathbb{Q} \otimes \Lambda_{\mathbb{Q}} = \mathbb{Z}[\underbrace{\psi_1, \psi_2, \ldots}_{\text{monoid } \mathbb{N}^+}]$$

Can do the real picture: not over $\mathbb{Z}$,

$$\begin{tikzcd} \mathbb{F}_p[t]\text{-schemes} \arrow[d, bend right] \arrow[d, bend left] \\ \mathbb{F}_p\text{-schemes} \arrow[u] \end{tikzcd}$$

2016.06.24

Witt vectors and semirings:

First recall big Witt vectors and λ-rings / $\mathbb{Z}$. Context: $A = \mathbb{Z}$; a pre-Λ-str. on A-alg. R is a family of commuting Frob. lifts ψ_p, one for each prime number p. The general Witt vector construction produces a $\mathbb{Z}$-plethory Λ and a Witt vector functor W. In this case,

$$\Lambda\text{-ring} = \lambda\text{-ring} \quad (\text{Grothendieck})$$

but this fact requires proof. The proof uses:

<u>Thm</u> (Wilkerson, Joyal) If $R/\mathbb{Z}$ is flat, then every pre-Λ-structure is a λ-ring structure. //

To extend the above to semirings, we wish to find a "model" for $\Lambda_{\mathbb{Z}}$ over $\mathbb{N} = \{0, 1, 2, \dots\}$. This means a semiring $\Lambda_{\mathbb{N}}$ together with semiring homomorphisms

$$\Lambda_{\mathbb{N}} \xrightarrow{\Delta^+, \Delta^-} \Lambda_{\mathbb{N}} \otimes_{\mathbb{N}} \Lambda_{\mathbb{N}}$$

$$\Lambda_{\mathbb{N}} \xrightarrow{\varepsilon^+, \varepsilon^\times} \mathbb{N}$$

$$\Lambda_{\mathbb{N}} \circ \Lambda_{\mathbb{N}} \longrightarrow \Lambda_{\mathbb{N}}$$

and a ring homomorphism

$$\Lambda_{\mathbb{N}} \otimes_{\mathbb{N}} \mathbb{Z} \longrightarrow \Lambda_{\mathbb{Z}}$$

that is an isomorphism of $\mathbb{Z}$-plethories.

Rmk K_0(reasonable cat.) = λ-ring; in this situation, would expect semiring of isom. classes to have a canonical λ-semiring str. To prove this, one should probably look at all Schur functors acting on this semiring."

Preparation on symmetric functions:

Fact: $\Lambda_{\mathbb{Z}}$ = the ring of symm. fcts.

$$= \lim_{n} \mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n}$$

where the limit is formed in the category of graded rings and $\deg(x_i) = 1$. By Newton,

$$\Lambda_{\mathbb{Z}} = \mathbb{Z}[e_1, e_2, e_3, \ldots]$$

where e_n is the n'th elementary symm. function

$$e_n = x_1 x_2 \cdots x_n + \text{all perm.}$$

For example,

$$e_1 = x_1 + x_2 + x_3 + \cdots$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$$

We note that

$$e = e_1$$

$$\psi_p = x_1^p + x_2^p + \cdots$$

$$S_p = \frac{(x_1^p + x_2^p + \cdots) - (x_1 + x_2 + \cdots)^p}{p} \quad \text{coeff. in } \mathbb{Z} \checkmark$$

Now, as a $\mathbb{Z}$-module, $\Lambda_{\mathbb{Z}}$ has a basis with respect to which all plethory structure constants are in $\mathbb{N}$, and hence $\Lambda_{\mathbb{Z}}$ descends to $\Lambda_{\mathbb{N}}$ defined to be the $\mathbb{N}$-span of said basis. (Rmk: $\mathbb{Z}$ is not flat over $\mathbb{N}$; the zero ring is the only ring flat over $\mathbb{N}$.)

Define a partition λ to be a weakly decreasing sequence in $\mathbb{N}$, the terms of which tend to zero, e.g.

$$\lambda = (5,5,4,2,1,1,0,0,\ldots)$$

The associated monomial fct. is

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots + \text{perm.}$$

For instance,

$$e_n = m_{(1,\ldots,1,0,\ldots)} \qquad n \ 1\text{'s}.$$

$$\psi_n = m_{(n,0,0,\ldots)}$$

Easy fact: $(m_\lambda)_{\lambda \text{ part.}}$ is a $\mathbb{Z}$-basis of $\Lambda_{\mathbb{Z}}$.

$$\Lambda_{\mathbb{Z}} \xrightarrow{\Delta^+} \Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}$$

$$f(x_1,x_2,\ldots) \longmapsto f(x_1\otimes 1, 1\otimes x_1, x_2\otimes 1, 1\otimes x_2, \ldots)$$

$$\Lambda_{\mathbb{Z}} \xrightarrow{\Delta^\times} \Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}$$

$$f(x_1,x_2,\ldots) \longmapsto f(\ldots, x_i\otimes x_j, \ldots)$$

The definition of Δ^+ (resp. $\Delta^\times$)

uses a choice of a bijection $\mathbb{N} \amalg \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ (resp. $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$); which one we choose does not matter, since we consider symm. functions.

Ex Calc. that ψ_p is ring-like:

$$\Delta^+(\psi_p) = \Delta^+(x_1^p + x_2^p + \cdots)$$

$$= (x_1 \otimes 1)^p + (1 \otimes x_1)^p + (x_2 \otimes 1)^p + (1 \otimes x_2)^p + \cdots$$

$$= \psi_p + \psi_p$$

$$\Delta^\times(\psi_p) = \Delta^\times(x_1^p + x_2^p + \cdots)$$

$$= \sum_{i,j} (x_i \otimes x_j)^p = (\sum_i x_i^p) \otimes (\sum_j x_j^p)$$

$$= \psi_p \otimes \psi_p . \quad //$$

Finally,

$$\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}} \xrightarrow{\circ} \Lambda_{\mathbb{Z}}$$

$$f \otimes g \longmapsto f(y_1, y_2, \cdots)$$

if $g = y_1 + y_2 + \cdots$ with y_j monomials in the x_i's.

All structure maps on $\Lambda_{\mathbb{Z}}$ descend to the sub-semiring

$$\Lambda_{\mathbb{N}} \subset \Lambda_{\mathbb{Z}}$$

defined to be the $\mathbb{N}$-span of the monomial basis $(m_\lambda)_{\lambda \text{ part}}$. So for R a semiring, define

$$W(R) = \mathrm{Hom}_{\mathbb{N}\text{-alg}}(\Lambda_{\mathbb{N}}, R).$$

We note that Newton's theorem fails: as a $\mathbb{N}$-algebra, $\Lambda_{\mathbb{N}}$ is not free. So the underlying set of $W(R)$ is not a product of copies of R unless R is a $\mathbb{Z}$-algebra. The semirings

$$W(\mathbb{R}_{\geq 0}), \; W(\mathbb{N}), \; W(\mathbb{B})$$

are all nice semirings, the latter two of which are countable. In addition, $W(\mathbb{N})$ is an integral domain,

$$W(\mathbb{N}) \hookrightarrow \mathcal{O}^{\text{hol}}(\mathrm{Re}(s) > 1).$$

What is $W(\mathbb{R}_{\geq 0}^{\text{trop}})$?