CANONICAL LIFTS AND δ-STRUCTURES

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Abstract. We extend the Serre–Tate theory of canonical lifts of ordinary abelian varieties to arbitrary families of ordinary abelian varieties parameterised by a p-adic formal scheme S. We show that the canonical lift is the unique lift to $W(S)$ which admits a δ-structure in the sense of Joyal and Buium.

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Introduction

Let p be a prime number. Serre and Tate proved that any ordinary abelian variety A over a perfect field k of characteristic p admits a distinguished lift to an abelian scheme $\widetilde{A}$ over $W(k)$, the ring of Witt vectors with entries in k. It is in fact the unique lift, up to unique isomorphism, which admits an endomorphism φ reducing to the Frobenius morphism modulo p and lying over the Frobenius endomorphism of $W(k)$.

The purpose of this paper is to extend this to ordinary abelian schemes over any p-adic base S:

Theorem. *Let A be an ordinary abelian scheme over S. Then A has a unique lift to an abelian scheme $\widetilde{A}$ over $W(S)$ admitting a $\delta_{W(S)}$-structure compatible with its group structure.*

The meaning of the terms in this theorem will be explained precisely in the body of the paper. But we can say a few words now informally. The δ-structures referred to are those of Joyal and Buium, generalised to our setting, and a $\delta_{W(S)}$-structure means a δ-structure compatible with the canonical δ-structure on $W(S)$. To a first approximation, a δ-structure is equivalent to a lift of the Frobenius endomorphism, but δ-structures are better in that they repair certain problems that Frobenius lifts have in the presence of p-torsion. (They also happen to be a p-typical analogue of the λ-ring structures of algebraic K-theory, and so have other names that reflect this, such as p-typical λ-structures, Λ_p-structures, and θ^p-structures, but we will not make use of this.)

The term p-adic in the theorem means that the map $S \to \mathrm{Spec}(\mathbf{Z})$ factors through $\mathrm{Spf}(\mathbf{Z}_p) \to \mathrm{Spec}(\mathbf{Z})$. So if S is affine, it means that p is nilpotent on S. In general, we allow S to be any sheaf of sets on the category of affine schemes with respect to the étale topology. So S can be any p-adic formal scheme. Note however that the passage from S affine to S arbitrary is purely formal. So the reader can take S to be affine with no real loss in generality.

Finally, the Witt vector construction $W(S)$ refers to the ind-object $\mathrm{colim}_n\, W_n(S)$. For example, if $S = \mathrm{Spec}(R)$, then $W(S)$ is the ind-scheme $\mathrm{colim}_n\, \mathrm{Spec}(W_n(R))$. In that case, an abelian scheme over the $W(S)$ can be identified with a compatible family of abelian schemes over the truncated Witt vector rings $W_n(R)$, as n varies.

A consequence of the theorem above is the following one:

Theorem. *The category of ordinary abelian schemes over S is equivalent to the category of abelian schemes over $W(S)$ equipped with a $\delta_{W(S)}$-structure compatible with the group structure.*

This confirms the philosophy (explicitly stated in [4] and implicit in much of Buium's work going back to [9]) that a $\delta_{W(S)}$-structure can be viewed as descent data for the non-existent map $W(S) \to S$.

The study of canonical lifts in families began with the papers of Finotti [13], answering a question of Mazur and Tate, and Erdoğan [11]. In our earlier paper [7], we observed that in the case of elliptic curves, the canonical lift construction extends rather formally to arbitrary families. But the approach there was different. There we worked throughout in the universal case, where the base S was the modular curve (with some level structure), and did not consider δ-structures except in some remarks. The non-algebraicity of the moduli space for abelian varieties of dimension $g > 1$ prevents such an approach from working here, at least as easily. So instead we stay over the given base S and single out the canonical lift using δ-structures. The moduli space and its smoothness do appear, but only briefly, in the proof of theorem 5.2.2.

1. δ-structures and Witt vectors

The primary purpose of this section is to define δ-structures on abelian schemes over Witt vector ind-schemes, or more generally on any sheaf of sets. This is done in 1.7. Before that, we recall the basic notions we will need in the theory of Witt vectors and δ-rings (also called Λ_p-rings, θ^p-rings, and p-typical λ-rings) from [15],[9],[8],[5],[6].

1.1. **δ-rings.** A δ-ring is a pair (R, δ) where R is a ring and $\delta\colon R \to R$ is a map satisfying the following identities:

(1) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$
(2) $\delta(xy) = x^p\delta(y) + \delta(x)y^p + p\delta(x)\delta(y)$
(3) $\delta(1) = 0$

A homomorphism of δ-rings $(R, \delta) \to (R', \delta')$ is a homomorphism $\alpha\colon R \to R'$ such that $\delta' \circ \alpha = \alpha \circ \delta$. This structure was introduced by Joyal [15] and later by Buium [9], independently and with different purposes in mind.

If (R, δ) is a δ-ring, then the map $\varphi\colon R \to R$ defined by

$$\varphi(x) = x^p + p\delta(x) \tag{1.1.1}$$

is a Frobenius lift, i.e., a ring homomorphism which reduces modulo p to the p-th power map. If R is p-torsion free, then the two structures determine each other: given any Frobenius lift $\varphi' \colon R \to R$, there exists a unique δ-ring structure on R such that $\varphi = \varphi'$; and a ring homomorphism between two p-torsion free δ-rings is a morphism of δ-rings if and only if it commutes with the Frobenius lifts.

But in the presence of p-torsion, a Frobenius lift is not a well-behaved structure—for instance, the category of rings with Frobenius lift does not have equalisers. The δ-ring structure repairs such flaws and gives an intelligent generalisation of the structure of a Frobenius lift to all rings. Some evidence for this is the fact that the forgetful functor $(R, \delta) \mapsto R$ from δ-rings to rings admits both a right adjoint

$$R \mapsto (W(R), \delta_{W(R)}) \tag{1.1.2}$$

and a left adjoint

$$R \mapsto (J(R), \delta_{J(R)}). \tag{1.1.3}$$

In fact, more is true, this forgetful functor is comonadic, meaning that the category of δ-rings agrees with the category of coalgebras for W, viewed as a comonad on the category of rings. The forgetful functor is also monadic: δ-rings are algebras for the monad J.

1.2. **Witt vectors.** The ring $W(R)$ is in fact the usual ring of p-typical Witt vectors with entries in R, and the corresponding Frobenius lift φ is the Witt vector Frobenius, more commonly denoted F. For formal reasons, $W(R)$ has the following description, which is not the traditional one. As sets, we have $W(R) = R \times R \times R \times \cdots$ and $\delta(x_0, x_1, \dots) = (x_1, x_2, \dots)$. The ring structure is given by laws

$$\begin{aligned}(x_0, x_1, \dots) + (y_0, y_1, \dots) &= (S_0, S_1, \dots)\\ (x_0, x_1, \dots) \cdot (y_0, y_1, \dots) &= (P_0, P_1, \dots)\end{aligned}$$

where $S_n = S_n(x_0, y_0, \dots, x_n, y_n)$ is the polynomial with integer coefficients giving the Leibniz rule for addition for the operation $\delta^{\circ n}$, in the sense that

$$\delta^{\circ n}(x + y) = S_n(x, y, \delta(x), \delta(y), \dots, \delta^{\circ n}(x), \delta^{\circ n}(y)),$$

and P_n is similarly the polynomial giving the Leibniz rule for multiplication for $\delta^{\circ n}$. Such polynomials exist because we can apply the basic Leibniz rules (1)–(3) above repeatedly, and they can be seen to be unique by reducing to the case of p-torsion free rings, where a δ-structure is equivalent to a Frobenius lift. The additive and multiplicative neutral elements are $(0, 0, 0, \dots)$ and $(1, 0, 0, \dots)$.

There is a canonical isomorphism between $W(R)$ as defined above and Witt's original construction, as defined for example in Serre's book [20]. This follows from the fact that the original construction satisfies the universal property of ours. For this fact, one can see p. 215 of Lazard's book [18], keeping in mind that it is enough to restrict to rings R which are p-torsion free since both functors are represented by rings which are p-torsion free—in fact by polynomial rings. Beware however that the canonical isomorphism between our construction and Witt's is not the identity! It is the identity on the components x_0 and x_1 but not on x_2 and higher. One could say that there are two different coordinate systems on the same functor—the ones above, which we call the *Buium–Joyal components*, and the traditional ones, which we call the *Witt components*. The Buium–Joyal components are directly connected to a simple universal property, as above, and hence are usually better for conceptual purposes. For example, the comonad structure map (or coplethysm or Artin–Hasse map)

$$\Delta\colon W(R) \to W(W(R))$$

does not have a simple explicit description in terms of the Witt components, but in terms of the Buium–Joyal components it does:

$$\Delta\colon (x_0, x_1, \dots) \mapsto ((x_0, x_1, \dots), (x_1, x_2, \dots), (x_2, x_3, \dots), \dots).$$

On the other hand, the Witt components are closer to the Verschiebung operator and are sometimes more convenient for computations.

1.3. **Truncations.** It also follows from the construction of $W(R)$ above that for any integer $n \geq 0$, the quotient

$$W_n(R) = R^{n+1}$$

of $W(R)$ consisting of truncated vectors $(x_0, \dots, x_n)$ is a quotient ring. Indeed, the Leibniz rules for $\delta^{\circ n}$ depend only on the operators $\delta^{\circ i}$ for $i \leq n$. Then $W(R)$ is naturally identified with the limit of the projective system of rings

$$\cdots \longrightarrow W_n(R) \xrightarrow{\tau} \cdots \xrightarrow{\tau} W_1(R) \xrightarrow{\tau} W_0(R)$$

given by the truncation maps $\tau\colon (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{n-1})$.

For $n \geq 0$ the Verschiebung ideal $V^{n+1}W(R)$ is defined to be the kernel of the truncation map:

$$0 \to V^{n+1}W(R) \to W(R) \to W_n(R) \to 0.$$

It can also be expressed as the image of the n-th iterate of the usual Verschiebung operator $V\colon W(R) \to W(R)$, but since it is not effortless to define V in terms of the Buium–Joyal components and since we will not need it, we can ignore this.

The operator $\delta\colon W(R) \to W(R)$ descends to the truncations but only at the expense of a shift in degree: we have a set map $\delta\colon W_n(R) \to W_{n-1}(R)$ given by

$$\delta(x_0, \dots, x_n) = (x_1, \dots, x_n).$$

Similarly, the Frobenius map $\varphi\colon W(R) \to W(R)$ descends to a ring homomorphism $W_n(R) \to W_{n-1}(R)$ given by $\varphi(x) = \tau(x)^p + p\delta(x)$.

The comonad structure map also descends to the truncations in a degree-shifting sense. It becomes a family of maps

$$\Delta\colon W_{m+n}(R) \to W_n(W_m(R)) \tag{1.3.1}$$

which send a Witt vector $(x_0, \dots, x_{m+n})$ to

$$\big((x_0, x_1, \dots, x_m), (x_1, x_2, \dots, x_{m+1}), \dots, (x_n, x_{n+1}, \dots, x_{m+n})\big). \tag{1.3.2}$$

1.4. **Ghost and coghost maps.** Given a δ-ring (R, δ), we have an associated pair (R, φ) of a ring with an endomorphism, where φ is the Frobenius lift defined in (1.1.1). This defines a functor from the category of δ-rings to the category of rings with an endomorphism.

The forgetful functor from the category of rings with endomorphism to rings also has a right adjoint $R \mapsto (\Pi(R), \varphi_{\Pi(R)})$, where $\Pi(R)$ is $R \times R \times \cdots$ with the product ring structure and where $\varphi_{\Pi(R)}$ is defined by

$$\varphi_{\Pi(R)}\colon \langle z_0, z_1, \dots \rangle \mapsto \langle z_1, z_2, \dots \rangle.$$

The induced map of rings with endomorphism

$$w\colon W(R) \to \Pi(R)$$

is the so-called ghost map. In coordinates, it sends a Witt vector $(x_0, x_1, \dots)$ to $\langle Z_0, Z_1, \dots \rangle$, where $Z_n = Z_n(x_0, \dots, x_n)$ is the integral polynomial which expresses $\varphi^{\circ n}$ in terms of the iterates of δ:

$$\varphi^{\circ n} = Z_n(\mathrm{id}, \delta, \delta^{\circ 2}, \dots, \delta^{\circ n}).$$

Once again, such a polynomial exists because we can apply the Leibniz rules (1)–(3) repeatedly, and one can show it is unique by considering the p-torsion free case. Finally, the ghost map descends to a homomorphism on the truncations:

$$w_n \colon W_n(R) \to R^{n+1},$$

where the target has the product ring structure.

1.5. **Witt vectors and jet spaces of sheaves.** The functor W_n on rings defines a functor on the category Aff of affine schemes, which we also denote by W_n:

$$\mathrm{Aff} \xrightarrow{W_n} \mathrm{Aff}$$
$$\mathrm{Spec}(B) \mapsto \mathrm{Spec}(W_n(B)).$$

To pass to non-affine schemes, we need to know this functor is well behaved with respect to localisation. This is provided by the following theorem:

Theorem 1.5.1. *If a morphism $R \to R'$ of rings is étale, then so is the induced map $W_n(R) \to W_n(R')$. If in addition $R \to S$ is an arbitrary ring map, then the canonical map $W_n(R') \otimes_{W_n(R)} W_n(S) \to W_n(R' \otimes_R S)$ is an isomorphism.*

For a proof, see 9.2 and 9.4 of [5]. We will only need this for rings in which p is nilpotent, in which case it was proved earlier in the appendix of Langer–Zink [17].

Let now equip Aff with the étale topology (SGA4 [1], exp. VII) and let Sh denote the corresponding category of sheaves of sets. Given any sheaf $X \in \mathrm{Sh}$, define the n-th arithmetic jet space (or Greenberg transform) $J^n(X) \colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$ by

$$J^n(X) \colon T \mapsto X(W_n(T)).$$

This is in fact a sheaf:

Theorem 1.5.2. *The presheaf $J^n(X)$ on Aff is a sheaf in the étale topology. The resulting functor $J^n \colon \mathrm{Sh} \to \mathrm{Sh}$ has a left adjoint $W_n \colon \mathrm{Sh} \to \mathrm{Sh}$. The functor W_n agrees with the usual Witt vector functor on the category of affine schemes:*

$$W_n(\mathrm{Spec}\, R) \xrightarrow{\sim} \mathrm{Spec}(W_n(R))$$

This theorem follows by general sheaf theory from theorem 1.5.1.

Theorem 1.5.3. *If X is a scheme (or an algebraic space), then so are $J^n(X)$ and $W_n(X)$.*

Proof. See 12.1, 15.1, 15.6 of [6]. □

We pass to $n = \infty$, by taking a limit:

$$J(X)(T) := \lim_n \big(J^n(X)(T)\big) = \lim_n X(W_n(T))$$

So for a ring C, a C-point of $J(X)$ is a compatible family of $W_n(C)$-points of X. In the affine case, $X = \mathrm{Spec}\, R$, we have

$$J(\mathrm{Spec}\, R)(\mathrm{Spec}\, C) = \lim_n \mathrm{Hom}(R, W_n(C)) = \mathrm{Hom}(R, W(C)) = \mathrm{Hom}(J(R), C),$$

where $J(R)$ denotes the free δ-ring on R, as in (1.1.3). Thus we have a canonical identification

$$J(\operatorname{Spec} R) = \operatorname{Spec}(J(R)),$$

and so the functor J we just defined on sheaves agrees with the one already defined on rings.

The truncation maps $\tau\colon W_{n+1}(R) \to W_n(R)$ on rings induce functorial projections

$$u\colon J^{n+1}(X) \to J^n(X), \tag{1.5.1}$$

for $X \in \mathrm{Sh}$. They in turn induce functorial maps $W_n(X) \to W_{n+1}(X)$ for $X \in \mathrm{Sh}$, and hence an inductive system

$$W_0(X) \to W_1(X) \to \cdots .$$

We then define

$$W(X) = \operatorname*{colim}_n W_n(X).$$

Observe that it does not matter whether we take this colimit in the category of sheaves or presheaves, since every object of Aff is quasi-compact and quasi-separated in the étale topology. Also, we emphasise that generally one has

$$W(\operatorname{Spec} R) \neq \operatorname{Spec}(W(R)),$$

unlike the case for W_n for n finite. For example, if $R = \mathbf{Z}/p\mathbf{Z}$, this becomes the familiar fact $\operatorname{Spf}(\mathbf{Z}_p) \neq \operatorname{Spec}(\mathbf{Z}_p)$.

Similarly, the maps $\varphi\colon W_n(R) \to W_{n-1}(R)$ induce maps $\varphi\colon J_n(X) \to J_{n-1}(X)$ whose limit

$$\varphi_{J(X)}\colon J(X) \to J(X) \tag{1.5.2}$$

under the projections u is a lift of the Frobenius. We also obtain a lift of the Frobenius on the Witt vectors

$$\varphi_{W(X)}\colon W(X) \to W(X).$$

The ghost maps $w_n\colon W_n(C) \to C^{n+1}$ induce maps on sheaves

$$w_n\colon \coprod_{n+1} X \to W_n(X)$$

which we call the also ghost maps. In the colimit, they become a map

$$w\colon \coprod_{\mathbf{N}} X \to W(X).$$

The map $X \to W(X)$ obtained by restricting w along the inclusion $X \to \coprod_{\mathbf{N}} X$ corresponding to $0 \in \mathbf{N}$ is called the first ghost component.

Finally, we also have *coghost* maps which are dual to the ghost maps:

$$\gamma_n\colon J^n(X) \to X^{n+1},$$

On C-points, γ_n is defined to be the map

$$J^n(X)(C) = X(W_n(C)) \xrightarrow{X(w_n)} X(C^{n+1}) = X^{n+1}(C).$$

The coghost maps are compatible with the projections u and hence pass to a map

$$\gamma\colon J(X) \to X^{\mathbf{N}}$$

in the limit.

Theorem 1.5.4 (cf. 15.2, 15.3 of [6]). *Let $Y = \operatorname{colim}_i Y_i$ be an ind-algebraic space and let $X \to Y$ be a morphism which is representable (by algebraic spaces) and étale. Then for all $m, n \leq \infty$, the following hold:*

(i) *The induced map $W_n(X) \to W_n(Y)$ is representable and étale.*

(ii) *The induced map*
$$W_n(X) \to W_{n+m}(X) \times_{W_{n+m}(Y)} W_n(Y)$$
is an isomorphism.

(iii) *For all morphisms $Y' \to Y$ where Y' is an algebraic space the natural map*
$$W_n(X \times_Y Y') \to W_n(X) \times_{W_n(Y)} W_n(Y')$$
is an isomorphism.

Proof. First observe that the case where $m, n < \infty$ implies the general case. Indeed, (ii) and (iii) follow from the finite case by taking colimits; then (i) then follows from (ii) and (iii) and the finite case. So it is enough to assume $m, n < \infty$.

Second observe that if X, Y, Y' are algebraic spaces, this the theorem is 15.2 and 15.3 of [6].

Now write $X_i = X \times_Y Y_i$. Then X_i is an algebraic space because X is representable over Y.

(iii) If Y' is quasi-compact and quasi-separated, then $Y' \to Y$ factors through $Y_i \to Y$ for some i. So we have
$$W_n(X \times_Y Y') = W_n(X_i \times_{Y_i} Y') = W_n(X_i) \times_{W_n(Y_i)} W_n(Y'),$$
since the result holds for algebraic spaces. Hence we obtain the commutative diagram
$$\begin{array}{ccc} W_n(X \times_Y Y') & \longrightarrow & W_n(X) \times_{W_n(Y)} W_n(Y') \\ \| & & \| \\ W_n(X_i) \times_{W_n(Y_i)} W_n(Y') & \longrightarrow & (W_n(X) \times_{W_n(Y)} W_n(Y_i)) \times_{W_n(Y_i)} W_n(Y'). \end{array}$$
Therefore, it is enough to show that
$$W_n(X_i) \to W_n(X) \times_{W_n(Y)} W_n(Y_i)$$
is an isomorphism. However, this is nothing but the colimit of the maps
$$W_n(X_i) \to W_n(X_j) \times_{W_n(Y_j)} W_n(Y_i)$$
for $j \geq i$, which are isomorphisms by 15.2 of [6].

For general Y'_λ, we can reduce to the case where Y' is quasi-compact and quasi-separated by taking an affine cover $Y'_\lambda \to Y'$. For we then have
$$\begin{array}{rcl} W_n(X \times_Y Y') \times_{W_n(Y')} W_n(Y'_\lambda) & = & W_n(X \times_Y Y'_\lambda) \\ & \xrightarrow{\sim} & W_n(X) \times_{W_n(Y)} W_n(Y'_\lambda) \\ & = & (W_n(X) \times_{W_n(Y)} W_n(Y')) \times_{W_n(Y')} W_n(Y'_\lambda) \end{array}$$
and so the map
$$W_n(X \times_Y Y') \longrightarrow W_n(X) \times_{W_n(Y)} W_n(Y')$$
is an isomorphism.

(ii) The map in question is the filtered colimit over i of the maps
$$W_n(X_i) \to W_{n+m}(X_i) \times_{W_{n+m}(Y_i)} W_n(Y_i)$$

which are isomorphisms by 15.4 of [6].

(i) The family $(W_n(Y_i) \to W_n(Y))_i$ is a cover in Sh and a morphism being representable by algebraic spaces and étale is local in the étale topology, so that it is enough to show the claim after base change along each $W_n(Y_i) \to W_n(Y)$. The map

$$W_n(X) \times_{W_n(Y)} W_n(Y_i) \to W_n(Y_i)$$

under the isomorphism

$$W_n(X) \times_{W_n(Y)} W_n(Y_i) \xrightarrow{\sim} W(X_i)$$

of (iii) is just the natural map $W_n(X_i) \to W_n(Y_i)$ which is étale by 15.4 of [6]. $\square$

1.6. **δ-structures on sheaves.** The monad structure on the ring functor $R \mapsto J(R)$ of (1.1.3) prolongs naturally to a comonad structure on the sheaf functor $J\colon \mathrm{Sh} \to \mathrm{Sh}$, as we now explain. The truncated comonad maps of (1.3.1) induce maps between iterated truncated jet spaces:

$$\nu_{m,n}\colon J^{m+n}(X) \to J^m(J^n(X)) \tag{1.6.1}$$

and hence in the limit

$$\nu\colon J(X) \to J(J(X))$$

This is the comultiplication map $J \to J \circ J$ for the comonad structure. The co-unit is given by the projection $u\colon J(X) \to J^0(X) = X$. We also note that since W is the left adjoint of J, it inherits a monad structure from the comonad structure on J.

We define a δ-structure on a sheaf X to be a co-action of the comonad J on X. This means a map $\alpha\colon X \to J(X)$ such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & J(X) \arrow[d, "J(\alpha)"] \\ J(X) \arrow[r, "\nu"] & J(J(X)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\mathrm{id}_X"'] & J(X) \arrow[d, "u"] \\ & X \end{tikzcd}$$

This definition could equivalently be given as an action of the monad W on X and we shall often use this definition instead.

If X is affine, this definition of δ-structure agrees with the one in section 1.1 above: there is a canonical bijection between the set of δ-structures on $\mathrm{Spec}(R)$ and the set of δ-structures on R. The category of δ-sheaves and δ-morphisms will be denoted by Sh^δ.

Finally, if X is a δ-sheaf, writing $\varphi_X\colon X \to X$ for the composition

$$X \xrightarrow{\alpha} J(X) \xrightarrow{\varphi_{J(X)}} J(X) \xrightarrow{u} X.$$

we find that $\varphi_X\colon X \to X$ is a lift of the Frobenius. (Note that when $X = J(Y)$, this map agrees with the previously defined map $\varphi_{J(Y)}$.) Therefore, every δ-sheaf is equipped with a lift of the Frobenius and all δ-morphisms are compatible with these lifts.

1.7. **Relative δ-structures.** Let S be a sheaf with a δ-structure and denote by Sh^{δ}_S the slice category of Sh^{δ} over S. Just as Sh^{δ} is comonadic over Sh, Sh^{δ}_S is comonadic over Sh_S, although in this case not by definition. The comonad is

$$X \mapsto J_S(X) := J(X) \times_{J(S)} S$$

where the implicit map $\alpha\colon S \to J(S)$ is the δ-structure map on S. Since both α and the map $J(X) \to J(S)$ are morphisms of δ-sheaves, $J_S(X)$ inherits a canonical δ-structure and is the fibre product in the category of δ-sheaves. As in the absolute setting, we have truncations

$$J^n_S(X) = J^n(X) \times_{J^n(S)} S$$

and comultiplication maps

$$\nu_{m,n}\colon J^{m+n}_S(X) \longrightarrow J^m_S J^n_S(X) \tag{1.7.1}$$

induced by the absolute comultiplication maps (which we abusively also denote $\nu_{m,n}$).

It is also true that Sh^{δ}_S is monadic over Sh. The monad is just $X \mapsto W(X)$ again, but here $W(X)$ is viewed as an object of Sh^{δ}_S via the composition

$$W(X) \longrightarrow W(S) \longrightarrow S$$

with the (adjunct) δ-structure map $W(S) \to S$. So with some abuse of notation, we will still use W to denote this monad on Sh^{δ}_S.

Given a δ_S-sheaf X, the Frobenius lift $\varphi_X\colon X \to X$ does not lie over the identity map on S but over the Frobenius lift $\varphi_S\colon S \to S$. Therefore, we obtain a relative Frobenius lift

$$\varphi_{X/S}\colon X \to \varphi_S^*(X) \tag{1.7.2}$$

which is a morphism in Sh^{δ}_S.

The absolute coghost maps $\gamma_n\colon J^n(S) \to S^{n+1}$ and $\gamma_n\colon J^n(X) \to X^{n+1}$ induce relative coghost maps

$$\gamma_{X/S,n}\colon J^n_S(X) = J^n(X) \times_{J^n(S)} S \to X^{n+1} \times_{S^{n+1}} S = \prod_{i=0}^{n} \varphi_S^{\circ i*}(X),$$

and taking the limit of these maps, a relative coghost map in the infinite-length setting:

$$\gamma_{X/S}\colon J_S(X) \to X^{\mathbf{N}} \times_{S^{\mathbf{N}}} S. \tag{1.7.3}$$

Since the forgetful functor $\mathrm{Sh}^{\delta}_S \to \mathrm{Sh}_S$ is both monadic and comonadic, all limits and colimits exist in Sh^{δ}_S, and the underlying S-sheaf of a limit or colimit can be computed in Sh_S. In particular, a group structure on an object $X \in \mathrm{Sh}^{\delta}_S$ is the same as a group structure on the underlying S-sheaf such that the structure morphisms (multiplication, identity, and inverse) are morphisms of δ_S-sheaves.

2. Further properties of δ-structures and Witt vectors

The purpose of this section is to establish some basic results about δ-structure and Witt vectors used in later sections. It can safely be skipped and referred back to as needed.

2.1. **Relative δ-structures are δ-local with respect to the base.** Let S be a δ-sheaf and let X be an S-sheaf. Then for each δ_S-sheaf S' we can consider the set

$$\delta_{X/S}(S') := \{\delta_{S'}\text{-structures on } X \times_S S'\}.$$

If $S'' \to S'$ is a morphism of δ_S-sheaves and $X \times_S S'$ is equipped with a $\delta_{S'}$-structure then the fibre product $(X \times_S S') \times_{S'} S'' = X \times_S S''$ is a $\delta_{S''}$-sheaf, or in other words, we are given a $\delta_{S''}$-structure on $X \times_S S''$. This gives a map

$$\delta_{X/S}(S') \to \delta_{X/S}(S'')$$

which makes the assignment $S' \mapsto \delta_{X/S}(S')$ a presheaf on Sh^{δ}_{S}.

Lemma 2.1.1. *The functor $\delta_{X/S}$ is a sheaf (for the canonical topology on Sh^{δ}_{S}).*

Proof. Let $S'' \to S'$ be an epimorphism of δ_S-sheaves and write $S''' = S'' \times_{S'} S''$. Then

$$X \times_S S''' \rightrightarrows X \times_S S'' \longrightarrow X \times_S S'$$

is a coequaliser diagram in the category of S-sheaves.

If $X \times_S S'''$ is given a $\delta_{S'''}$-structure and $X \times_S S''$ is given a $\delta_{S''}$-structure such that the two maps $X \times_S S''' \rightrightarrows X \times_S S''$ are $\delta_{S''}$ maps then the S'-sheaf underlying the coequaliser of the two maps $X \times_S S''' \rightrightarrows X \times_S S''$ in the category of $\delta_{S'}$-sheaves is $X \times_S S'$, because $\mathrm{Sh}^{\delta}_{S'}$ is comonadic over $\mathrm{Sh}_{S'}$. Hence, $X \times_S S'$ admits a unique $\delta_{S'}$-structure making the map $X \times_S S'' \to X \times_S S'$ a $\delta_{S'}$-morphism. This is equivalent to the functor $\delta_{X/S}$ being a sheaf for the canonical topology on Sh^{δ}_{S}. □

2.2. **δ-structures on p-adic sheaves.** Let us say that a sheaf $S \in \mathrm{Sh}$ is *p-adic* if the structure map $S \to \mathrm{Spec}(\mathbf{Z})$ factors through

$$\mathrm{Spf}(\mathbf{Z}_p) = \operatorname*{colim}_n \mathrm{Spec}(\mathbf{Z}/p^n) \subset \mathrm{Spec}(\mathbf{Z}).$$

For example, an affine scheme $\mathrm{Spec}(R)$ is p-adic if and only if p is nilpotent in R.

Proposition 2.2.1. *If S is a p-adic scheme, then so is $W_n(S)$ and the natural maps*

$$W_n(S) \to W_{n+m}(S)$$

are nilpotent immersions.

Proof. Using theorem 1.5.4, we may take an affine cover of S and assume that $S = \mathrm{Spec}(R)$ where p is nilpotent on R and then show that the kernels of the truncation maps $W_{n+m}(R) \to W_n(R)$ are nilpotent. It is further enough to take $m = 1$. Then the kernel $VW_n(R)$ of $W_{n+1}(R) \to W_n(R)$ satisfies $(VW_n(R))^2 \subset pW_n(R)$, and so it is enough to show that p is nilpotent in $W_n(R)$. However, as R is a $\mathbf{Z}/p^{i+1}\mathbf{Z}$-algebra for some i, and $\mathbf{Z}/p^{i+1}\mathbf{Z} = W_i(\mathbf{F}_p)$ it follows that $W_n(R)$ is a $W_n(W_i(\mathbf{F}_p))$-algebra. Therefore it is enough to show that p is nilpotent in $W_n(W_i(\mathbf{F}_p))$. But the comonad comultiplication map

$$W_{n+i}(\mathbf{F}_p) \to W_n(W_i(\mathbf{F}_p))$$

makes $W_n(W_i(\mathbf{F}_p))$ an algebra over $W_{n+i}(\mathbf{F}_p) = \mathbf{Z}/p^{n+i+1}\mathbf{Z}$, in which p is nilpotent. □

For any sheaf S let Et_S denote the category of relatively representable étale algebraic spaces over S. If S also has a δ-structure write Et^{δ}_{S} for the category of relatively representable étale δ_S-sheaves.

Proposition 2.2.2. *If S is a p-adic ind-scheme, the functor*

$$\mathrm{Et}^{\delta}_{W(S)} \longrightarrow \mathrm{Et}_S, \quad Z \mapsto Z \times_{W(S)} S$$

is an equivalence of categories with quasi-inverse W.

Proof. First, observe that we may assume S is a p-adic scheme because if $S = \mathrm{colim}_i\, S_i$, the categories in question are the 2-limits of the categories Et_{S_i} and $\mathrm{Et}^{\delta}_{W(S_i)}$ respectively.

Now the functor in question factors:

$$\mathrm{Et}^{\delta}_{W(S)} \hookrightarrow \mathrm{Et}_{W(S)} \to \mathrm{Et}_S.$$

The second functor is an equivalence because of the equivalences

$$\mathrm{Et}_{W_n(S)} \xrightarrow{\sim} \mathrm{Et}_S$$

for all $n \geq 0$, which follow from proposition 2.2.1. Further $W\colon \mathrm{Et}_S \to \mathrm{Et}_{W(S)}$ is a right quasi-inverse of the functor $\mathrm{Et}_{W(S)} \to \mathrm{Et}_S$, by theorem 1.5.4. But W has essential image in the subcategory $\mathrm{Et}^{\delta}_{W(S)}$. Therefore $\mathrm{Et}^{\delta}_{W(S)} \hookrightarrow \mathrm{Et}_S$ is an equivalence and W is a two-sided quasi-inverse. □

Proposition 2.2.3. *Let $S = \mathrm{Spec}(A)$, where A is a ring in which p is nilpotent. Then the category of finite locally free $\delta_{W(S)}$-schemes is equivalent to the category of finite locally free $\delta_{W(A)}$-algebras.*

Proof. As $W(A) = \lim_n W_n(A)$ and the transition maps are surjective with nilpotent kernels, by proposition 2.2.1, it follows that the categories of finite locally free $W(S)$-schemes and finite locally free $W(A)$-schemes are equivalent, with the $W(A)$-scheme $\mathrm{Spec}(B)$ corresponding to the $W(S)$-scheme $\mathrm{colim}_n\, \mathrm{Spec}(W_n(A) \otimes_{W(A)} B)$ and the $W(S)$-scheme T corresponding to the $W(A)$-scheme $\mathrm{Spec}(\lim_n B_n)$, where $\mathrm{Spec}(B_n) = T \times_{W(S)} W_n(S)$. (We leave the argument to the reader.)

Now, to give a $\delta_{W(A)}$-structure on B is equivalent to giving a $W(A)$-morphism $B \to W(B)$ satisfying certain properties. However, this is also equivalent to giving a family of morphisms

$$B_{m+n} \to W_n(B_m)$$

for $n, m \geq 0$ which in turn is equivalent to giving morphisms

$$W_n(T_m) \to T_{n+m}$$

for $n, m \geq 0$ whose colimit defines a $W(S)$-morphism

$$W(T) \to T.$$

Tracing through these equivalences, we see that to give a δ_S-structure on T is equivalent to giving a $\delta_{W(A)}$-structure on B.

□

Lemma 2.2.4. *Let S be a p-adic sheaf and let $\widetilde{G}$ and $\widetilde{H}$ be two $\delta_{W(S)}$-groups over $W(S)$. If $\widetilde{H}$ is relatively representable and étale over $W(S)$, then the natural map*

$$\mathrm{Hom}_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G}) \to \mathrm{Hom}_S(H, G)$$

from $\delta_{W(S)}$-homomorphisms to S-homomorphisms is bijective.

Proof. Any map $f\colon H \to G = \widetilde{G} \times_{W(S)} S$ lifts by adjunction to a unique $\delta_{W(S)}$-map $\widetilde{f}\colon W(H) \to \widetilde{G}$. However, $\widetilde{H} = W(H)$ and the functor W commutes with étale fibre products (by proposition 2.2.2), from which it follows that $\widetilde{f}\colon \widetilde{H} \to \widetilde{G}$ is a homomorphism if and only if $f\colon H \to G$ is a homomorphism. □

2.3. **The sufficiency of J^1 for ind-affine sheaves.** Putting a δ-ring structure on a ring is equivalent to putting an action of the comonad W on it. This is true nearly by definition, namely our definition of W. The purpose of this section is to prove the analogous result for ind-affine sheaves, where it appears to require proving something.

We will also work in the relative setting. So fix a δ-sheaf S. Let $(\mathrm{Aff}_{S,\mathrm{ind}})^\delta$ denote the full subcategory of Sh_S^δ consisting of the objects which are ind-affine. Let $\mathscr{C}$ denote the category of pairs (X, β), where X is an ind-affine sheaf over S and $\beta\colon X \to J_S^1(X)$ is a section of the projection $u\colon J_S^1(X) \to X$ and where a morphism $(X, \beta) \to (X', \beta')$ is a morphism $f\colon X \to X'$ of Sh_S compatible with the sections in the evident sense: $\beta' \circ f = J_S^1(f) \circ \beta$.

Consider the functor

$$(\mathrm{Aff}_{S,\mathrm{ind}})^\delta \longrightarrow \mathscr{C}, \quad X \mapsto (X, \alpha_1) \tag{2.3.1}$$

where α_1 is the composition

$$\alpha_1\colon X \xrightarrow{\alpha} J_S(X) \xrightarrow{u} J_S^1(X)$$

and where α is the structure map of the δ_S-structure on X.

Proposition 2.3.1. *The functor (2.3.1) is an equivalence of categories.*

One might say that W, viewed as a monad on ind-affine sheaves, is freely generated by the pointed functor W_1. Before we prove it, we will need some preliminary results.

Lemma 2.3.2. *For any $n \geq 1$ and any ring R, the diagram*

$$W_{n+1}(R) \xrightarrow{\Delta} W_n W_1(R) \underset{\Delta \circ W_n(\tau)}{\overset{\tau}{\rightrightarrows}} W_{n-1} W_1(R)$$

is an equaliser diagram.

Proof. This is immediate in Buium–Joyal components. Consider a Witt vector

$$z = \big((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\big) \in W_n W_1(R).$$

Then we have

$$\tau(z) = \big((x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\big)$$

and

$$\Delta \circ W_n(\tau)(z) = \Delta(x_0, \ldots, x_n) = \big((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\big).$$

These two Witt vectors are equal if and only if

$$y_0 = x_1, \quad y_1 = x_2, \quad \ldots, \quad y_{n-1} = x_n.$$

But by the explicit description of $\Delta\colon W_{n+1}(R) \to W_n W_1(R)$ given in (1.3.2), this is precisely the condition for z to lie in the image of Δ and, second, Δ is a monomorphism. □

Remark 2.3.3. The map $\Delta\colon W_{n+1}(R) \to W_nW_1(R)$ even has a functorial retraction. In ghost components, it is given by

$$\big\langle \langle a_0, b_0\rangle, \dots, \langle a_n, b_n\rangle \big\rangle \mapsto \langle a_0, a_1, \dots, a_{n-1}, a_n, b_n\rangle.$$

Proposition 2.3.4. *If X is ind-affine, the diagram*

$$J_S^{n+1}(X) \xrightarrow{\nu_{1,n}} J_S^1 J_S^n(X) \underset{\nu_{1,n-1}\circ u}{\overset{J_S^1(u)}{\rightrightarrows}} J_S^1 J_S^{n-1}(X)$$

is an equaliser diagram.

Proof. For any affine S-scheme $\mathrm{Spec}(R)$, the diagram of R-points is

$$X(W_{n+1}(R)) \xrightarrow{\Delta} X(W_nW_1(R)) \underset{\Delta\circ W_n(\tau)}{\overset{\tau}{\rightrightarrows}} X(W_{n-1}W_1(R)) \; .$$

It is enough to show that this is an equaliser diagram. If X is affine, this is an immediate consequence of lemma 2.3.2. If X is ind-affine, the result follows formally from the affine case and the fact that for filtered colimits we have

$$(\operatorname*{colim}_i X_i)(C) = \operatorname*{colim}_i \big(X_i(C)\big)$$

for any ring C and the fact that finite limits are preserved by filtered colimits. $\square$

Proof of proposition 2.3.1. First consider the category $\mathscr{C}'$ consisting of pairs (X, β), where $X \in \mathrm{Sh}_S$ and β is any morphism $X \to J_S^1(X)$, and where a morphism $(X,\beta) \to (X', \beta')$ is any morphism $X \to X'$ compatible with β and β'. (This is the category of so-called co-algebras for the functor J_S^1.) Then $\mathscr{C}$ is a full subcategory of $\mathscr{C}'$, and $\mathscr{C}'$ is comonadic over Sh_S with comonad $X \mapsto \prod_n (J_S^1)^{\circ n}(X)$. Therefore it is enough to show that for any object (X, β) of $\mathscr{C}$ and for any $n \geq 1$, the composition

$$\beta_n\colon X \xrightarrow{\beta} J_S^1(X) \xrightarrow{J_S^1(\beta)} J_S^1 J_S^1(X) \longrightarrow \cdots \longrightarrow (J_S^1)^{\circ n}(X)$$

factors through the monomorphism

$$\nu_n\colon J_S^n(X) \longrightarrow (J_S^1)^{\circ n}(X)$$

obtained by iterating the maps $\nu_{1,m-1}\colon J_S^m(X) \to J_S^1 J_S^{m-1}(X)$ starting at $m = n$ and going down to $m = 2$. We do this by induction, the case $n = 1$ being clear. So assume β_n factors through ν_n, yielding a map $\alpha_n\colon X \to J_S^n(X)$.

First, let β'_{n+1} denote the composition

$$X \xrightarrow{\beta} J_S^1(X) \xrightarrow{J_S^1(\alpha_n)} J_S^1 J_S^n(X),$$

and let

$$\nu'_{n+1}\colon J_S^{n+1}(X) \longrightarrow J_S^1 J_S^n(X)$$

denote the comultiplication map $\nu_{1,n}$. When composed with the monomorphism $J_S^1(\nu_n)\colon J_S^1 J_S^n(X) \to (J_S^1)^{\circ(n+1)}(X)$, the map β'_{n+1} becomes β_{n+1} and ν'_{n+1} becomes ν_{n+1}. Therefore we have $\nu'_n \circ \alpha_n = \beta'_n$, since $\nu_n \circ \alpha_n = \beta_n$.

Now, to prove that β_{n+1} factors through ν_{n+1}, it is certainly enough to prove that β'_{n+1} factors through ν'_{n+1}. This will then follow from proposition 2.3.4 by the universal property of equalisers, once we verify the equation

$$J_S^1(u) \circ (J_S^1(\alpha_n) \circ \beta) = \nu'_n \circ u \circ (J_S^1(\alpha_n) \circ \beta)$$

But this holds because both sides equal β'_n:

$$J^1_S(u) \circ J^1_S(\alpha_n) \circ \beta = J^1_S(\alpha_{n-1}) \circ \beta = \beta'_n$$

and

$$\nu'_n \circ u \circ J^1_S(\alpha_n) \circ \beta = \nu'_n \circ \alpha_n \circ u \circ \beta = \nu'_n \circ \alpha_n = \beta'_n.$$

$\square$

2.4. **δ-structures on ind-affine sheaves.** Let S be a ind-affine sheaf. We say that S is *p-torsion free* if it is of the form $\operatorname{colim}_i \operatorname{Spec}(R_i)$, where the pro-abelian group $(R_i[p])_{i\in I}$ is isomorphic to the zero pro-abelian group, where $R_i[p]$ denotes the group of p-torsion elements in R_i. This is equivalent to saying that for any $i \in I$, there exists a $j \geq i$ such that the map $R_j[p] \to R_i[p]$ is zero.

Proposition 2.4.1. *The forgetful functor from the category of p-torsion free ind-affine δ-sheaves to the category of p-torsion free ind-affine sheaves equipped with a lift of the Frobenius is an equivalence.*

Proof. Let $S = \operatorname{colim}_{i\in I} \operatorname{Spec}(R_i)$, where $(R_i)_{i\in I}$ is a p-torsion free pro-ring. First we show that the functor is essentially surjective, that any Frobenius lift on S comes from a unique δ-structure on S.

Let $\overline{W}_1(R_i) \subset R_i \times R_i$ denote the image of the ghost map $W_1(R_i) \to R_i \times R_i$. So we have $\overline{W}_1(R_i) = \{\langle x_0, x_1\rangle \in R_i^2 \mid x_1 \equiv x_0^p \bmod pR_i\}$. Therefore sections of the projection $\overline{W}_1(R_i) \to R_i$ are in natural bijection to Frobenius lifts on R_i.

We now show that this also holds for pro-rings. Suppose we have a Frobenius lift φ on $(R_i)_i$. So given any index $i \in I$, there is an index $j \geq i$ and a map $\varphi_{i,j} \colon R_j \to R_i$ such that for all $x \in R_j$, we have $\varphi_{i,j}(x) \equiv \bar{x}^p \bmod pR_i$, where $\bar{x}$ denotes the image of x under the structure map $R_j \to R_i$. Therefore the ring homomorphism $R_j \to R_i \times R_i$ defined by $x \mapsto \langle \bar{x}, \varphi_{i,j}(x)\rangle$ has image in $\overline{W}_1(R_i)$, and hence on the pro-rings it defines a map $s\colon (R_i)_i \to (\overline{W}_1(R_i))_i$ which is a section of the projection onto the left-hand component. Observe further that its composition $(R_i)_i \to (\overline{W}_1(R_i))_i \to (R_i)_i$ with the projection onto the right-hand component is indeed the given Frobenius lift φ and that s is the only section with this property, since $\overline{W}_1(R_i)$ is a subset of $R_i \times R_i$. In this way, sections of the projection $(W_1(R_i))_i \to (R_i)_i$ are in natural bijection to Frobenius lifts on $(R_i)_i$.

Since the analogous result for δ-structures and W_1 is true by proposition 2.3.1, all that remains is to show that the map of pro-rings $(W_1(R_i))_i \to (\overline{W}_1(R_i))_i$ is a pro-isomorphism. So for any $j \geq i$, consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R_j[p] \arrow[r, "V"] \arrow[d] & W_1(R_j) \arrow[r, "w_1"] \arrow[d] & \overline{W}_1(R_j) \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & R_i[p] \arrow[r, "V"] & W_1(R_i) \arrow[r, "w_1"] & \overline{W}_1(R_i) \arrow[r] & 0,
\end{tikzcd}$$

where $V(x) = (0, x)$. As $(R_i)_i$ is p-torsion free, for any index i there exists an index $j \geq i$ such that the map $R_j[p] \to R_i[p]$ is zero. It follows that for such j there exists a unique map $\overline{W}_1(R_j) \to W_1(R_i)$ such that both the triangles in the diagram above commute. It therefore defines a morphism of pro-rings $(\overline{W}_1(R_i))_i \to (W_1(R_i))_i$ which is an inverse of the map $(W_1(R_i))_i \to (\overline{W}_1(R_i))_i$ on both the left and the right. $\square$

Remark 2.4.2. If $S = \operatorname{colim}_i \operatorname{Spec}(R_i)$ is a p-torsion free ind-affine sheaf then:

(i) $W(S)$ is a p-torsion free ind-affine sheaf. Indeed, it is enough to show that the pro-ring $(W_n(R_i))_{i\in I}$ is p-torsion free for each n, but this is true because the kernel of the n-th ghost map
$$w_n\colon W_n(R) \to \Pi_n(R)$$
is the set of Witt vectors with Witt components $(x_0, x_1, \dots, x_n)$ such that $p^i x_i = 0$ for $0 \le i \le n$. (Note that it does not matter whether we use the Buium–Joyal components or the Witt components here. This is because the statement admits a coordinate-free formulation. It is equivalent to the element lying in the kernel of the map $W_n(R) \to W_j(R/(p^j\text{-torsion}))$ for all $j \le n$. The proof, however, does seem to be easier in the Witt coordinates.)

(ii) If $X \to S$ is relatively representable, flat and affine then X is a p-torsion free ind-affine sheaf. Indeed, if $X = \operatorname{colim}\operatorname{Spec}(B_i)$, where $\operatorname{Spec}(B_i) = X \times_S \operatorname{Spec}(R_i)$, then each B_i is flat over R_i and so $B_i[p] = R_i[p] \otimes_{R_i} B_i$ and hence for each i there is a $j \ge i$ such that $R_i[p] \otimes_{R_i} B_i \to R_i[p] \otimes_{R_i} B_i$ is zero.

Proposition 2.4.3. *Let $S = \operatorname{colim} S_i$ be an ind-affine δ-sheaf, let $X \to S$ be a relatively affine δ_S-sheaf and let E be a δ_S-equivalence relation on X in the category of* fpqc*-sheaves.*[1] *If the equivalence relation E is one of the following:*

(a) *smooth,*

(b) *finite locally free,*

then the quotient X/E in the category of fpqc*-sheaves over S admits a unique δ_S-structure compatible with the quotient map $X \to X/E$.*

Proof. (a) If the equivalence relation is smooth then the quotient X/E in the category of fpqc sheaves coincides with the quotient in the category of étale sheaves which then coincides with the quotient in the category of δ_S-sheaves.

(b) Now assume that the equivalence relation is finite locally free. Write $E_i = E \times_S S_i$, $X_i = X \times_S S_i$, and X/E and X_i/E_i for the quotients in the category of fpqc sheaves. Then $X/E \times_S S_i = X_i/E_i$ is an affine scheme over S_i (by usual descent, cf. Corollaire 7.6 Exposé VIII [3]) and hence X/E is a relatively affine scheme over S. It remains to show that X/E admits a unique δ_S-structure for which the map $X \to X/E$ is a δ_S-map.

As the functor W_n preserves equalisers of rings and $E \rightrightarrows X$ is a δ_S-equivalence relation, for each i, n, there is a $j \ge i$ and a diagram of solid arrows
$$\begin{tikzcd} W_n(E_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W_n(X_i) \arrow[r] \arrow[d] & W_n(X_i/E_i) \arrow[d, dotted] \\ E_j \arrow[r, shift left] \arrow[r, shift right] & X_j \arrow[r] & X_j/E_j \end{tikzcd}$$
whose rows are coequalisers in the category of (absolutely) ind-affine sheaves over S and which is commutative in the evident sense. Therefore, we obtain a unique morphism $W_n(X_i/E_i) \to X_j/E_j$, as shown, compatible with the rest of the diagram. Moreover, the diagrams above are compatible as i and j vary, so that we may take

[1] We note that as S and $W(S)$ are ind-affine sheaves it follows that they are sheaves on Aff for the fpqc topology and that all relatively affine sheaves (for the étale topology) over them are also sheaves on Aff the fpqc topology.

the colimit to obtain a morphism $\rho_n\colon W_n(X/E) \to X/E$. These morphisms are compatible as n varies and hence define a morphism $\rho\colon W(X/E) \to X/E$.

To show that ρ is a δ_S-structure on X/E, it is enough to check that the ρ_n satisfy a certain associativity property (which will be recalled below). To do this, consider the diagram

$$\begin{tikzcd}
W_n(W_m(E)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & W_n(W_m(X)) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & W_n(W_m(X/E)) \arrow[d, shift left] \arrow[d, shift right] \\
W_{n+m}(E) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W_{n+m}(X) \arrow[r] \arrow[d] & W_{n+m}(X/E) \arrow[d] \\
E \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & X/E.
\end{tikzcd}$$

defined as follows. The horizontal morphisms are the obvious ones, the lower vertical maps $\rho_{m+n}\colon W_{m+n}(*) \to *$ come from the δ_S-structure or, in the case of $* = X/E$, were constructed above. The vertical parallel pairs are given, on the left, by the coplethysm map (1.3.1), and on the right by

$$W_n(W_m(*)) \overset{\text{inclusion}}{\to} W_{n+m}(W_m(*)) \overset{W_{n+m}(\rho_m)}{\to} W_{n+m}(*).$$

The associativity property we need to show is that the two compositions down the right column agree for every m and n.

However, the two compositions down each of the other columns agree because X and E are δ_S-sheaves. Additionally, the rows are still coequalisers in the category of absolutely ind-affine sheaves over S, as they are colimits of such. Then because the diagram commutes (in the evident sense), it follows from a diagram chase that the map $\rho_{m+n}\colon W_{n+m}(X/E) \to X/E$ is equalised by the pair of maps above it. $\square$

2.5. **Affineness of the coghost map.** We collect here a handful of affineness results concerning the coghost maps. The main application of these results is to prove certain uniqueness results for δ-structures on abelian schemes.

Lemma 2.5.1. *Let X be a scheme with the property that every finite set of points of X is contained in an open affine sub-scheme of X. Then for each $n \geq 0$ the coghost map*

$$\gamma_n\colon J^n(X) \to X^{n+1}$$

is affine.

Proof. The property satisfied by X implies that there is an open affine cover $(X_i)_{i\in I}$ of X such that $(X_i^{n+1})_{i\in I}$ is an open cover of X^{n+1}. However, the diagram

$$\begin{tikzcd}
J^n(X_i) \arrow[r, "\gamma_n"] \arrow[d] & X_i^{n+1} \arrow[d] \\
J^n(X) \arrow[r, "\gamma_n"] & X^{n+1}
\end{tikzcd} \tag{2.5.1}$$

is cartesian by proposition 12.2 of [6] (where it is also assumed that the open immersions $X_i \to X$ are closed but the proof that the diagram (2.5.1) is cartesian does not use this assumption) and the top morphisms is affine. As $(X_i^{n+1})_{i\in I}$ is a cover of X^{n+1} it follows that the bottom row of (2.5.1) is affine. $\square$

Lemma 2.5.2. *Let $S = \operatorname{colim}_{i \in I} S_i$ be an ind-affine sheaf equipped with a δ-structure and let $f \colon X \to S$ be sheaf over S. If, setting $X_i = X \times_S S_i$, the coghost maps*

$$\gamma_n \colon J^n(X_i) \to X_i^{n+1}$$

are affine for all i then the relative coghost map

$$\gamma_{X/S,n} \colon J^n_S(X) \to X^{n+1} \times_{S^{n+1}} S$$

is affine.

Proof. The morphism $\gamma_{X/S,n}$ is affine if and only if the morphisms

$$\gamma_{X/S,n} \times_S S_i$$

are affine for each $i \in I$. Fixing such an i, since S_i is quasi-compact and quasi-separated there is some $j \in I$ such that the composition $S_i \to S \to J^n(S)$ factors through $J^n(S_j) \to J^n(S)$. Therefore, we have

$$J^n_S(X) \times_S S_i = J^n(X) \times_{J^n(S)} S_i = J^n(X_j) \times_{J^n(S_j)} S_i.$$

It also follows that $S_i \to S \to S^{n+1}$ factors through $S_j \to S^{n+1}$ and hence we also have

$$X^{n+1} \times_{S^{n+1}} S_i = X_j^{n+1} \times_{S_j^{n+1}} S_i.$$

We can now express $\gamma_{X/S,n} \times_S S_i$ as the composition along the top row in the diagram

$$\begin{tikzcd}
J^n(X_j) \times_{J^n(S_j)} S_i \arrow[r] \arrow[d] & J^n(X_j) \times_{S_j^n} S_i \arrow[r] \arrow[d] & X_j^n \times_{S_j^n} S_i \\
J^n(S_j) \arrow[r, "\Delta"] & J^n(S_j) \times_{S_j^n} J^n(S_j) &
\end{tikzcd}$$

The square in this diagram is cartesian, which shows that the top left arrow is affine. Since the top right arrow is affine by hypothesis, their composition $\gamma_{X/S,n} \times_S S_i$ is affine, and hence so is $\gamma_{X/S,n}$. $\square$

We end this section with a peculiar application of the above affineness results which will have applications to the existence and uniqueness of δ-structures on abelian schemes.

Proposition 2.5.3. *Let S be a δ-sheaf and suppose we have the following:*

(a) *X is an anti-affine[2] δ_S-sheaf equipped with a δ_S-point $S \to X$,*
(b) *G is a δ_S-group such that the relative coghost map*

$$\gamma_{G/S} \colon J_S(G) \to G^{\mathbf{N}} \times_{S^{\mathbf{N}}} S$$

of (1.7.3) is affine, and

(c) *$f \colon X \to G$ is an S-pointed morphism.*

Then f is a δ_S-morphism if (and only if) it is compatible with the relative Frobenius lifts on X and G.

[2] An S-sheaf X is anti-affine if every morphism from X to a relatively affine S-sheaf factors through the structure map $X \to S$.

Proof. Let the δ_S-structures on X and G be given by the S-morphisms

$$\alpha_{X/S}\colon X \to J_S(X) \quad \text{and} \quad \alpha_{G/S}\colon G \to J_S(G).$$

Since $f\colon X \to G$ commutes with the relative Frobenius maps on X and G, the difference

$$g := \alpha_{G/S} \circ f - J_S(f) \circ \alpha_{X/S}\colon X \to J_S(G)$$

factors through the kernel of the relative coghost homomorphism $\gamma_{G/S}$.

As X and G are equipped with δ_S-points so are $J_S(X)$ and $J_S(G)$, and the morphisms composing g are S-pointed, hence so is g. However, the kernel of $\gamma_{G/S}$ is S-affine by hypothesis and X is S-anti-affine, so that g factors through the structure morphism $X \to S$:

$$X \to S \to J_S(G).$$

As the morphism g is S-pointed, it follows that $g = 0$. So we have $\alpha_{G/S} \circ f = J_S(f) \circ \alpha_{X/S}$ and thus f is a δ_S-morphism. $\square$

3. Group schemes, torsors and extensions with δ-structures

Let S be an object of Sh, assumed to be ind-affine and p-adic beginning in section 3.2. The sole purpose of this section is to prove corollary 3.2.8, which relates extensions of étale group schemes over S to extensions of their lifts to $W(S)$.

3.1. **Background on torsors and extensions.** Let G and H be a pair of relatively flat and affine commutative group schemes over S. We will write $e_G\colon S \to G$ and $\mu_G\colon G \times_S G \to G$ for the identity and multiplication maps (and similarly for all other groups). Then S, G and H are all sheaves for the fpqc topology, and we denote by $\mathscr{T}or_S(G)$ the fibred category of G-torsors over Aff_S for the fpqc topology. It is a stack for the fpqc topology. We recall the following constructions for torsors (for those over S, with the obvious extension to any base).

(i) If $f\colon G \to H$ is a homomorphism, we have a morphism of stacks

$$\mathscr{T}or_S(G) \to \mathscr{T}or_S(H)\colon X \mapsto X \otimes_G H := (X \times_S H)/G$$

where G acts on $X \times_S H$ via the restriction of the action of $G \times_S H$ along

$$G \to G \times_S H : g \mapsto (g, f(g^{-1})).$$

(ii) If G and H are a pair of sheaves of abelian groups over S then the morphism

$$\mathscr{T}or_S(G \times_S H) \to \mathscr{T}or_S(G) \times_S \mathscr{T}or_S(H) : X \mapsto (X/H, X/G)$$

is an equivalence of stacks.

(iii) If X is a G-torsor then the dual $X^\vee$ of X is the G-torsor whose underlying sheaf is X with G acting via the inverse of the given action. It is canonically isomorphic to the construction $X \mapsto X \otimes_G G$ in (i) for the homomorphism $G \to G : g \mapsto g^{-1}$.

(iv) The monoidal structure on $\mathscr{T}or_S(G)$ is given by

$$\begin{aligned} &\mathscr{T}or_S(G) \times_S \mathscr{T}or_S(G) \to \mathscr{T}or_S(G) \\ &\qquad\qquad (X_1, X_2) \mapsto X_1 \otimes_G X_2 := (X_1 \times_S X_2)/G \end{aligned}$$

where the quotient by G is via the restriction of the action of $G \times_S G$ along $G \mapsto G \times_S G : g \mapsto (g, g^{-1})$ and the G-action on the quotient is via either of the (equal) residual actions on X_1 and X_2. This monoidal structure is symmetric and for all pairs of G-torsors X_1, X_2 we have canonical isomorphisms

$$\underline{\mathrm{Hom}}_S^G(X_1, X_2) \xrightarrow{\sim} X_2 \otimes_G X_1^\vee.$$

3.1.1. Let $f\colon G \to H$ be a homomorphism with relatively representable affine and flat kernel $\ker(f) = K$. We define $\mathscr{T}or_S[f]$ to be the fibred category whose sections are given by the category of pairs (X, ρ) where X is a G-torsor and $\rho\colon H \to X \otimes_G H$ is a trivialisation, i.e., an isomorphism of H-torsors. There is a natural functor

$$\mathscr{T}or_S(K) \to \mathscr{T}or_S[f] : X \mapsto (X \otimes_K G, \rho)$$

where ρ is the trivialisation induced by

$$X \otimes_K H = (X \times_S H)/K = (X/K) \times_S H = H.$$

This functor is an equivalence of stacks with quasi-inverse given by sending (X, ρ) to the equaliser of the two morphisms $X \rightrightarrows X \otimes_G H$ given by the compositions

$$X \xrightarrow{X \times_S e_H} X \times_S H \to X \otimes_G H \quad \text{and} \quad X \to S \xrightarrow{e_H} H \xrightarrow{\rho} X \otimes_G H.$$

3.1.2. An extension of H by G is a short exact sequence of commutative group schemes over S

$$0 \to G \to E \to H \to 0.$$

A morphism of extensions $E \to E'$ is a homomorphism $E \to E'$ compatible with the identity maps on G and H. This yields the moduli stack of extensions $\mathscr{E}xt_S(H, G)$ over Aff_S.

The group structure on H induces on Aff_H the structure of a symmetric monoidal stack over Aff_S: the product of T_1 and T_2 is the Cartesian product $T_1 \times_S T_2$ viewed as an object of Aff_H via the composition $T_1 \times_S T_2 \to H \times_S H \to H$ with the product map. So we may also consider the Hom stack

$$\mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_S(G))$$

whose fibre over S is the category of symmetric monoidal morphisms

$$e\colon \mathrm{Aff}_H \to \mathscr{T}or_S(G)$$

over Aff_S.

There is a natural equivalence of stacks (cf. proposition 1.4.23 of exposé XVIII in [2])

$$\mathscr{E}xt_S(H, G) \to \mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_S(G))$$

sending an extension

$$0 \to G \to E \to H \to 0$$

to the symmetric monoidal functor $\eta\colon \mathrm{Aff}_H \to \mathscr{T}or_S(G)$ whose value on $T \to H$ is the G-torsor $E \times_H T$ (the symmetric monoidal structure on this functor is induced by the commutative group structure on E). A quasi-inverse is given by sending a symmetric monoidal functor $\eta\colon \mathrm{Aff}_H \to \mathscr{T}or_S(G)$ to the G-torsor $E := \eta(\mathrm{id}_H\colon H \to H)$, which inherits the structure of an extension of G by H via the symmetric monoidal structure on η. Indeed, the addition law on E is obtained via the composition

$$E \times_S E \xrightarrow{\sim} p_1^*(E) \times_{(H \times_S H)} p_2^*(E) \to p_1^*(E) \otimes_G p_2^*(E) \xrightarrow{\sim} \mu_H^*(E) \to E,$$

where p_i are the projections $H \times_S H \to H$ and μ_H is the multiplication. The rest we leave to the reader.

3.2. **Torsors and extensions with δ-structures.** Let $\widetilde{G}/W(S)$ be a flat and affine group scheme over $W(S)$ such that either of the following two conditions hold:

(a) $\widetilde{G}$ is smooth over $W(S)$,
(b) $\widetilde{G}$ is finite locally free over $W(S)$.

Write G for the pull-back $\widetilde{G} \times_{W(S)} S$ along the first ghost component. (Later we will primarily consider groups $\widetilde{G}$ which are canonical lifts of a given group G, but for now $\widetilde{G}$ can be an arbitrary lift satisfying the properties above, although this is some abuse of notation).

Now assume that $\widetilde{G}$ has a $\delta_{W(S)}$-structure. A $\delta_{W(S)}$-$\widetilde{G}$-torsor is a $\widetilde{G}$-torsor over $W(S)$ equipped with a $\delta_{W(S)}$-structure compatible with the $\widetilde{G}$-action. We denote by $\mathscr{T}or_{\delta_{W(S)}}(\widetilde{G})$ the fibred category over Aff_S whose fibre over an affine S-scheme T is the groupoid of $\delta_{W(T)}$-$\widetilde{G}_{W(T)}$-torsors. The fibred category $\mathscr{T}or_{\delta_{W(S)}}(\widetilde{G})$ is a stack for the étale topology and the constructions (i)–(iv) of section 3.1 for usual torsors work just as well for the fibred categories of $\delta_{W(S)}$-torsors using proposition 2.4.3 combined with the assumptions (a) and (b) above.

We denote by $g_{\widetilde{G}}$ the symmetric monoidal functor

$$\begin{aligned} \mathscr{T}or_{\delta_{W(S)}}(\widetilde{G}) &\xrightarrow{g_{\widetilde{G}}} \mathscr{T}or_S(G) \\ X &\mapsto X \times_{W(S)} S. \end{aligned}$$

Proposition 3.2.1. *The functor $g_{\widetilde{G}}$ is fully faithful. If G/S is smooth, then it is an equivalence.*

Proof. If $\widetilde{X}_1$ and $\widetilde{X}_2$ are a pair of $\delta_{W(S)}$-$\widetilde{G}$-torsors, then we have

$$\begin{aligned} \mathrm{Hom}^{\widetilde{G}}_{\delta_{W(S)}}(\widetilde{X}_1, \widetilde{X}_2) &= \mathrm{Hom}^{\widetilde{G}}_{\delta_{W(S)}}(\widetilde{G}, \widetilde{X}_2 \otimes_{\widetilde{G}} \widetilde{X}_1^{-1}) \\ &= \mathrm{Hom}_{\delta_{W(S)}}(W(S), \widetilde{X}_2 \otimes_{\widetilde{G}} \widetilde{X}_1^{-1})) \\ &= \mathrm{Hom}_S(S, X_2 \otimes_G X_1^{-1}) \\ &= \mathrm{Hom}^G_S(G, X_2 \otimes_G X_1^{-1}) \\ &= \mathrm{Hom}^G_S(X_1, X_2) \end{aligned}$$

whence the full faithfulness.

If G is smooth, then with respect to the étale topology, every G-torsor is locally trivial so that the fully faithful functor $g_{\widetilde{G}}$ is locally essentially surjective. As both $\mathscr{T}or_{\delta_{W(S)}}(\widetilde{G})$ and $\mathscr{T}or_S(G)$ are stacks for the étale topology it follows that $g_{\widetilde{G}}$ is an equivalence. □

3.2.2. Consider an exact sequence of relatively flat and affine $\delta_{W(S)}$-group schemes

$$0 \to \widetilde{K} \to \widetilde{G} \xrightarrow{\widetilde{f}} \widetilde{H},$$

and denote its pull-back along the first ghost component by

$$0 \to K \to G \xrightarrow{f} H.$$

We define $\mathscr{T}or_{\delta_{W(S)}}[\widetilde{f}]$ to be the fibred category over Aff_S whose fibre over an affine S-scheme T is the groupoid of pairs $(\widetilde{X}, \widetilde{\rho})$ where $\widetilde{X}$ is a $\widetilde{G}$-torsor over $W(T)$ and $\widetilde{\rho}\colon \widetilde{H} \to \widetilde{X} \otimes_{\widetilde{G}} \widetilde{H}$ is an isomorphism of $\delta_{W(S)}$-$\widetilde{H}$-torsors together with the obvious notion of morphism. We denote by $g_{\widetilde{f}}$ the functor

$$\mathscr{T}or_{\delta_{W(S)}}[\widetilde{f}] \xrightarrow{g_{\widetilde{f}}} \mathscr{T}or_S[f]$$
$$(\widetilde{X}, \widetilde{\rho}) \mapsto (X, \rho),$$

where ρ denotes the pull-back of $\widetilde{\rho}$ along the first ghost component map $S \to W(S)$.

Proposition 3.2.3. *If $g_{\widetilde{G}}$ is an equivalence of categories, then so is $g_{\widetilde{f}}$.*

Proof. Let $(\widetilde{X}_1, \widetilde{\rho}_1)$ and $(\widetilde{X}_2, \widetilde{\rho}_2)$ be a pair of objects of $\mathscr{T}or_{\delta_{W(S)}}[\widetilde{f}]$ over some affine S-scheme. Given a morphism $\theta\colon (X_1, \rho_1) \to (X_2, \rho_2)$, the full faithfulness of $g_{\widetilde{G}}$ implies that there exists a unique $\widetilde{\theta}\colon \widetilde{X}_1 \to \widetilde{X}_2$ lifting θ. The faithfulness of $g_{\widetilde{H}}$ shows that $\widetilde{\theta}$ is in fact a morphism $\widetilde{\theta}\colon (\widetilde{X}_1, \widetilde{\rho}_1) \to (\widetilde{X}_2, \widetilde{\rho}_2)$, and so $g_{\widetilde{f}}$ is fully faithful.

Now let (X, ρ) be a section of $\mathscr{T}or_S[f]$. Since $g_{\widetilde{G}}$ is an equivalence, there exists a unique $\widetilde{X}$ lifting X. As $g_{\widetilde{H}}$ is fully faithful, it follows that there exists a unique $\widetilde{\rho}\colon \widetilde{H} \to \widetilde{X} \otimes_{\widetilde{G}} \widetilde{H}$ lifting ρ. Therefore $g_{\widetilde{f}}$ is essentially surjective and hence an equivalence. □

Proposition 3.2.4. *If $g_{\widetilde{G}}$ is an equivalence of categories, then so is*

$$g_{\widetilde{K}}\colon \mathscr{T}or_{\delta_{W(S)}}(\widetilde{K}) \to \mathscr{T}or_S(K).$$

Proof. As in 3.1.1, we can define a functor

$$\mathscr{T}or_{\delta_{W(S)}}(\widetilde{K}) \to \mathscr{T}or_{\delta_{W(S)}}[\widetilde{f}] : \widetilde{X} \mapsto (\widetilde{X} \otimes_{\widetilde{K}} \widetilde{G}, \widetilde{\rho}),$$

and as before it is an equivalence. We then obtain a 2-commutative diagram

$$\begin{tikzcd}
\mathscr{T}or_{\delta_{W(S)}}(\widetilde{K}) \arrow[r, "\sim"] \arrow[d, "g_{\widetilde{K}}"'] & \mathscr{T}or_{\delta_{W(S)}}[\widetilde{f}] \arrow[d, "g_{\widetilde{f}}"] \\
\mathscr{T}or_S(K) \arrow[r, "\sim"] & \mathscr{T}or_S[f].
\end{tikzcd}$$

The top and bottom arrows are equivalences and the right arrow is an equivalence by proposition 3.2.3, so that left arrow is also an equivalence. □

3.2.5. Let $\widetilde{G}$ be a flat affine $\delta_{W(S)}$-group scheme over $W(S)$ satisfying either of the conditions (a) and (b) of section 3.2, and let H be an affine étale group scheme over S. Then proposition 2.2.2 implies $\widetilde{H} := W(H)$ is the unique affine étale $\delta_{W(S)}$-group scheme lifting H.

Denote by $\mathscr{E}xt_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G})$ the fibred category over Aff_S whose fibre over an affine S-scheme T is the groupoid of $\delta_{W(T)}$-extensions of $\widetilde{G}$ by $\widetilde{H}$, by which we mean short exact sequences (in the sense of sheaves in the fpqc topology) of $\delta_{W(T)}$-group schemes

$$0 \to \widetilde{G}_{W(T)} \to \widetilde{E} \to \widetilde{H}_{W(T)} \to 0,$$

with morphisms $\delta_{W(T)}$-morphisms inducing the identity on $\widetilde{G}$ and $\widetilde{H}$. Denote by

$$\mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_{\delta_{W(S)}}(\widetilde{G}))$$

the fibred category whose fibre over an affine S-scheme T is the groupoid of symmetric monoidal functors (over Aff_S)

$$\eta\colon \mathrm{Aff}_{H_T} \to \mathscr{T}or_{\delta_{W(S)}}(\widetilde{G}).$$

Such a functor η is determined up to equivalence by its value at the universal point

$$\widetilde{E} = \eta(H_T \to H_T),$$

which is a $\delta_{W(H_T)}$-$\widetilde{G}$-torsor $\widetilde{E}$ over $W(H_T)$, together with the $\delta_{W(H_T\times_T H_T)}$-$\widetilde{G}$-isomorphism

$$W(p_1)^*(\widetilde{E}) \otimes_{\widetilde{G}} W(p_2)^*(\widetilde{E}) \xrightarrow{\sim} W(\mu)^*(\widetilde{E}), \tag{3.2.1}$$

coming from the monoidal structure on η.

By theorem 1.5.4, we have

$$W(H_T \times_T H_T) = W(H_T) \times_{W(T)} W(H_T) = \widetilde{H}_{W(T)} \times_{W(T)} \widetilde{H}_{W(T)}.$$

Under this identification $W(p_1)$, $W(p_2)$ and $W(\mu)$ correspond to the usual projections and multiplication map $\widetilde{H}_{W(T)} \times_{W(T)} \widetilde{H}_{W(T)} \to \widetilde{H}_{W(T)}$ respectively, and the isomorphism (3.2.1) corresponds to a $\delta_{\widetilde{H}_{W(T)}\times_{W(T)}\widetilde{H}_{W(T)}}$-$\widetilde{G}$-isomorphism

$$p_1^*(\widetilde{E}) \otimes_{\widetilde{G}} p_2^*(\widetilde{E}) \xrightarrow{\sim} \mu^*(\widetilde{E}).$$

This induces the structure of a $\delta_{W(T)}$-extension of $\widetilde{H}_{W(T)}$ by $\widetilde{G}_{W(T)}$ on $\widetilde{E}$ and so (as for usual torsors) we obtain an equivalence of stacks

$$\mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_{\delta_{W(S)}}(\widetilde{G})) \to \mathscr{E}xt_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G}).$$

Theorem 3.2.6. *If $g_{\widetilde{G}}$ is an equivalence, then*

$$\mathscr{E}xt_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G}) \to \mathscr{E}xt_S(H, G)$$

is also an equivalence.

Proof. We have the 2-commutative diagram

$$\begin{tikzcd} \mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_{\delta_{W(S)}}(\widetilde{G})) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathscr{E}xt_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G}) \arrow[d] \\ \mathscr{H}om_S^{\otimes}(H, \mathscr{T}or_S(G)) \arrow[r, "\sim"] & \mathscr{E}xt_S(H, G) \end{tikzcd}$$

in which the left arrow (composition with the equivalence $g_{\widetilde{G}}$) is an equivalence. The bottom arrow is an equivalence, as explained in 3.1.2. Similarly, the top arrow is an equivalence, as just explained in 3.2.5. □

Remark 3.2.7. This theorem can be viewed as a derived analogue of lemma 2.2.4.

Corollary 3.2.8. *Let H be an affine étale group scheme over S, and let $\widetilde{G}$ be a flat and affine $\delta_{W(S)}$-group scheme satisfying* (a) *or* (b) *of section 3.2. If $\widetilde{G}$ is isomorphic to the kernel of a $\delta_{W(S)}$-homomorphism of flat and affine $\delta_{W(S)}$-group schemes $\widetilde{G_1} \to \widetilde{G_2}$ with G_1 smooth over S, then the functor*

$$\mathscr{E}xt_{\delta_{W(S)}}(\widetilde{H}, \widetilde{G}) \to \mathscr{E}xt_S(H, G)$$

is an equivalence.

Proof. This follows immediately from proposition 3.2.4, proposition 3.2.1, and theorem 3.2.6. □

4. Canonical lifts of ordinary p-groups

4.1. **δ-structures on étale and multiplicative p-groups.** By a p-group over a p-adic sheaf S we mean any finite locally free group scheme G locally of p-power degree. We say that G is étale if $G \to S$ is étale, and we say that G is multiplicative if G is the Cartier dual of an étale p-group.

By an ordinary p-group over S we mean a p-group E for which there exists a short exact sequence of sheaves of groups

$$0 \to G \to E \to H \to 0$$

where G is a multiplicative p-group scheme and H an étale p-group scheme. (See remark 4.1.1 for an alternative characterisation.)

If E is an ordinary p-group over S, then G and H are uniquely determined (as sub and quotient groups of E) and are compatible with base change. Indeed, the inclusion $G \to E$ is both an open and closed immersion (being the pull back of the open and closed immersion $S \to H$ along $E \to H$) and is therefore smallest open and closed sub-sheaf containing the identity section of E. We will write $G = E_{\mathrm{mult}}$ and $H = E_{\text{ét}}$ and note that the assignments $E \mapsto E_{\mathrm{mult}}$ and $E \mapsto E_{\text{ét}}$ are functorial in ordinary p-groups E to multiplicative and étale p-groups respectively.

It follows from the above that if E is a p-group over S which is fpqc locally an ordinary p-group then it is itself an ordinary p-group and the fibred category $\mathscr{O}rd$ over $\mathrm{Aff}_{\mathrm{Spf}(\mathbf{Z}_p)}$, whose fibre over each p-adic affine scheme S is the groupoid of ordinary p-groups, is a stack for the fpqc topology.

We note that every homomorphism $G \to E$, from a multiplicative p-group to an ordinary p-group, factors through the inclusion $E_{\mathrm{mult}} \to E$. In particular, there are no non-trivial homomorphisms from a multiplicative p-group to an étale p-group.

Remark 4.1.1. A p-group G over S is ordinary in the sense above if and only if:

(i) for all $s \in S$ the p-group G_s is ordinary, and
(ii) the degree of $(G_s)_{\mathrm{mult}}$ is a locally constant function on S.

As follows from lemma 4.8 of [19]. In particular, if A/S is an abelian scheme such that A_s is ordinary for all $s \in S$ then $A[p^n]$ is ordinary for all $n \geq 0$.

Proposition 4.1.2. *Let S be p-adic sheaf.*

(i) *Every étale (resp. multiplicative) p-group over $W(S)$ admits a unique $\delta_{W(S)}$-structure compatible with its group structure.*
(ii) *Every homomorphism of such groups is a $\delta_{W(S)}$-homomorphism.*
(iii) *The category of étale (resp. multiplicative) $\delta_{W(S)}$-p-groups is equivalent, via base change along the first ghost component, to the category of étale (resp. multiplicative) p-groups over S.*

Proof. (i): For étale groups this follows from proposition 2.2.2.

For multiplicative groups, we first note that there is one and only one way to lift a multiplicative group G/S to a multiplicative group $\widetilde{G}/W(S)$ (forgetting about $\delta_{W(S)}$-structures). Indeed, the only possible option is to take the Cartier dual of the unique lift of the Cartier dual of G (which is an étale group). In particular, it follows that every multiplicative p-group $\widetilde{G}/W(S)$ is, after base change along a morphism $W(S') \to W(S)$ with $S' \to S$ étale and surjective, of the form

$$\prod_{i=1}^{r} \mu_{p^{n_i}}. \tag{4.1.1}$$

Using the étale local nature of $\delta_{W(S)}$-structures on sheaves over $W(S)$, as given by lemma 2.1.1, we may assume that $\widetilde{G}$ is of the form (4.1.1) and show that it admits a unique $\delta_{W(S)}$-structure compatible with its group structure. It is enough to show existence and uniqueness locally in the étale topology so that we may also assume that $S = \mathrm{Spec}(A)$ is affine. By proposition 2.2.3 we can instead show that the $W(A)$-Hopf algebra

$$R := \bigotimes_{i=1}^{r} W(A)[T]/(T^{p^{n_i}} - 1) = W(A)[T_1, \ldots, T_r]/(T_1^{p^{n_1}} - 1, \ldots, T_1^{p^{n_r}} - 1)$$

admits a unique $\delta_{W(A)}$-structure compatible with its $W(A)$-Hopf algebra structure. It is clear that R admits at least one such $\delta_{W(A)}$-structure given by $\delta_R(T_i) = 0$ for $i = 1, \ldots, r$. So we will show that this is the only one.

Consider a $\delta_{W(A)}$-structure δ_R on R. Set $P = \Pi_{i=1}^{r}\{0, \ldots, p^{n_i} - 1\}$ and for $I = (a_1, \ldots, a_r) \in P$ write $T^I = T_1^{a_1} \cdots T_r^{a_r}$ so that $\{T^I : I \in P\}$ forms a basis for R as $W(A)$-module. The operator δ_R is completely determined by its values

$$\delta_R(T_i) = \sum_{I \in P} a_{I,(i)} T^I$$

on the elements T_i for $1 \leq i \leq r$. The compatibility of δ_R with the Hopf algebra structure on R is by definition equivalent to the equality of set maps

$$\Delta_R \circ \delta_R = \delta_{R \otimes_{W(A)} R} \circ \Delta_R,$$

which is in turn equivalent to the equalities for $1 \leq i \leq r$:

$$\Delta_R(\delta_R(T_i)) = \delta_{R \otimes_{W(A)} R}(\Delta_R(T_i)).$$

These equalities expand to

$$\Delta_R(\sum_{I \in P} a_{I,(i)} T^I) = \delta_{R \otimes_{W(A)} R}(T_i \otimes T_i)$$

and then, applying the product rule to $\delta_{R \otimes_{W(A)} R}((T_i \otimes 1)(1 \otimes T_i))$, to

$$\sum_{I \in P} a_{I,(i)} T^I \otimes T^I = \sum_{I \in P} a_{I,(i)} \left(T_i^p \otimes T^I + T^I \otimes T_i^p\right) + \sum_{I,J \in P} p a_{I,(i)} a_{J,(i)} T^I \otimes T^J.$$

Equating the coefficients of $T^I \otimes T^I$ we find that $a_{I,(i)} = p a_{I,(i)}^2$ unless $I = (0, \ldots, p, \ldots, 0)$ with p in the ith position, where we find that $a_{I,(i)} = 2a_{I,(i)} + p a_{I,(i)}^2$. Regardless, we find that $a_{I,(i)}(1 \pm p a_{I,(i)}) = 0$ which, as p is topologically nilpotent, in $W(R)$ shows that $a_{I,(i)} = 0$. Hence $\delta_R(T_i) = 0$ for $1 \leq i \leq r$ and the only $\delta_{W(A)}$-structure on R compatible with its Hopf algebra structure is the one with $\delta(T_i) = 0$.

(ii): By additivity we are reduced to showing that all $W(S)$-group scheme homomorphisms

$$\mu_{p^m} \to \mu_{p^n}$$

are $\delta_{W(S)}$-morphisms or, again using proposition 2.2.3, that every Hopf algebra homomorphism

$$\theta\colon W(A)[T]/(T^{p^n}-1) \to W(A)[T]/(T^{p^m}-1)$$

is a $\delta_{W(A)}$-homomorphism. As $\delta(T)=0$ (on both sides) it is enough to show that $\delta(\theta(T))=0$.

As θ is a Hopf algebra homomorphism, we must have

$$\theta(T) = \sum_{i=1}^{p^n} a_i T^i$$

where the $a_i \in W(A)$ are orthogonal idempotents (i.e. $a_i a_j = \delta_{ij} a_i$ and $a_i^2 = a_i$). As $W(A)[T]/(T^{p^n}-1)$ is p-adically complete and the a_i are orthogonal and idempotent, it follows from lemma 4.1.3 below that

$$\delta(\theta(T)) = \delta\left(\sum_{i=1}^{p^n} a_i T^i\right) = \sum_{i=1}^{p^n} \delta(a_i T^i) = \sum_{i=1}^{p^n} a_i \delta(T^i) = 0$$

(as $\delta(T)=0$, it follows immediately that $\delta(T^i)=0$). Therefore, θ is a $\delta_{W(A)}$-homomorphism.

(iii): Since the functor is essentially surjective, by (i), it is enough to prove it is full and faithful. The category étale p-groups over $W(S)$ is equivalent, by proposition 2.2.2, to that over S. The same is true for multiplicative p-groups by Cartier duality. Therefore for any étale (resp. multiplicative) p-groups G and H over $W(S)$, every homomorphism $G\times_{W(S)} S \to H \times_{W(S)} S$ lifts to a unique homomorphism $G \to H$. By (ii), it is necessarily a $\delta_{W(S)}$-morphism. □

Lemma 4.1.3. *Let R be a p-adically complete δ-ring.*

(i) *If $r \in R$ is idempotent, then $\delta(r)=0$ and $\delta_R(rr') = r\delta(r')$ for all $r' \in R$.*
(ii) *If $r_1, r_2 \in R$ satisfy $r_1 r_2 = 0$, then $\delta_R(r_1+r_2) = \delta_R(r_1) + \delta_R(r_2)$.*

Proof. (i) As $\delta(rr') = r^p\delta(r') + r'^p\delta(r) + p\delta(r)\delta(r')$ and $r^p = r$ it is enough to show that $\delta(r)=0$. For this, note that $\delta(r) = \delta(r^2) = 2r^p\delta(r) + p\delta(r)^2$ and hence $\delta(r)(2r-1+p\delta(r)) = 0$. But $2r-1$ is a unit as $(2r-1)^2 = 1$ and hence so is $2r-1+p\delta(r)$ as R is p-adically complete. Therefore $\delta(r)=0$.

(ii) This follows immediately from the addition law for δ. □

4.1.4. Given (for now) a multiplicative or étale p-group G over S, we refer to the unique $\delta_{W(S)}$-p-group over $W(S)$ lifting G as its canonical lift. We will typically denote it $\widetilde{G}$. This does conflict with our earlier convention of allowing $\widetilde{G}$ to denote any lift of G to $W(S)$, but we believe this will not cause any confusion.

4.2. **δ-structures on ordinary p-groups.** By an ordinary $\delta_{W(S)}$-p-group we mean an ordinary p-group $\widetilde{E}$ over $W(S)$ equipped with a $\delta_{W(S)}$-structure compatible with its group structure. In this case, the multiplicative and étale group schemes $\widetilde{G} := \widetilde{E}_{\mathrm{mult}}$ and $\widetilde{H} := \widetilde{E}_{\text{ét}}$ are both uniquely $\delta_{W(S)}$-groups, by proposition 4.1.2. Further, the morphisms in the short exact sequence

$$0 \to \widetilde{G} \to \widetilde{E} \to \widetilde{H} \to 0$$

are $\delta_{W(S)}$-morphisms. Indeed, $\widetilde{G} \to \widetilde{E}$ is an open immersion (of p-adic sheaves), and $\widetilde{E} \to \widetilde{H}$, being the quotient of $\widetilde{E}$ by the finite locally free $\delta_{W(S)}$-group $\widetilde{G}$, is a $\delta_{W(S)}$-morphism, by proposition 2.4.3. We denote by $\mathscr{O}rd_\delta$ the fibred category over $\mathrm{Aff}_{\mathrm{Spf}(\mathbf{Z}_p)}$ whose fibre over a p-adic affine scheme S is the groupoid of ordinary $\delta_{W(S)}$-p-group schemes. The étale-local nature of $\delta_{W(S)}$-structures shows that this is a stack for the étale topology.

Lemma 4.2.1. *Let G and H be a multiplicative and an étale p-group over $\mathrm{Spf}(\mathbf{Z}_p)$ respectively. Then base change along the first ghost component induces an equivalence:*

$$\mathscr{E}xt_{\delta_{W(\mathrm{Spf}(\mathbf{Z}_p))}}(\widetilde{H},\widetilde{G}) \to \mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z}_p)}(H,G). \tag{4.2.1}$$

Proof. Since the source and target of the functor are stacks, it is enough to show equivalence locally on the base $\mathrm{Spf}(\mathbf{Z}_p)$. Therefore we may assume $G=\prod_{i=1}^n \mu_{p^{r_i}}$ and hence

$$\widetilde{G} = \prod_{i=1}^{n} \mu_{p^{r_i}/W(\mathrm{Spf}(\mathbf{Z}_p))}.$$

Then $\widetilde{G}$ is the kernel of a homomorphism between relatively affine and smooth group schemes

$$0 \to \widetilde{G} \to \mathbf{G}^n_{\mathrm{m}/W(\mathrm{Spf}(\mathbf{Z}_p))} \xrightarrow{(p^{r_1},\dots,p^{r_n})} \mathbf{G}^n_{\mathrm{m}/W(\mathrm{Spf}(\mathbf{Z}_p))}.$$

Further if $\mathbf{G}_{\mathrm{m}/W(\mathrm{Spf}(\mathbf{Z}_p))}$ is given its usual $\delta_{W(S)}$-structure, these maps are $\delta_{W(S)}$-equivariant, by proposition 4.1.2. Corollary 3.2.8 then implies the functor (4.2.1) is an equivalence. □

Proposition 4.2.2. *The functor*

$$\mathscr{O}rd_\delta \to \mathscr{O}rd : \widetilde{E}/W(S) \mapsto E/S$$

induced by pull-back along the first ghost component is an equivalence of groupoids.

Proof. First, as G and H vary over all multiplicative and étale p-groups over $\mathrm{Spf}(\mathbf{Z}_p)$ the maps

$$\big(\mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z}_p)}(H,G) \to \mathscr{O}rd\big)_{(G,H)}$$

form a cover of $\mathscr{O}rd$. Indeed, an ordinary p-group over a p-adic affine scheme S is locally an extension of a split étale group H by a split multiplicative group G, and all such groups defined over $\mathrm{Spf}(\mathbf{Z}_p)$. The claim will then follow if we can show that for each pair (G,H), the map $\mathscr{E} \to \mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z})}(H,G)$ is an equivalence where $\mathscr{E}$ is the fibre product

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z}_p)}(H,G) \arrow[d] \\ \mathscr{O}rd_\delta \arrow[r] & \mathscr{O}rd. \end{tikzcd}$$

The sections of $\mathscr{E}$ over a p-adic affine scheme S are triples

$$(\widetilde{E}, E', \rho\colon \widetilde{E}\times_{W(S)} S \xrightarrow{\sim} E')$$

where $\widetilde{E}$ is an ordinary $\delta_{W(S)}$-p-group and E'/S is an extension of G_S by H_S. To show that

$$\mathscr{E} \to \mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z}_p)}(H,G)$$

is an equivalence we will instead show that the natural map

$$\mathscr{E}xt_{\delta_{W(\mathrm{Spf}(\mathbf{Z}_p))}}(\widetilde{H},\widetilde{G}) \to \mathscr{E} : \widetilde{E} \mapsto (\widetilde{E}, \widetilde{E}\times_{W(S)} S, \mathrm{id}_{\widetilde{E}\times_{W(S)}S})$$

is an equivalence, from which the claim follows, as the composition

$$\mathscr{E}xt_{\delta_{W(\mathrm{Spf}(\mathbf{Z}_p))}}(\widetilde{H},\widetilde{G}) \to \mathscr{E} \to \mathscr{E}xt_{\mathrm{Spf}(\mathbf{Z}_p)}(H,G)$$

is isomorphic to the equivalence (4.2.1).

If $(\widetilde{E}, E', \rho\colon \widetilde{E}\times_{W(S)} S \xrightarrow{\sim} E')$ is a section of $\mathscr{E}$ over S, then we claim that $\widetilde{E}$ admits a unique structure of an extension of $\widetilde{G}_{W(S)}$ by $\widetilde{H}_{W(S)}$ lifting the corresponding structure on $\widetilde{E}\times_{W(S)} S$ induced by ρ. But this is obvious, as the isomorphisms

$$\widetilde{E}_{\text{ét}} \times_{W(S)} S \xrightarrow{\sim} E'_{\text{ét}} = H_S \quad\text{and}\quad \widetilde{E}_{\text{mult}} \times_{W(S)} S \xrightarrow{\sim} E'_{\text{mult}} = G_S$$

induced by ρ lift uniquely to isomorphisms

$$\widetilde{E}_{\text{ét}} \xrightarrow{\sim} \widetilde{H}_{W(S)} \quad\text{and}\quad \widetilde{E}_{\text{mult}} \xrightarrow{\sim} \widetilde{G}_{W(S)}$$

by proposition 4.1.2, which gives $\widetilde{E}$ the structure of a $\delta_{W(S)}$-extension of $\widetilde{H}_{W(S)}$ by $\widetilde{G}_{W(S)}$ and so the map in question is essentially surjective.

Moreover, if

$$(\widetilde{f}, f')\colon (\widetilde{E}_1, E'_1, \rho_1) \to (\widetilde{E}_2, E'_2, \rho_2)$$

is any isomorphism of sections of $\mathscr{E}$ over S then it is easy to check that the induced map of $\delta_{W(S)}$-p-groups

$$\widetilde{f}\colon \widetilde{E}_1 \to \widetilde{E}_2$$

is a morphism of the corresponding $\delta_{W(S)}$-extensions of $\widetilde{H}_{W(S)}$ by $\widetilde{G}_{W(S)}$ and so the functor in question is also full and therefore an equivalence. □

Theorem 4.2.3. *For each p-adic affine scheme S, the* category *of ordinary $\delta_{W(S)}$-p-groups is equivalent, via base change along the first ghost component, to the category of ordinary p-groups over S.*

Proof. By proposition 4.2.2, this functor induces an equivalence of the groupoids. In particular, it is essentially surjective, and so we need only show that it is fully faithful.

First, the categories under consideration are exact, with admissible epimorphisms those which are epimorphisms of fpqc sheaves, and the functor, pull-back along the first ghost component, is exact. Hence, fixing a pair $\widetilde{E}$ and $\widetilde{E}'$ of ordinary $\delta_{W(S)}$-p-groups we obtain a morphism of exact sequences where the rightmost column denotes the groups of Yoneda extensions:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\delta_{W(S)}}(\widetilde{E}'_{\text{ét}},\widetilde{E}) & \rightarrowtail & \mathrm{Hom}_{\delta_{W(S)}}(\widetilde{E}',\widetilde{E}) & \to & \mathrm{Hom}_{\delta_{W(S)}}(\widetilde{E}'_{\text{mult}},\widetilde{E}) & \to & \mathrm{Ext}_{\delta_{W(S)}}(\widetilde{E}'_{\text{ét}},\widetilde{E}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_S(E'_{\text{ét}},E) & \rightarrowtail & \mathrm{Hom}_S(E',E) & \longrightarrow & \mathrm{Hom}_S(E'_{\text{mult}},E) & \longrightarrow & \mathrm{Ext}_S(E'_{\text{ét}},E).
\end{array}$$

By proposition 2.2.4, the first vertical arrow is bijective; by proposition 4.1.2, the third arrow is bijective (noting that any $\delta_{W(S)}$-homomorphism $\widetilde{E}'_{\text{mult}} \to \widetilde{E}$ factors uniquely as a $\delta_{W(S)}$-homomorphism $\widetilde{E}'_{\text{mult}} \to \widetilde{E}_{\text{mult}}$, and any homomorphism $E'_{\text{mult}} \to E$ factors uniquely as a homomorphism $E_{\text{mult}} \to E_{\text{mult}}$); and by corollary 3.2.8, the fourth vertical arrow is bijective. Hence by the Five Lemma, the second vertical arrow is bijective and we are done. □

5. Canonical lifts of ordinary abelian schemes

5.1. **δ-structures on abelian schemes.** Before we consider canonical lifts of ordinary abelian schemes we will need several lemmas on δ-structures and abelian schemes over p-adic ind-schemes S.

Lemma 5.1.1. *Let S be a p-adic ind-scheme, let $f\colon A \to B$ be a homomorphism of abelian schemes over S and let $S_0 \to S$ be a nilpotent immersion. If $f \times_S S_0 = 0$ then $f = 0$. In other words, $\underline{\mathrm{Hom}}_S(A, B)$ is formally unramified.*

Proof. We may assume that S is an affine scheme on which p is nilpotent. If $f \times_S S_0 = 0$ then it follows that the homomorphism $f\colon A \to B$ factors through the formal completion of the identity section of B:

$$A \to \widehat{B} \to B.$$

However, $\widehat{B}$ is the colimit of the infinitesimal neighbourhoods of the zero section in B, each of which is affine over S, so that as A is quasi-compact over S the homomorphism $A \to B$ factors through one of these infinitesimal neighbourhoods. But these neighbourhoods are all S-affine, and as A is an S-abelian scheme, it is anti-affine. Therefore this a map factors further through the structure map $A \to S$. In other words, it is the zero homomorphism. □

Proposition 5.1.2. *Let S be an ind-affine δ-sheaf, and let A and B be a pair of abelian schemes over S with δ_S-structures (not necessarily compatible with the group structures). Let $f\colon A \to B$ be an S-homomorphism. If f commutes with the relative Frobenius lifts on A and B, then it is a δ_S-morphism.*

Proof. Write $S = \mathrm{colim}_{i\in I} S_i$ as a filtered colimit of affine schemes. By theorem 1.9 of chapter I of [12], for each $i \in I$ the S_i-scheme $B \times_S S_i$ satisfies the hypotheses of lemma 2.5.1 so that we may apply lemma 2.5.2 and deduce that the relative coghost homomorphism

$$\gamma_{B/S,n}\colon J^n_S(B) \to B^{n+1} \times_{S^{n+1}} S$$

is affine. Taking the limit over n, we see for general reasons that

$$\gamma_{B/S}\colon J_S(B) \to B^{\mathbf{N}} \times_{S^{\mathbf{N}}} S$$

is also affine.

At the same time, A is a relative abelian scheme over S and hence is S-anti-affine. The claim then follows from proposition 2.5.3. □

Corollary 5.1.3. *Let S be an ind-affine δ-sheaf, and let A/S be a relative abelian scheme over S. Then the following hold:*

(i) *Two δ_S-structures on A are equal if and only if their relative Frobenius lifts $\varphi_{A/S}$ (defined in (1.7.2)) agree.*
(ii) *A δ_S-structure on A is compatible with the group law if and only if the relative Frobenius lift $\varphi_{A/S}$ is a group homomorphism.*

Proof. (i) Apply proposition 5.1.2 to the morphism $\mathrm{id}\colon A \to A$, where the source has one δ_S-structure and the target has the other.

(ii) Apply proposition 5.1.2 to the group law $A \times_S A \to A$. □

5.2. **Canonical lifts.** We now fix an ind-affine p-adic sheaf S. As the category of ordinary $\delta_{W(S)}$-p-groups is equivalent to the category of ordinary p-groups over S by theorem 4.2.3, it follows that the corresponding categories of ind-objects are also equivalent. In particular, the category of ordinary $\delta_{W(S)}$-p-divisible groups is equivalent to the category of ordinary p-divisible groups over S. This allows us to define canonical lifts of ordinary abelian schemes using the theorem of Serre–Tate, which we recall below for convenience.

Let $S_0 \to S$ be a nilpotent thickening of p-adic sheaves and denote by $D_{S_0/S}$ the category whose objects are triples (G, A_0, h) where G/S is a p-divisible group, A_0/S_0 is an abelian variety and $h\colon G \times_S S_0 \xrightarrow{\sim} A_0[p^\infty]$ is an isomorphism. A morphism $(G, A_0, h) \to (G', A'_0, h')$ is a homomorphism $G \to G'$ and a homomorphism $A_0 \to A'_0$ compatible with h and h' in the evident sense.

Theorem 5.2.1 (Serre–Tate)**.** *The functor from the category of abelian schemes over S to $D_{S_0/S}$ given by*

$$A/S \mapsto (A[p^\infty], A \times_S S_0, \mathrm{id})$$

is an equivalence of categories.

Proof. When S is affine, see 1.2.1 of [16] or the appendix of [10]. The general case then follows for formal reasons. □

Now, if A/S is an ordinary abelian scheme there exists a unique ordinary abelian scheme $\widetilde{A}/W(S)$ with the property that $\widetilde{A}[p^\infty]/W(S)$ is the unique ordinary $\delta_{W(S)}$-p-divisible group lifting $A[p^\infty]/S$. We call $\widetilde{A}/W(S)$ the canonical lift of A/S.

It also follows from theorem 5.2.1 that the abelian scheme $\widetilde{A}/W(S)$ admits a unique lift of the Frobenius

$$\varphi_{\widetilde{A}/W(S)}\colon \widetilde{A} \to \varphi^*_{W(S)}(\widetilde{A})$$

which is a homomorphism.

Theorem 5.2.2. *There is a unique $\delta_{W(S)}$-structure on $\widetilde{A}/W(S)$ compatible with its group structure. Moreover, $\widetilde{A}/W(S)$ is the unique deformation of A/S admitting a $\delta_{W(S)}$-structure compatible with its group structure.*

Proof. As $\widetilde{A}/W(S)$ admits a lift of the relative Frobenius which is a group homomorphism, uniqueness of a $\delta_{W(S)}$-structure on $\widetilde{A}$ inducing this relative Frobenius lift follows from corollary 5.1.3, which also shows that such a $\delta_{W(S)}$-structure must be compatible with the group law.

Moreover, any deformation of A/S along $S \to W(S)$ which admits a $\delta_{W(S)}$-structure compatible with the group law induces a $\delta_{W(S)}$-structure on the corresponding p-divisible group, which must therefore be isomorphic to the p-divisible group of $\widetilde{A}/W(S)$ in a unique way, and so such a deformation must also be isomorphic to $\widetilde{A}$ in a unique way.

It remains to show that there exists a $\delta_{W(S)}$-structure on $\widetilde{A}/W(S)$ inducing the given relative Frobenius lift. For this we use the fact that moduli stack of abelian schemes is locally finitely presented and formally smooth (for formal smoothness see 8.5.24.(a) of [14]). Local finite presentation allows us to assume that $S = \mathrm{Spec}(\mathbf{Z}_p[T_1, \dots, T_r]/I)$ where I is an ideal, and formal smoothness allows us to assume instead that $S = \operatorname{colim}_n \operatorname{Spec}(\mathbf{Z}_p[T_1, \dots, T_r]/I^n)$, noting that now S is a p-torsion free ind-affine sheaf.

As the transition maps in the system $S = \operatorname{colim}_n \operatorname{Spec}(\mathbf{Z}_p[T_1, \dots, T_r]/I^n)$ are nilpotent immersions, it follows that we can find an representable open cover $(\widetilde{A}_i \to \widetilde{A})_i$ where each $\widetilde{A}_i$ is a relatively affine and smooth $W(S)$-scheme, in particular they are p-torsion free ind-affine sheaves, as explained in remark 2.4.2. The Frobenius lift $\varphi_{\widetilde{A}/S}$ restricts to each $\widetilde{A}_i$ (because $\widetilde{A}$ is p-adic and the Frobenius map is topologically the identity) and equips each $\widetilde{A}_i$ with a unique $\delta_{W(S)}$-structure by proposition 2.4.1. Because of this uniqueness, the $\delta_{W(S)}$-structures glue to give a $\delta_{W(A)}$-structure on $\widetilde{A}$ itself which by construction induces the given Frobenius lift. □

Theorem 5.2.3. *Let S be a p-adic scheme. Then the category of ordinary $\delta_{W(S)}$-abelian schemes is equivalent, via base change along the first ghost component, to the category of ordinary abelian schemes over S.*

Proof. It follows from the remarks above that the functor is essentially surjective. Moreover, it is faithful by lemma 5.1.1, as $S \to W(S)$ is a nilpotent immersion.

For fullness, let $f\colon A \to B$ be a homomorphism of ordinary abelian schemes over S. The restriction of f to the associated p-divisible groups lifts to a unique $\delta_{W(S)}$-morphism between the associated canonical lifts (of the p-divisible groups). By the Serre–Tate theorem (5.2.1), it follows that f itself lifts to a unique morphism

$$\widetilde{f}\colon \widetilde{A} \to \widetilde{B}.$$

As the restriction of $\widetilde{f}$ to the p-divisible groups is a $\delta_{W(S)}$-homomorphism it follows that

$$\widetilde{f} \circ \varphi_{\widetilde{A}} - \varphi_{\widetilde{A}'} \circ \varphi^*_{W(S)}(\widetilde{f})\colon \widetilde{A} \to \varphi^*_{W(S)}(\widetilde{B})$$

is trivial on the p-divisible groups and is therefore trivial itself. Hence $\widetilde{f}$ commutes with the Frobenius lifts on $\widetilde{A}$ and $\widetilde{B}$ is a $\delta_{W(S)}$-homomorphism, by lemma 5.1.2. □

5.3. **Duality.** Let S be a p-adic sheaf and let A/S be an ordinary abelian scheme. It is easily checked that kernel of the relative Frobenius

$$\varphi_{\widetilde{A}/W(S)} : \widetilde{A} \to \varphi^*_{W(S)}(\widetilde{A})$$

is $\widetilde{A}[p]_{\mathrm{mult}} \subset \widetilde{A}[p]$ so that we can factor the multiplication-by-p map as

$$\widetilde{A} \xrightarrow{\varphi_{\widetilde{A}/W(S)}} \varphi^*_{W(A)}(\widetilde{A}) \xrightarrow{v_{\widetilde{A}/W(S)}} \widetilde{A}.$$

It is immediate from the definition that the homomorphism

$$v_{\widetilde{A}/W(S)} : \varphi^*_{W(S)}(\widetilde{A}) \to \widetilde{A}$$

lifts the relative Verschiebung homomorphism. The dual of the relative Verschiebung is the relative Frobenius, and hence the dual of $v_{\widetilde{A}/W(S)}$

$$v^{\vee}_{\widetilde{A}/W(S)} : \widetilde{A}^{\vee} \to \varphi^*_{W(S)}(\widetilde{A}^{\vee})$$

defines a lift of the the relative Frobenius on the dual abelian scheme $\widetilde{A}^{\vee}$.

Corollary 5.3.1. *The canonical lift functor $A \mapsto \widetilde{A}$ is compatible with duality. That is, the canonical lift of the dual of A is canonically isomorphic to the dual of the canonical lift of A.*

Proof. By the uniqueness of theorem 5.2.2, it is enough to show that for an abelian scheme A/S, the dual $\widetilde{A}^\vee$ of the canonical lift admits a $\delta_{W(S)}$-structure compatible with its group structure. As the canonical lift and the dual are compatible with base change we may, as in the proof of theorem 5.2.2, assume that S is a p-torsion free ind-affine sheaf. We now need only show that $\widetilde{A}^\vee$ admits a lift of the relative Frobenius which is compatible with its group structure, but by the remarks above we may take $\varphi_{\widetilde{A}^\vee/W(S)} := v^\vee_{\widetilde{A}/W(S)}$. $\square$

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