Math. Ann. 329, 1–30 (2004) **Mathematische Annalen**

DOI: 10.1007/s00208-003-0490-1

Conductors and the moduli of residual perfection

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Received: 10 February 2003 / Revised version: 22 July 2003 /
Published online: 17 February 2004 –

Abstract. Let A be a complete discrete valuation ring with possibly imperfect residue field. The purpose of this paper is to give a notion of conductor for Galois representations over A which agrees with the classical Artin conductor when the residue field is perfect. The definition rests on two results of perhaps wider interest: there is a moduli space that parametrizes the ways of modifying A so that its residue field is perfect, and any Galois-theoretic object over A can be recovered from its pullback to the (residually perfect) discrete valuation ring corresponding to the generic point of this moduli space. Finally, I show that this conductor extends the non-logarithmic variant of Kato's conductor to representations of rank greater than one.

Introduction

Let A be a complete discrete valuation ring of residue characteristic $p > 0$. If the residue field of A is perfect, there is a satisfactory theory [26, IV, VI] of ramification over A. For example, let ρ be a Galois representation over A, which is a continuous action of an absolute Galois group of the fraction field of A on a finite-rank complex (for now) vector space. Then there is a non-negative integer, the Artin conductor of ρ, that measures the extent to which ρ is ramified. If, on the other hand, we allow the residue field of A to be imperfect, ramification over A is still quite mysterious. This prevents us from understanding, say, ramification in codimension one of local systems on arithmetic surfaces.

The work in this paper began with the observation that much about ramification over A can be understood by simply changing base to various extensions with perfect residue field and of relative ramification index one. The first main point (1.4) is that such extensions make up the (perfect-field-valued) points of a natural representable moduli problem. The *universal residual perfection* A^{u} of A is the A-algebra corresponding to the representing object itself. It is not a discrete valuation ring – but, in a sense, only because its residue ring is not a field (and actually not even noetherian). The *generic residual perfection* A^{g} of A is the A-algebra corresponding to the fraction field of the representing object. It is a complete discrete valuation ring with perfect residue field. Both A^{u} and A^{g} are, of course, unique up to unique isomorphism.

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It may be useful to keep a geometric analogue in mind. If we think of complete residually perfect discrete valuation rings as being like germs of curves, then it is natural to regard A^{u} as the universal jet on A transverse to the maximal ideal and A^{g} as the generic jet.

There is an explicit description (1.10) of these rings in the first section. For example, if $A = \mathbf{F}_p(x)[\![y]\!]$, then we have

$$A^{\mathrm{u}} \cong \mathbf{F}_p(\bar{x})[u_1, u_2, \ldots]^{p^{-\infty}}[\![y]\!] \text{ and}$$
$$A^{\mathrm{g}} \cong \mathbf{F}_p(\bar{x}, u_1, u_2, \ldots)^{p^{-\infty}}[\![y]\!],$$

where the A-algebra structures are determined by the data $x \mapsto \bar{x} + u_1 y + u_2 y^2 + \cdots$ and $y \mapsto y$. If $A = \widehat{\mathbf{Z}[x]_{(p)}}$, we have

$$A^{\mathrm{u}} \cong W\big(\mathbf{F}_p(\bar{x})[u_1, u_2, \ldots]^{p^{-\infty}}\big) \text{ and}$$
$$A^{\mathrm{g}} \cong W\big(\mathbf{F}_p(\bar{x}, u_1, u_2, \ldots)^{p^{-\infty}}\big),$$

where W denotes the functor of Witt vectors and the A-algebra structures are given by sending x to $(\bar{x}, u_1^p, u_2^{p^2}, \ldots)$.

In the second section, I give some properties of A^{u} and A^{g}. The most important is that the fraction field of A is algebraically closed in the fraction field of A^{g}. And hence the second main point: a Galois representation ρ over A is determined by its pullback $\rho|_{A^{\mathrm{g}}}$ to A^{g}. Therefore any invariant of ρ, such as a measure of ramification, can be recovered (in principal) from $\rho|_{A^{\mathrm{g}}}$.

Second, I give some evidence that in defining "non-logarithmic" (or "Artin") conductors for Galois representations over A, the most simple-minded way of proceeding along these lines is correct: the conductor $\mathrm{ar}(\rho)$ of ρ should be the classical Artin conductor of $\rho|_{A^{\mathrm{g}}}$. (Logarithmic, or "Swan", conductors are more subtle. See below.) Taking this as the definition, we have the following result.

Theorem (A). *Let ρ and ρ' be Galois representations over A.*

(1) $\mathrm{ar}(\rho)$ *is a non-negative integer.*
(2) $\mathrm{ar}(\rho \oplus \rho') = \mathrm{ar}(\rho) + \mathrm{ar}(\rho')$
(3) If ρ is trivialized by a residually separable extension of A, then $\mathrm{ar}(\rho)$ *agrees with the classical Artin conductor of ρ.*
(4) $\mathrm{ar}(\rho)$ *is zero if and only if ρ is unramified.*
(5) $\mathrm{ar}(\rho)$ *equals the codimension of the subspace of inertia invariants if and only if ρ is tame.*

This theorem is an elementary consequence of the basic properties of A^{g} proved in section 2. The proof of a slightly stronger version is written down in section 3.

For Galois representations of rank one, Kato [17] has introduced a logarithmic conductor. As first observed in the work [20] of Matsuda (who credits it to T. Saito), this conductor has a non-logarithmic variant, which we denote $\mathrm{ar}_{\mathrm{K}}(\cdot)$. The second part of the paper is devoted to the proof of the following result.

Theorem (B). *Let K and K^{g} denote the fraction fields of A and A^{g}. If χ is a class in $H^1(K, \mathbf{Q}/\mathbf{Z})$ and χ' is its image in $H^1(K^{\mathrm{g}}, \mathbf{Q}/\mathbf{Z})$, then* $\mathrm{ar}_{\mathrm{K}}(\chi) = \mathrm{ar}_{\mathrm{K}}(\chi')$.

For present purposes, the interest in this result is the following:

Corollary. *Let ρ be a rank-one Galois representation over A. Then the non-logarithmic Kato conductor of ρ agrees with* $\mathrm{ar}(\rho)$.

The intuitive reason why the theorem should be true is that the order of any differential form on A should remain unchanged when the form is pulled back to A^{g} and that conductors of rank-one representations are essentially pole orders of differential 1-forms. When A is of equal characteristic, this has meaning and, once the necessary foundations are laid, is essentially a proof. In fact, the observation that there are residually perfect extensions with this property is what led to the definition of the general Artin conductor. In mixed characteristic, however, this provides little more than motivation, and most of this paper is spent pushing it through to a real proof.

Kato's original, logarithmic conductor is not, however, always invariant under pullback to A^{g}, and so the naive logarithmic analogue of $\mathrm{ar}(\cdot)$ does not necessarily agree with Kato's conductor. For some brief thoughts on logarithmic conductors for representations of higher rank, see 3.3.

Of course, there is an "upper" filtration (3.5) of the absolute Galois group corresponding to this paper's conductor, and it satisfies the Hasse-Arf property simply because residually separable extensions do. Abbes and Saito [1] also have a non-logarithmic $\mathbf{Q}_{\geqslant 0}$-indexed upper filtration. It is tempting to hope the two agree (shifted by one); a proof of this would imply the interesting result that Abbes and Saito's filtration also satisfies the Hasse-Arf property. Boltje-Cram-Snaith [4] and Zhukov [30] also have approaches to non-abelian ramification theory; the relations with them are even more mysterious.

Let me briefly describe the remaining sections. In section 4, I recall Kato's theory, prove some basic results, and define the Kato-Artin conductor. The proof of theorem B when A is of equal characteristic is in section 5. It uses Matsuda's refinement [20] of Kato's refined Swan conductor. Because the proof in equal characteristic is so much simpler than the proof in mixed characteristic, I encourage the reader to read it first. The basic technique in mixed characteristic is to use Kato's description [17, 4.1] (following Bloch-Kato [3]) of certain graded pieces of cohomology groups in terms of explicit K-theoretic symbols and then understand how these symbols behave under pullback to A^{g}. Section 7 contains a commutative diagram that encodes this behavior, and section 8 gives the proof in mixed characteristic. In the final section, I prove the corollary above.

It is a pleasure to thank the Massachusetts Institute of Technology for its hospitality and the National Science Foundation for support while I was writing this paper.

Conventions

Throughout, p denotes a fixed prime number, and A denotes a discrete valuation ring, held fixed in each subsection, whose residue field has characteristic p. Its fraction field and residue field are denoted by K and k.

All other conventions and notation are quite common, but I will state them anyway:

All rings are commutative and contain 1, and all ring maps preserve 1. The fraction field of a domain R is denoted $\mathrm{Fr}(R)$. An extension of a field is a homomorphism to another field. An extension of a discrete valuation ring is an injective local homomorphism to another discrete valuation ring. In both cases, we usually refer to the target of the morphism, rather than the morphism itself, as the extension.

We say a ring A as above is of *mixed characteristic* if K has characteristic 0, and is of *equal characteristic* if K has characteristic p. Variants of the words *residue* and *generic* refer to the residue and fraction fields of A. We also use the same words to refer to extensions. For example, we might say B/A is residually purely inseparable or is generically Galois. An extension B/A is unramified (resp. tame) if it is finite and residually separable and its ramification index is one (resp. not divisible by p). The notations $e_{B/A}$, $f_{B/A}$, and $f^{\mathrm{sep}}_{B/A}$ denote the ramification index, residue degree, and separable residue degree of the extension B/A, and $U_A^{\bullet}$ denotes the filtration of A^* with $U_A^0 = A^*$ and $U_A^i = 1+\mathfrak{p}_A^i$ for positive integers i.

1. The moduli of residual perfection

The purpose of this section is to define the category CRP_A of complete residual perfections of A, to prove the objects are parametrized by a moduli space, and to give a concrete description of this moduli space. Our general reference for categorical terminology will be Mac Lane's book [19].

1.1. An $\mathbf{F}_p$-algebra R is *perfect* if the endomorphism $F : x \mapsto x^p$ of R is an isomorphism. The perfection $R^{\mathrm{pf}} = R^{p^{-\infty}}$ of an $\mathbf{F}_p$-algebra R is the universal perfect $\mathbf{F}_p$-algebra that R maps to. It is the colimit of the iterates of F. For any $\mathbf{F}_p$-algebra S, let $\mathsf{PerfAlg}_S$ be the full subcategory of the category of S-algebras whose objects are perfect.

1.2. Let CRP_A be the full subcategory of the category of A-algebras consisting of objects B such that B is flat (i.e., B is torsion-free as an A-module), B is complete with respect to the ideal $\mathfrak{p}_A B$, and $B/\mathfrak{p}_A B$ is perfect. (Note that the second condition forces morphisms to be continuous.) For an object $B \in \mathsf{CRP}_A$, let $\mathfrak{p}_B$ denote the ideal $\mathfrak{p}_A B$ (which will usually be prime), let $\overline{B}$ denote $B/\mathfrak{p}_B$, and if x is an element of B, let $\bar{x}$ denote its image in $\overline{B}$. If $f : B \to B'$ is a morphism in CRP_A, let $\bar{f}$ denote its reduction $\overline{B} \to \overline{B'}$. Let s_B denote the unique multiplicative section [26, II §4 Prop. 8] of the map $B \to \overline{B}$. Since $s_B(x)$ is the unique lift of x that has a p^m-th root for all integers m, every morphism $f : B \to B'$ in CRP_A satisfies

$$f \circ s_B = s_{B'} \circ \bar{f}. \tag{1.2.1}$$

1.3. If $\pi \in A$ is a uniformizer and x is an element of an object $B \in \mathsf{CRP}_A$, then there are unique elements $x_0, x_1, \cdots \in \overline{B}$ such that $x = s_B(x_0) + s_B(x_1)\pi + \cdots$. We call these the *coefficients of x (with respect to π)*. If $f : B \longrightarrow C$ is a morphism in CRP_A, then (1.2.1) implies that the coefficients of $f(x)$ are simply the images under $\bar{f}$ of the coefficients of x.

1.4. Theorem. The category CRP_A has an initial object A^{u}, and the functor

$$\mathsf{CRP}_A \to \mathsf{PerfAlg}_{\overline{A^{\mathrm{u}}}}, \quad B \mapsto \overline{B}$$

is an equivalence of categories.

In other words, the functor $\mathsf{CRP}_A \to \mathsf{PerfAlg}_{\overline{A}}$ defined by $B \mapsto \overline{B}$ is a representable moduli problem with universal object A^{u}.

It is a pleasant exercise to give a hands-off proof of this theorem using Freyd's method [19]. But we will need more precise information about A^{u}, and so we will give a quite explicit presentation of it in terms of a p-basis of k lifted to A.

1.5. Let $\pi \in A$ be a uniformizer, and let T be a lift to A of a p-basis of $\overline{A}$. (A good general reference for information on p-bases is EGA [10, Ch. 0 §21].) Let R_T be the polynomial algebra $\overline{A}[u_{t,j} \mid t \in T, j \in \mathbf{Z}_{>0}]$. We will see below that R_T^{pf} is naturally $\overline{A^{\mathrm{u}}}$, the moduli space we seek.

1.6. Lemma. Let Q be a residually perfect discrete valuation ring that is a subring of A with the property that A/Q is an extension of ramification index one. Let B be a Q-algebra that is complete with respect to an ideal I that contains the image of the maximal ideal of Q. Let n be a positive integer, and let $\varphi' : A \to B/I^n$ be a Q-linear homomorphism. For every $t \in T$, let $x_t \in B$ be a lift of $\varphi'(t)$. Then there is a unique Q-linear map $\varphi : A \to B$ such that $\varphi' = \varphi \mod I^n$ and $\varphi(t) = x_t$ for all $t \in T$.

Proof. Since B is complete with respect to I, it suffices by induction to prove the existence and uniqueness of φ modulo I^{n+1}. Let $\Omega^1_{A/Q}$ denote the A-module of Kähler differentials with respect to Q. Since A/Q is formally smooth [10, 19.7.1], some lift $A \to B/I^{n+1}$ of φ' exists, and so the set of such lifts is a torsor under

$$\begin{aligned}\mathrm{Hom}_A(\Omega^1_{A/Q}, I^n/I^{n+1}) &= \mathrm{Hom}_{\overline{A}}(\overline{A} \otimes_A \Omega^1_{A/Q}, I^n/I^{n+1}) \\ &= \mathrm{Hom}_{\overline{A}}(\bigoplus_{t\in T} \overline{A}dt, I^n/I^{n+1})\end{aligned}$$

with the obvious action. It follows that the image of T can be lifted arbitrarily and that any such lift determines $\varphi \mod I^{n+1}$.

1.7. Construction. *Functors* $U : \mathsf{CRP}_A \to \mathsf{PerfAlg}_{R_T^{\mathrm{pf}}}$ *and* $V : \mathsf{PerfAlg}_{R_T^{\mathrm{pf}}} \to \mathsf{CRP}_A$.

Let B be an object of CRP_A. For $t \in T$ and $j \in \mathbf{Z}_{>0}$, let $v_{t,j} \in \overline{B}$ denote the j-th coefficient (1.3) with respect to π of the image of t in B. The data $u_{t,j} \mapsto v_{t,j}$ gives $\overline{B}$ the structure of an R_T-algebra. Since $\overline{B}$ is perfect, it has a unique compatible R_T^{pf}-algebra structure; $U(B)$ is then $\overline{B}$ with this R_T^{pf}-algebra structure. It is easy to see this is functorial.

Let S be a perfect R_T^{pf}-algebra. If A is of equal characteristic, set $V(S) = S[\![\tilde{\pi}]\!]$, where $\tilde{\pi}$ is a free variable. Then $V(S)$ is an $\mathbf{F}_p[\![\tilde{\pi}]\!]$-algebra, and $\tilde{\pi} \mapsto \pi$ makes A into an $\mathbf{F}_p[\![\tilde{\pi}]\!]$-algebra. By 1.6, there is a unique $\mathbf{F}_p[\![\tilde{\pi}]\!]$-linear map $A \to V(S)$ such that for all $t \in T$,

$$t \mapsto \bar{t} + \sum_{j \geqslant 1} u_{t,j} \tilde{\pi}^j.$$

If A is of mixed characteristic, let C be the Cohen subring [7, pp. 82–83] of A determined by T. It is a complete discrete valuation ring of absolute ramification index one that contains T, and A/C is a finite residually trivial extension. Let W be the ring of Witt vectors [13, 0.1] with coefficients in S and let $s_W : S \to W$ denote the Teichmüller section. Again by 1.6 (taking $Q = \mathbf{Z}_p$), there is a unique map $C[X] \to W[\![\tilde{\pi}]\!]$ such that $X \mapsto \tilde{\pi}$ and for all $t \in T$, we have

$$t \mapsto s_W(\bar{t}) + \sum_{j \geqslant 1} s_W(u_{t,j}) \tilde{\pi}^j.$$

View A as a quotient of $C[X]$ using the map $X \mapsto \pi$, and put $V(S) = A \otimes_{C[X]} W[\![\tilde{\pi}]\!]$.

In either case, it is easy to see that V is a functor from $\mathsf{PerfAlg}_{R_T^{\mathrm{pf}}}$ to the category of A-algebras.

1.8. Proposition. The image of V is in CRP_A.

Proof. Let S be a perfect R_T^{pf}-algebra. Of the properties $V(S)$ is required to satisfy to be in CRP_A, the only one that is not immediately clear is flatness over A in mixed characteristic. To show this, it suffices to show the element $\pi \otimes 1 = 1 \otimes \tilde{\pi}$ is not a zero-divisor in $V(S)$.

Let $g(X)$ denote the Eisenstein polynomial that generates the kernel of the surjection $C[X] \to A$, and let $h(\tilde{\pi})$ denote its image in $W[\![\tilde{\pi}]\!]$. We will show $\tilde{\pi}$ is not a zero-divisor in the ring $V(S) = W[\![\tilde{\pi}]\!]/(h(\tilde{\pi}))$. Let $f_1(\tilde{\pi})$ and $f_2(\tilde{\pi})$ be elements of $W[\![\tilde{\pi}]\!]$ such that $\tilde{\pi} f_2(\tilde{\pi}) = h(\tilde{\pi}) f_1(\tilde{\pi})$.

Suppose $f_1(0) \neq 0$. Then $h(0)$, which is the image of $g(0)$ under the map $C \to W$, is a zero-divisor. But since $g(X)$ is an Eisenstein polynomial, this implies

p is a zero-divisor in W, which is impossible. Therefore, we have $f_1(0) = 0$ and, hence,

$$f_2(\tilde{\pi}) = h(\tilde{\pi})\frac{f_1(\tilde{\pi})}{\tilde{\pi}} \in h(\tilde{\pi})W[[\tilde{\pi}]].$$

Thus $f_2(\tilde{\pi})$ reduces to zero in $V(S)$, and so $\tilde{\pi}$ is not a zero-divisor in $V(S)$.

1.9. Construction. *Natural transformations* $\eta : 1 \to UV$ *and* $\varepsilon : VU \to 1$

Let B be an object of CRP_A and S be an object of $\mathsf{PerfAlg}_{R_T^{\mathrm{pf}}}$. Let $\eta(S)$ be the map

$$S \longrightarrow UV(S), \quad x \mapsto \overline{s_{V(S)}(x)}.$$

If A is of equal characteristic, let $\varepsilon(B)$ be the composite $VU(B) = \overline{B}[[\tilde{\pi}]] \longrightarrow B$ defined by $\tilde{\pi} \mapsto \pi$ and $b \mapsto s_B(b)$ for $b \in \overline{B}$. It is a homomorphism of rings because s_B is [26, II Prop. 8].

If A is of mixed characteristic, there is a unique map $W(\overline{B}) \longrightarrow B$ that reduces to the identity [26, II Prop. 10]. Then, $\tilde{\pi} \mapsto \pi$ determines a map $W(\overline{B})[[\tilde{\pi}]] \longrightarrow B$ and, hence, a map

$$\varepsilon(B) : VU(B) = V(\overline{B}) = A \otimes_{C[X]} W(\overline{B})[[\tilde{\pi}]] \longrightarrow B.$$

It is easy to see ε is natural in B and η is natural in S.

1.10. Theorem. $\langle V, U; \eta, \varepsilon\rangle$ is an adjoint equivalence between CRP_A and $\mathsf{PerfAlg}_{R_T^{\mathrm{pf}}}$.

The proof is nothing more than a straight-forward verification of the so-called triangular identities ($\varepsilon V \circ V\eta = 1$ and $U\varepsilon \circ \eta U = 1$) and is left to the reader.

1.11. Proposition. Let B be an object of CRP_A. If $\overline{B}$ is noetherian or a domain, then B is the same. If $\overline{B}$ is a field, then B is a discrete valuation ring.

Proof. Suppose $\overline{B}$ is noetherian. Since $\mathrm{gr}(B)$ is isomorphic to $\overline{B}[X]$, it is noetherian. This implies B is noetherian [2, 10.25].

Now suppose $\overline{B}$ is a domain and a and b are non-zero elements of B with $ab=0$. Since a uniformizer π of A is not a zero-divisor in B, we can assume $a, b \notin \pi B$, but this immediately contradicts the fact that $\overline{B}$ is a domain.

When $\overline{B}$ is a field, it is easy to see that every element outside $\mathfrak{p}_B$ is a unit. Thus B is a noetherian local ring whose maximal ideal is generated by a non-nilpotent element and, hence, a discrete valuation ring [26, I §2].

1.12. Because R_T^{pf} is the initial object of $\mathsf{PerfAlg}_{R_T^{\mathrm{pf}}}$, its image $V(R_T^{\mathrm{pf}})$ is the initial object of CRP_A, but we will denote it A^{u} to reflect the fact that it is independent of T (up to unique isomorphism). It is called the *universal residual perfection* of A. The *generic residual perfection* A^{g} of A is defined to be $V(\mathrm{Fr}(R_T^{\mathrm{pf}}))$. It is also independent of T. We will often denote its residue field $\overline{A^{\mathrm{g}}}$ by k^{g}.

It follows from the previous proposition that A^{u} is a domain (it is in fact integrally closed) and A^{g} is a discrete valuation ring. More generally, it follows that the points of A^{u} with values in perfect fields are the same as complete residually perfect extensions of A of ramification index one.

1.13. Note that if B is an extension of A of ramification index one, then CRP_B is a subcategory of CRP_A, and so there is a unique map $A^{\mathrm{u}} \to B^{\mathrm{u}}$. If it happens that T can be extended to a lift $T' \subset B$ of a p-basis for $\overline{B}$, this map is the same as the map associated by 1.10 to the obvious map $R_T \to R_{T'}$. We will consider the functoriality of the generic residual perfection in the next section.

1.14. It is not hard to show that the filtration

$$F_n R_T^{\mathrm{pf}} = \overline{A}[u_{t,j} \mid t \in T, 1 \leqslant j \leqslant n]^{\mathrm{pf}}$$

of R_T is independent of the choices of T and π. Though its role in this paper is small, this filtration is important and should not be ignored. (See 3.3.)

2. Properties

As in the previous section, A^{u} and A^{g} will denote the universal and generic residual perfections of A, and K^{g} will denote the fraction field of A^{g}.

2.1. Proposition. Let B be an extension of A. If $e_{B/A} = 1$ and B/A is residually separable, then there exists a unique map $A^{\mathrm{g}} \longrightarrow B^{\mathrm{g}}$ of A-algebras.

Note that we do not require that B/A be residually algebraic, and so B satisfies these assumptions if and only if B/A is formally smooth [10, 19.6.1,19.7.1].

Proof. By 1.4, it is enough to show there is a unique map $\overline{A^{\mathrm{g}}} \longrightarrow \overline{B^{\mathrm{g}}}$ of $\overline{A^{\mathrm{u}}}$ algebras. Since $\overline{A^{\mathrm{g}}}$ is the fraction field of $\overline{A^{\mathrm{u}}}$, there is clearly at most one. To show there is at least one, we just have to check that the map $\overline{A^{\mathrm{u}}} \to \overline{B^{\mathrm{g}}}$ is an inclusion. Since any p-basis for $\overline{A}$ can be extended to one for $\overline{B}$, this follows immediately from 1.10 and 1.13.

2.2. Proposition. Let B be a finite étale extension of A. If C is an object of CRP_A, then $C \otimes_A B$ is an object of CRP_B.

Proof. Since $B \otimes_A \overline{C}$ is finite étale over $\overline{C}$, which is perfect, it is perfect (by, for example, [10, 21.1.7]). Since $C \otimes_A B$ is a finitely generated free C-module, it is complete. Therefore, $C \otimes_A B$ is an object of CRP_B.

2.3. Proposition. Let B be a finite étale extension of A. Then the induced maps

$$B \otimes_A A^{\mathrm{u}} \to B^{\mathrm{u}} \quad \text{and} \quad B \otimes_A A^{\mathrm{g}} \to B^{\mathrm{g}}$$

of B-algebras are isomorphisms.

Proof. By 2.2, both $B \otimes_A A^{\mathrm{u}}$ and $B \otimes_A A^{\mathrm{g}}$ are in CRP_B, and so by 1.4, it is enough to show they are isomorphisms after tensoring with $\overline{B}$. Since $\overline{B}/\overline{A}$ is finite and separable, any p-basis for $\overline{A}$ is one for $\overline{B}$. Again, 1.10 and 1.13 complete the proof.

The rest of the results in this section are devoted to the proof of the following theorem.

2.4. Theorem. Fix a separable closure of K^{g}. Then the map $G_{K^{\mathrm{g}}} \to G_K$ of absolute Galois groups is surjective. The induced maps of inertia groups and wild inertia groups are also surjective.

2.5. Remark. It would be interesting to see if the analogous result is true for some motivic Galois group. For example, when A is of equal characteristic, is the functor from crystals (of whatever kind) on K to crystals on K^{g} fully faithful?

2.6. Lemma. K is algebraically closed in K^{g}.

Proof. If A is residually perfect, then $K^{\mathrm{g}} = K$ and the result is trivially true. Now assume A is not residually perfect. Let L/K be a finite subextension of K^{g}/K, and let B be the normalization of A in L. Since $e_{A^{\mathrm{g}}/A} = 1$, we have $e_{B/A} = 1$. Since $\overline{A}$ is separably closed in $\overline{A^{\mathrm{g}}}$, the extension B/A is residually purely inseparable. It is therefore enough to show it is residually separable.

Suppose there is an element $\bar{a} \in \overline{A} - (\overline{A})^p$ that has a p-th root in $\overline{B}$. We will derive a contradiction (working only modulo $\mathfrak{p}_A^2$). Let $a \in A/\mathfrak{p}_A^2$ be a lift of $\bar{a}$ and let $x \in B/\mathfrak{p}_B^2$ be a lift of $\sqrt[p]{\bar{a}}$. Then there is an element $y \in \overline{B}$ such that $a = x^p + \pi y$.

Let θ denote the map $B \longrightarrow A^{\mathrm{g}}/\mathfrak{p}_A^2 A^{\mathrm{g}}$, and let s denote the multiplicative section of the map $A^{\mathrm{g}}/\mathfrak{p}_A^2 A^{\mathrm{g}} \longrightarrow \overline{A^{\mathrm{g}}}$. Since the set $\{\bar{a}\}$ can be extended to a p-basis of $\overline{A}$, there is (by, say, 1.10) an element $u \in \overline{A^{\mathrm{g}}} - \overline{A}^{\mathrm{pf}}$ such that $\theta(a) = s(\bar{a}) + \pi u$. Take $v \in \overline{A^{\mathrm{g}}}$ such that $\theta(x) = s(\bar{x}) + \pi v$. Then we have

$$\begin{aligned} s(\bar{a}) + \pi u &= \theta(a) \\ &= \theta(x)^p + \pi y \\ &= (s(\bar{x}) + \pi v)^p + \pi y \\ &= s(\bar{x}^p) + \pi y \end{aligned}$$

and thus $u = y \in \overline{A}^{\mathrm{pf}}$. Since this cannot be, we have our contradiction.

2.7. A finite extension B of A is said to be *monogenic* if there is an element $x \in B$ that generates B as an A-algebra. For example, any finite extension that is generically and residually separable is monogenic [26, III §6 Prop. 12].

2.8. Lemma. Let B be a finite generically separable extension of A. Then B/A is monogenic if and only if $B \otimes_A A^{\mathrm{g}}$ is a discrete valuation ring. In this case, we have $f^{\mathrm{sep}}_{B\otimes_A A^{\mathrm{g}}/A^{\mathrm{g}}} = f^{\mathrm{sep}}_{B/A}$.

Proof. Suppose $B \otimes_A A^{\mathrm{g}}$ is a discrete valuation ring. Then we have

$$A^{\mathrm{g}} \otimes_A \Omega^2_{B/A} = \Omega^2_{B\otimes_A A^{\mathrm{g}}/A^{\mathrm{g}}} = 0,$$

where, as usual, $\Omega^2_{*/*}$ denotes the second exterior power of the module of relative Kähler differentials. Therefore, $\Omega^2_{B/A}$ is zero. The extension B/A is then monogenic by de Smit [8, 4.2].

Now suppose B/A is monogenic. By 2.6, the ring $B \otimes_A A^{\mathrm{g}}$ is a domain. By 2.3, it suffices to assume B/A is residually purely inseparable, in which case it is enough to show that $B \otimes_A A^{\mathrm{g}}$ is generated as an A^{g}-algebra by the root of an Eisenstein polynomial.

There is [5, Prop. 1] a finite extension A' of A such that $e_{A'/A} = f^{\mathrm{sep}}_{A'/A} = 1$ and $B \otimes_A A'$ is a discrete valuation ring with $f_{B\otimes_A A'/A'} = 1$. By Zorn's lemma, we can assume the residue field of A' is $\overline{A}^{\mathrm{pf}}$ at the expense of allowing A' to be integral, rather than finite, over A. Now let $x \in B$ be a generator of the A-algebra B and let $f(X) \in A[X]$ be its characteristic polynomial over A. Put

$$y = x \otimes 1 - 1 \otimes s_{A^{\mathrm{u}}}(\bar{x}) \in B \otimes_A A^{\mathrm{u}}.$$

Then y generates $B \otimes_A A^{\mathrm{u}}$ as an A^{u}-algebra, the image of y in $(B \otimes_A \overline{A^{\mathrm{u}}})_{\mathrm{red}} = \overline{A^{\mathrm{u}}}$ is zero, and the polynomial

$$g(X) = f(X + s_{A^{\mathrm{u}}}(\bar{x})) \in A^{\mathrm{u}}[X]$$

is the characteristic polynomial of y over A^{u}. We will show $g(X)|_{A^{\mathrm{g}}}$ is an Eisenstein polynomial.

First, note that since $\bar{y} = 0$, we have $g(X)|_{A^{\mathrm{g}}} \equiv X^n \mod \mathfrak{p}_{A^{\mathrm{g}}}$, where n is the degree of B/A. All that remains is to show $g(0)|_{A^{\mathrm{g}}} \notin \mathfrak{p}^2_{A^{\mathrm{g}}}$. Hence, it is enough to show $g(0)|_{A^{\mathrm{u}}} \notin \mathfrak{p}^2_A A^{\mathrm{u}}$ and, therefore, even $g(0)|_{A'} \notin \mathfrak{p}^2_{A'}$. But since $y|_{A'}$ generates $B \otimes_A A'$ as an A'-algebra and since $B \otimes_A A'$ is a discrete valuation ring, $g(X)|_{A'}$ is an Eisenstein polynomial. Therefore, $g(0)|_{A'} \notin \mathfrak{p}^2_{A'}$ and $g(X)|_{A^{\mathrm{g}}}$ is an Eisenstein polynomial.

2.9. Lemma. Let B be a finite, generically Galois extension of A. Let B' be the integral closure of the domain $B \otimes_A A^{\mathrm{g}}$. Then $f^{\mathrm{sep}}_{B'/A^{\mathrm{g}}} = f^{\mathrm{sep}}_{B/A}$.

Proof. By 2.3, it is enough to assume B/A is residually purely inseparable. Let G be the generic Galois group of B/A. Then 2.6 implies G is also the generic Galois group of B'/A^{g}. Let C' be the maximal étale subextension of B'/A^{g}. Then C' corresponds to a normal subgroup of G and, hence, to a generically Galois subextension C of B/A. Since the extension B/A is residually purely inseparable and $e_{C'/A} = 1$, the ramification index of C/A is one and its Galois group is a p-group.

Now let D be a monogenic subextension of C/A. Then $D \otimes_A A^{\mathrm{g}}$ is a discrete valuation ring by 2.8. Since $D \otimes_A A^{\mathrm{g}}/A^{\mathrm{g}}$ is a subextension of C'/A^{g}, it is étale. Now, the residue field of $D \otimes_A A^{\mathrm{g}}$ is $(\overline{D} \otimes_{\overline{A}} \overline{A^{\mathrm{g}}})_{\mathrm{red}}$. Since $\overline{D}$ is a finite purely inseparable extension of $\overline{A}$, this reduced quotient is $\overline{A^{\mathrm{g}}}$. Thus, the étale extension $D \otimes_A A^{\mathrm{g}}/A^{\mathrm{g}}$ is trivial, and therefore, so is D/A.

But since the generic Galois group of C/A is a p-group and since all extensions of degree p are monogenic, the only way for all monogenic subextensions of C/A to be trivial is if C/A itself is trivial. And this can happen only if C'/A^{g} is trivial.

Proof. (of 2.4) It follows from 2.6 that K is separably closed in K^{g}. The surjectivity of $G_{K^{\mathrm{g}}} \to G_K$ is just the translation of this into Galois theory.

The image of the inertia subgroup of $G_{K^{\mathrm{g}}}$ is then a closed normal subgroup N of G_K and is contained in the inertia subgroup of G_K. Let M be the corresponding extension of K. Now, for any finite extension B of A that is contained in M, we see, by lemma 2.9, that B/A is unramified. Therefore, N is the entire inertia group of K.

Since the wild inertia groups are the unique (pro-)p-Sylow subgroups of the corresponding inertia groups [29, Ch. I, 1.1], the surjectivity of the map between wild inertia groups follows immediately.

2.10. *Questions.* The maps $A \to A^{\mathrm{u}}$ and $A \to A^{\mathrm{g}}$ are faithfully flat. How can we best understand the descent data? The generic descent data? What can be said about the structure of the exact sequence

$$1 \to H \to G_{K^{\mathrm{g}}} \to G_K \to 1?$$

Since A^{g} is residually perfect, the classfield theory of Hazewinkel and Serre [12] gives a description of the abelianization of $G_{K^{\mathrm{g}}}$. Does this give a classfield theory describing the abelianization of G_K? When K is a higher local field, how does the classfield theory of K^{g} relate to Kato and Parshin's classfield theory [14,22] of K?

3. Conductors

3.1. Fix a field Λ whose characteristic is not p, and let ρ be a Galois representation over A, that is, a homomorphism $\rho : G \longrightarrow \mathrm{Aut}(V)$, where G is the Galois group of a finite generically Galois extension B of A and V is a finite-rank Λ-module. For any $i \in \mathbf{N}$, let G_i be the kernel of the map $G \to \mathrm{Aut}(B/\mathfrak{p}_B^{i+1})$. Define

$$\mathrm{ar}_{\mathrm{n}}^{B}(\rho) = e_{B/A}^{-1} \sum_{i \geqslant 0} |G_i| \operatorname{codim} V^{G_i},$$

where $|\cdot|$ denotes cardinality and $\operatorname{codim} V^{G_i}$ is the codimension of the subspace of G_i-invariants. We call $\mathrm{ar}_{\mathrm{n}}^{B}(\rho)$ the *naive Artin conductor of ρ with respect to B*. It is a non-negative rational number. If B/A is residually separable, then $\mathrm{ar}_{\mathrm{n}}^{B}(\rho)$ is left unchanged if we replace B with a larger generically Galois and residually separable extension of A. In this case, we will use the notation $\mathrm{ar}_{\mathrm{n}}(\rho)$.

Let L denote the fraction field of B. Then by 2.4, $L \otimes_K K^{\mathrm{g}}$ is a finite Galois extension of K^{g}, and its Galois group is canonically isomorphic to G. Let $\rho|_{A^{\mathrm{g}}}$ denote the resulting representation of $\mathrm{Gal}(L \otimes_K K^{\mathrm{g}}/K)$. Define the Artin conductor $\mathrm{ar}(\rho)$ of ρ to be $\mathrm{ar}_{\mathrm{n}}(\rho|_{A^{\mathrm{g}}})$. We have the following slightly stronger version of theorem A:

3.2. Theorem. Let A, ρ, G, V, and B be as above, and let ρ' be another Galois representation over A. Then we have the following:

(1) $\mathrm{ar}(\rho \oplus \rho') = \mathrm{ar}(\rho) + \mathrm{ar}(\rho')$.
(2) $\mathrm{ar}(\rho)$ is a non-negative integer.
(3) $\mathrm{ar}(\rho) = 0$ if and only if ρ is unramified, i.e., G_0 acts trivially on V.
(4) If B/A is monogenic, then $\mathrm{ar}(\rho) = \mathrm{ar}_{\mathrm{n}}^{B}(\rho)$.
(5) The following are equivalent:
 (a) ρ is tame, i.e., the p-Sylow subgroup of G_0 acts trivially on V
 (b) $\mathrm{ar}(\rho) = \operatorname{codim}(V^{G_0})$
 (c) $\mathrm{ar}(\rho) \leqslant \operatorname{codim}(V^{G_0})$

Proof. Statement 1 follows from the additivity of the naive Artin conductor. Statement 2 follows from the classical Hasse-Arf theorem [26, VI §2] (if the characteristic of Λ is not zero, see [27, 19.3]) and the non-negativity of the naive Artin conductor. Statement 3 follows from the statement about inertia groups in 2.4.

Now consider statement 4. Let G' be the generic Galois group of the extension $B \otimes_A A^{\mathrm{g}}/A^{\mathrm{g}}$. By 2.8, the tensor product $B \otimes_A A^{\mathrm{g}}$ is a discrete valuation ring. A short argument then shows that for all $i \in \mathbf{N}$, we have

$$G_i = G'_{ei} = \cdots = G'_{ei+e-1},$$

where $e = e_{B \otimes_A A^{\mathrm{g}}/B}$. Therefore, $\mathrm{ar}_{\mathrm{n}}^{B}(\rho) = \mathrm{ar}_{\mathrm{n}}^{B \otimes_A A^{\mathrm{g}}}(\rho|_{A^{\mathrm{g}}}) = \mathrm{ar}(\rho)$.

As for statement 5, if ρ is tame, then there is some generically Galois and residually separable extension B of A such that $\rho|_B$ is trivial. Since B/A is residually separable, it is monogenic, and so statement 4 implies $\mathrm{ar}(\rho) = m$. If, on the other hand, we have $\mathrm{ar}(\rho) \leqslant m$, then $\rho|_{A^{\mathrm{g}}}$ is tame, 2.4 therefore implies ρ is tame.

3.3. *Remark.* As mentioned in the introduction, logarithmic conductors are more subtle than non-logarithmic ones. For $\chi \in H^1(K, \mathbf{Q}/\mathbf{Z})$, let $\mathrm{sw}_{\mathrm{K}}(\chi)$ be Kato's Swan conductor [17, 2.3], which equals the logarithmic order of the pole of his refined Swan conductor. Then there can be classes χ (those that Matsuda says are *of type II* [20, 3.2.10]) such that $\mathrm{sw}_{\mathrm{K}}(\chi) = \mathrm{sw}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) + 1$. Because of this, the naive Swan analogue of $\mathrm{ar}(\cdot)$ does not always agree with Kato's Swan conductor. I believe that by taking into account the filtration $F_{\bullet}\overline{A^{\mathrm{u}}}$ of A^{u} (see 1.14) and using Kato's refined conductor in the context of perfect residue fields [15], one could give a good definition of a logarithmic conductor. Indeed, using the techniques of Matsuda (see section 5), it is easy to see how to recover the Kato-Swan conductor of χ from the refined Swan conductor of $\chi|_{A^{\mathrm{g}}}$ in equal characteristic.

3.4. *Remark.* It may also be worth mentioning that the conductor $\mathrm{ar}(\cdot)$ probably does not satisfy an induction formula. Let A'/A be a finite generically separable extension, ρ' a Galois representation over A', and ρ the induced representation over A. If $\overline{A}$ is perfect, then [26, VI §2]

$$\mathrm{ar}(\rho) = f_{A'/A}\mathrm{ar}(\rho) + \dim(\rho)v_A(\mathfrak{D}), \tag{3.4.1}$$

where $\mathfrak{D}$ denotes the discriminant of A'/A. In general, the refined conductor of ρ should be the norm, in some suitable sense, of the refined conductor of ρ'. When $\overline{A}$ is perfect, the conductor determines the refined conductor up to an element of A^*, and then 3.4.1 would follow from such a norm formula for refined conductors. But when $\overline{A}$ is not perfect, the refined conductor contains more information than the conductor together with a unit. In fact, even in the abelian case, the conductor of ρ' probably does even determine the conductor of ρ in general.

3.5. There is, however, an apparently satisfactory theory of the upper filtration. In the notation of 3.1, since G is naturally identified with $\mathrm{Gal}(L \otimes_K K^{\mathrm{g}}/K^{\mathrm{g}})$, it inherits a $\mathbf{Q}_{\geqslant 0}$-indexed upper filtration [26, IV §3]. This filtration is both stable under passage to quotients and has integral jumps on abelian quotients for the simple reason that the same facts are true for residually separable extensions. We also have the usual relation to conductors: following Katz [18, 1.1], there is a break decomposition $V = \oplus_x V(x)$, with respect to $G^{\bullet}$, and we have

$$\mathrm{ar}(\rho) = \mathrm{codim}\,(V^{G_0}) + \sum_{x \in \mathbf{Q}_{\geqslant 0}} x \dim V(x).$$

4. Kato's theory

The purpose of this section is to collect some results in Kato's theory [17]. Let us first recall the basics.

4.1. Let F be a field and let $n > 0$ and r be integers. If n is invertible in F^*, let $\mathbf{Z}/n(r)$ be the r-th Tate twist of the constant sheaf $\mathbf{Z}/n$ on the étale topology (of Grothendieck [11]) of F. If the characteristic of F is $p > 0$, write $n = mp^s$, where $p \nmid m$, and let $\mathbf{Z}/n(r)$ be the complex

$$\mathbf{Z}/m(r) \oplus W_s\Omega^r_{F,\log}[-r]$$

of abelian sheaves on $\mathrm{Spec}(F)_{\text{ét}}$. Here, $W_s\Omega^r_{F,\log}$ is the piece of degree r of the logarithmic part [13, I 5.7] of the de Rham-Witt complex $W_s\Omega^\bullet_F$ on $\mathrm{Spec}(F)_{\text{ét}}$.

For positive integers q, write $H^q_n(F) = H^q(F, \mathbf{Z}/n(q-1))$, and let $H^q(F)$ be the colimit of $H^q_n(F)$ over the integers n (ordered by divisibility). The natural map $H^q_n(F) \to H^q(F)$ is an isomorphism of $H^q_n(F)$ with the n-torsion of $H^q(F)$. I will usually identify the two without comment.

Let

$$h_F : F^* \longrightarrow H^1(F, \mathbf{Z}/n(1))$$

be the connecting homomorphism of the Kummer triangle

$$\mathbf{Z}/n(1) \longrightarrow \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \longrightarrow \mathbf{Z}/n(1)[1].$$

(When n is a power of the characteristic of F, the existence of such a triangle follows from the theory of the de Rham-Witt complex [13, I 3.23.2, I 5.7.1].) Let $K^{\mathrm{M}}_r(F)$ denote the r-th Milnor K-group [21] of F. Then there is a homomorphism

$$K^{\mathrm{M}}_r(F) \longrightarrow H^r(F, \mathbf{Z}/n(r)),$$

also denoted h_F, sending $\{x_1, \ldots, x_r\}$ (for $x_1, \ldots, x_r \in F^*$) to the cup product

$$h_F(x_1) \cup \cdots \cup h_F(x_r).$$

For $\chi \in H^q_n(F)$, let $\{\chi, x_1, \ldots, x_r\}$ denote $\chi \cup h_F(\{x_1, \ldots, x_r\})$. Taking the colimit over integers n, we get a pairing

$$H^q(F) \otimes K^{\mathrm{M}}_r(F) \longrightarrow H^{q+r}(F).$$

We will use $\{\chi, x_1, \ldots, x_r\}$ to denote the image of $\chi \otimes \{x_1, \ldots, x_r\}$ under this pairing as well.

If the characteristic of F is p, let $\xi_s : W_s\Omega^{q-1}_F \twoheadrightarrow H^q_{p^s}(F)$ denote the higher Artin-Schreier maps [17, 1.3].

4.2. For any non-negative integer n, let $\mathrm{fil}_n H^q(K)$ be the subgroup of classes χ that have the property $\{\chi|_{A'}, 1 + \mathfrak{p}^n_A \mathfrak{p}_{A'}\} = 0$ for every henselian extension A' of A. This gives [17, 2.2, 6.3] an exhaustive increasing filtration of $H^q(K)$. The *Kato-Swan conductor* (or *logarithmic Kato conductor*) of a class $\chi \in H^q(K)$ is the smallest integer n such that $\chi \in \mathrm{fil}_n H^q(K)$. It is denoted $\mathrm{sw}_{\mathrm{K}}(\chi)$.

4.3. Let $\tilde{A}$ denote the henselization of the localization of the polynomial algebra $A[T]$ at the ideal generated by $\mathfrak{p}_A$. Then $\tilde{A}$ is a henselian discrete valuation ring, and for any uniformizer π of A, we have [17, 6.3]

$$\chi \in \mathrm{fil}_n H^q(K) \iff \{\chi|_{\tilde{A}}, 1+\pi^{n+1}T\} = 0.$$

We will denote the fraction field of $\tilde{A}$ by $\tilde{K}$ and the residue field by $\tilde{k}$.

4.4. The map $H^q(k) \to H^q(K)$ extends naturally to an exact sequence

$$0 \longrightarrow H^q(k) \longrightarrow \mathrm{fil}_0 H^q(K) \longrightarrow H^{q-1}(k) \longrightarrow 0.$$

Given a uniformizer π of A, the map $\psi \mapsto \{\psi, \pi\}$ is a splitting [17, 6.1].

4.5. The reduction map

$$(A - \{0\})/U_A^1 \longrightarrow k$$

gives k the structure of a log ring [16, 1.1]. Let $\omega_k^1 = \omega_{k/\mathbf{Z}}^1$ denote the k-module of absolute Kähler differentials with respect to this log structure [16, 1.7]. For $q \in \mathbf{N}$, let ω_k^q denote the q-th exterior power of ω_k^q. There is a natural exact sequence

$$0 \longrightarrow \Omega_k^q \longrightarrow \omega_k^q \xrightarrow{\text{res}} \Omega_k^{q-1} \longrightarrow 0.$$

(Of course, $\Omega_k^\bullet$ means $\Omega_{k/\mathbf{Z}}^\bullet$.) Given a uniformizer π of A, the map $\eta \mapsto \eta \wedge \mathrm{dlog}(\pi)$ is a splitting.

4.6. There is a unique map $\lambda_A : \omega_k^{q-1} \longrightarrow \mathrm{fil}_0 H_p^q(K)$ that gives rise to a map of sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_k^{q-1} \arrow[r] \arrow[d, "\xi_1"'] & \omega_k^{q-1} \arrow[r] \arrow[d, "\lambda_A"'] & \Omega_k^{q-2} \arrow[r] \arrow[d, "\xi_1"'] & 0 \\
0 \arrow[r] & H_p^q(k) \arrow[r] & \mathrm{fil}_0 H_p^q(K) \arrow[r] & H_p^{q-1}(k) \arrow[r] & 0
\end{tikzcd}$$

respecting the splittings in 4.4 and 4.5 (for every uniformizer π of A).

The following theorem [17, 5.1, 5.2, 5.3] is fundamental.

4.7. Theorem. Let n be a positive integer. Then for any class $\chi \in \mathrm{fil}_n H^q(K)$, there is a unique element $\eta \in \mathfrak{p}_A^{-n} \otimes_A \omega_k^q$ such that for every henselian extension A' of A and every element z of $\mathfrak{p}_A^n A'$, we have

$$\{\chi, 1+z\} = \lambda_{A'}(z\eta).$$

Furthermore, the function $\chi \mapsto \eta$ induces an injective homomorphism

$$\kappa_n : \mathrm{gr}_n H^q(K) \longrightarrow \mathfrak{p}_A^{-n} \otimes \omega_k^q.$$

I will typically write $\kappa_n(\chi)$ for the image under κ_n of the graded class of χ. It is called the *refined Swan conductor of* χ.

4.8. Let us now consider the non-logarithmic analogue of the Kato-Swan conductor. Let χ be a class in $H^1(K, \mathbf{Q}/\mathbf{Z})$ and put $n = \mathrm{sw}_{\mathrm{K}}(\chi)$. Define

$$\mathrm{ar}_{\mathrm{K}}(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is unramified} \\ 1 & \text{if } \chi \text{ is tame and ramified} \\ n & \text{if } \chi \text{ is ramified and } \kappa_n(\chi) \in \mathfrak{p}_A^{-n} \otimes \Omega_k^1 \\ n+1 & \text{if } \chi \text{ is ramified and } \kappa_n(\chi) \notin \mathfrak{p}_A^{-n} \otimes \Omega_k^1 \end{cases}$$

We call $\mathrm{ar}_{\mathrm{K}}(\chi)$ the *Kato-Artin conductor* of χ. It is the natural non-logarithmic analogue of Kato's Swan conductor. As mentioned in the introduction, when $\mathrm{sw}_{\mathrm{K}}(\chi)$ is not zero, $\mathrm{ar}_{\mathrm{K}}(\chi)$ can be viewed as the order of the pole of $\kappa_n(\chi)$ in the usual sense and $\mathrm{sw}_{\mathrm{K}}(\chi)$ can be viewed as the order in the logarithmic sense.

Matsuda [20, 3.2.5] has used what is essentially the same conductor. His is one less than $\mathrm{ar}_{\mathrm{K}}(\chi)$ except when χ is unramified, in which case both are zero.

Some facts

The rest of this section contains some propositions we will need in the proof of theorem B in mixed characteristic. All the proofs are straightforward.

4.9. Proposition. Let χ be a class in $H^1(K)$. If $\mathrm{ar}_{\mathrm{K}}(\chi)$ and $\mathrm{sw}_{\mathrm{K}}(\chi)$ have the same value, then it is a multiple of p.

Proof. [17, 5.4]

Let A' be an extension of A of ramification index e, let K' denote its fraction field, and let k' denote its residue field. Let n, q, and s be positive integers.

4.10. Proposition. The following diagram commutes:

$$\begin{tikzcd} \mathrm{gr}_{en} H^q(K') \arrow[r, "\kappa_{en}"] & \mathfrak{p}_{A'}^{en} \otimes_{A'} \omega_{k'}^q \\ \mathrm{gr}_n H^q(K) \arrow[r, "\kappa_n"] \arrow[u] & \mathfrak{p}_A^n \otimes_A \omega_k^q \arrow[u] \end{tikzcd}$$

Proof. Use the uniqueness statement in 4.7.

4.11. Corollary. If the extension A'/A is tame, then for any class $\chi \in H^q(K)$, we have $\mathrm{sw}_{\mathrm{K}}(\chi|_{A'}) = e\,\mathrm{sw}_{\mathrm{K}}(\chi)$

Proof. The map $\Omega_k^\bullet \to \Omega_{k'}^\bullet$ is injective.

4.12. Proposition. If the extension K'/K is finite and Galois with group G and its degree is not a multiple of p, the natural maps

$$\begin{gathered} H^q_{p^s}(K) \longrightarrow H^q_{p^s}(K')^G \\ \mathrm{fil}_{n-1} H^q_{p^s}(K) \longrightarrow \mathrm{fil}_{en-1} H^q_{p^s}(K')^G \end{gathered}$$

are isomorphisms.

Proof. Because the order of G is relatively prime to p, the groups $H^i(G, H^j_{p^s}(K'))$ are zero for $i > 0$. The existence of a spectral sequence

$$H^i(G, H^j_{p^s}(K')) \Rightarrow H^{i+j}_{p^s}(K),$$

implies the first map is an isomorphism. The second then is by 4.11.

4.13. Proposition. The exact sequences of 4.4 form a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^q_{p^s}(k') \arrow[r] & \mathrm{fil}_0 H^q_{p^s}(K') \arrow[r] & H^{q-1}_{p^s}(k') \arrow[r] & 0 \\ 0 \arrow[r] & H^q_{p^s}(k) \arrow[r] \arrow[u] & \mathrm{fil}_0 H^q_{p^s}(K) \arrow[r] \arrow[u] & H^{q-1}_{p^s}(k) \arrow[r] \arrow[u, "e"] & 0, \end{tikzcd} \tag{4.13.1}$$

where the left two vertical maps are the canonical maps and the rightmost vertical map is the canonical map multiplied by e.

Proof. Apply 4.4.

4.14. Proposition. Suppose the extension A'/A is tame and generically Galois with group G. Then its inertia group G_0 acts trivially on $\mathrm{fil}_0 H^q_{p^s}(K')$.

Proof. Assume, as we may, that A'/A is residually trivial, and consider (4.13.1). Because A'/A is tame and $k' = k$, the outer vertical morphisms are isomorphisms. Therefore the inner vertical map is, too. Because G acts trivially on $\mathrm{fil}_0 H^q_{p^s}(K)$, it acts trivially on $\mathrm{fil}_0 H^q_{p^s}(K')$.

5. The proof in equal characteristic

In this section, we use Matsuda's refinement [20] of Kato's refined Swan conductor to prove theorem B in equal characteristic. Again, let us first recall the basics.

Assume A is of equal characteristic.

5.1. For any positive integer s, let $W_s(K)$ denote the group of Witt vectors of K of length s. The Verschiebung maps form an inductive system of abelian groups

$$W_1(K) \xrightarrow{V} W_2(K) \xrightarrow{V} W_3(K) \xrightarrow{V} \cdots$$

As in Fontaine [9], let $CW(K)$ be its colimit. For example, if $\mathbf{F}_p$ denotes the finite field of p elements, then $CW(\mathbf{F}_p) = \mathbf{Q}_p/\mathbf{Z}_p$. The maps $W_s(K) \to \prod_{-\mathbf{N}} K$ defined by

$$(a_{-s+1}, \ldots, a_0) \mapsto (\ldots, 0, a_{-s+1}, \ldots, a_0)$$

induce a bijection between $CW(K)$ and the set of elements $(\ldots, a_{-1}, a_0)$ such that $a_{-i} = 0$ for sufficiently large i. I will typically use this identification without comment.

5.2. The Frobenius endomorphisms of the groups $W_s(K)$ extend to an endomorphism of $CW(K)$. Call it F. For any separable closure K^{sep} of K, we have an Artin-Schreier sequence of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules

$$0 \longrightarrow CW(\mathbf{F}_p) \longrightarrow CW(K^{\mathrm{sep}}) \xrightarrow{F-1} CW(K^{\mathrm{sep}}) \longrightarrow 0.$$

It is easy to show that this sequence induces a surjection

$$\xi : CW(K) \longrightarrow H^1(K, CW(\mathbf{F}_p)) = H^1(K, \mathbf{Q}_p/\mathbf{Z}_p)$$

with kernel $(F-1) \cdot CW(K)$.

5.3. Let $\varphi : CW(K) \to \Omega^1_K$ denote the homomorphism defined by

$$(\ldots, a_{-1}, a_0) \longmapsto -\sum_{i \in \mathbf{N}} a_{-i}^{p^i - 1} da_{-i}.$$

It appears that this map was first considered by Serre [25]; it has a nice interpretation in terms of the de Rham-Witt complex [13, I 3.12].

5.4. Define the following filtrations indexed by non-negative integers n:

$$\begin{aligned} \mathrm{fil}_n CW(K) &= \{(\ldots, a_{-1}, a_0) \mid \forall i \in \mathbf{N}\ p^i v_A(a_{-i}) \geqslant -n\}, \\ \mathrm{fil}_n \Omega^1_K &= \mathfrak{p}_A^{-n} \cdot \mathrm{dlog}(K^*), \text{ and} \\ \mathrm{fil}_n H^1(K, \mathbf{Q}_p/\mathbf{Z}_p) &= \xi(\mathrm{fil}_n CW(K)). \end{aligned}$$

The filtration on fixed length Witt vectors was apparently first considered by Schmid [24] in the residually perfect case and (independently) Brylinski [6] in the residually imperfect case. It is immediate that ξ and φ preserve these filtrations. It is easy to check that φ does not factor through ξ but that $\mathrm{gr}\,\varphi$ does factor through $\mathrm{gr}\,\xi$. Matsuda remarked [20, 3.2.2] that we have even more:

5.5. Proposition. Let n be a non-negative integer. Then $\mathrm{fil}_n\varphi/\mathrm{fil}_{[n/p]}\varphi$ factors through $\mathrm{fil}_n\xi/\mathrm{fil}_{[n/p]}\xi$.

(Here, $[x]$ is the greatest integer that is at most x.)

5.6. Denote by ϕ_n the resulting homomorphism

$$\mathrm{fil}_n H^1(K,\mathbf{Q}_p/\mathbf{Z}_p)/\mathrm{fil}_{[n/p]}H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) \to \mathrm{fil}_n\Omega^1_K/\mathrm{fil}_{[n/p]}\Omega^1_K$$

and by $\mathrm{gr}\,\phi$ the induced map $\mathrm{gr}\,H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) \to \mathrm{gr}\,\Omega^1_K$ of associated graded modules. The following theorem of Kato's [17, 2.5, 3.2, 3.7] shows how to use $\mathrm{gr}\,\phi$ to compute the refined Swan conductor of a Galois character given a representation of it as a Witt vector.

5.7. Theorem. For every positive integer n, we have

$$\mathrm{fil}_n H^1(K,\mathbf{Q}/\mathbf{Z}) \cap H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) = \mathrm{fil}_n H^1(K,\mathbf{Q}_p/\mathbf{Z}_p),$$

where $H^1(K,\mathbf{Q}_p/\mathbf{Z}_p)$ is viewed as the p^∞-torsion subgroup of $H^1(K,\mathbf{Q}/\mathbf{Z})$. Furthermore, under the natural identification $\mathrm{gr}_n\Omega^1_K = \mathfrak{p}_A^{-n}\otimes_A \omega^1_k$, the restriction of κ_n to $\mathrm{gr}_n H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) \subseteq \mathrm{gr}_n H^1(K,\mathbf{Q}/\mathbf{Z})$ coincides with $\mathrm{gr}_n\phi$.

5.8. Proposition. For non-negative integers $m \leqslant n$, the diagram

$$\begin{tikzcd}
\mathrm{fil}_n H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) \arrow[r, "\phi_n"] & \mathrm{fil}_n\Omega^1_K/\mathrm{fil}_{[n/p]}\Omega^1_K \\
\mathrm{fil}_m H^1(K,\mathbf{Q}_p/\mathbf{Z}_p) \arrow[r, "\phi_m"] \arrow[u] & \mathrm{fil}_m\Omega^1_K/\mathrm{fil}_{[m/p]}\Omega^1_K \arrow[u]
\end{tikzcd}$$

commutes.

Proof. Clear.

5.9. Proposition. Let A' be an extension of A of ramification index e, and let K' denote its fraction field. If $\chi \in H^1(K,\mathbf{Q}_p/\mathbf{Z}_p)$, then

$$\phi_{\mathrm{sw_K}(\chi)}(\chi)|_{A'} = \phi_{\mathrm{sw_K}(\chi|_{A'})}(\chi|_{A'}) \bmod \mathrm{fil}_{[e\,\mathrm{sw_K}(\chi)/p]}\Omega^1_{K'}.$$

Proof. By 5.8.

5.10. Finally, for any non-negative integer n, put $\mathrm{fil}'_n\Omega^1_K = \mathfrak{p}_A^{-n}\cdot\Omega^1_A \subset \Omega^1_K$. This filtration measures the order of the pole in the usual sense, whereas $\mathrm{fil}_\bullet\Omega^1_K$ measures it in the logarithmic sense. The two filtrations are intertwined:

$$\cdots \subseteq \mathrm{fil}_n\Omega^1_K \subseteq \mathrm{fil}'_{n+1}\Omega^1_K \subseteq \mathrm{fil}_{n+1}\Omega^1_K \subseteq \cdots.$$

Matsuda [20, 3.1] has given non-logarithmic variants of the other filtrations in 5.4, but we will not need them here. (Note, however, that our indexing of $\mathrm{fil}'_\bullet\Omega^1_K$ differs from Matsuda's by one.)

5.11. Proposition. Let χ be a class in $H^1(K, \mathbf{Q}_p/\mathbf{Z}_p)$. Then $\mathrm{ar}_{\mathrm{K}}(\chi)$ is the smallest integer n satisfying $\chi \in \mathrm{fil}'_n H^1(K, \mathbf{Q}_p/\mathbf{Z}_p)$.

Proof. By 5.7.

5.12. Lemma. For $n \geqslant 1$, the natural map $\mathrm{gr}'_n \Omega^1_K \longrightarrow \mathrm{gr}'_n \Omega^1_{K^{\mathrm{g}}}$ is injective.

Proof. Because we have $\mathrm{gr}'_n \Omega^1_K = \Omega_A \otimes_A \mathfrak{p}_A^{-n}/\mathfrak{p}_A^{-n+1}$, it is enough to show the map

$$k \otimes_A \Omega^1_A \longrightarrow k \otimes_A \Omega^1_{A^{\mathrm{g}}}$$

is injective.

Let T be a lift to A of a p-basis for k, and let π be a uniformizer of A. Then the set $dT \cup \{d\pi\}$ is a basis for the k-module $k \otimes_A \Omega^1_A$. To show injectivity, it is enough to check that the image of $dT \cup \{d\pi\}$ is k-linearly independent. But $\Omega^1_{A^{\mathrm{g}}} = A^{\mathrm{g}} d\pi$, and so it is enough to show, in the notation of 1.5, that $\{1\} \cup \{u_{t,1} \mid t \in T\}$ is k-linearly independent in $k^{\mathrm{g}} = \mathrm{Fr}(R_T)$. This follows from the definition of R_T.

We can now prove theorem B in equal characteristic.

Proof. First suppose χ is in $H^1(K, \mathbf{Q}_p/\mathbf{Z}_p)$. Put $n = \mathrm{sw}_{\mathrm{K}}(\chi)$ and $m = \mathrm{sw}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}})$. Then $\mathrm{ar}_{\mathrm{K}}(\chi)$ is either n or $n+1$. We will treat these two sub-cases separately. If $\mathrm{ar}_{\mathrm{K}}(\chi) = n+1$, then $\kappa_n(\chi) \notin \mathfrak{p}_A^{-n} \otimes \Omega^1_k$. The naturality (4.10) of the maps $\kappa_\bullet$ therefore implies $\kappa_n(\chi|_{A^{\mathrm{g}}}) \notin \mathfrak{p}_{A^{\mathrm{g}}}^{-n} \otimes \Omega^1_{k^{\mathrm{g}}}$ and, so, $\mathrm{ar}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) = n+1$.

Now consider the second sub-case, when $\mathrm{ar}_{\mathrm{K}}(\chi)$ is n. By 4.9, we have $n \geqslant 2$; so, for $n = 2$, it is enough to show $\chi|_{A^{\mathrm{g}}}$ is not tame. This follows from 2.4. If $n \geqslant 3$, we have $[n/p] \leqslant n-2$ and, hence,

$$\mathrm{fil}_{[n/p]}\Omega^1_{K^{\mathrm{g}}} \subseteq \mathrm{fil}_{n-2}\Omega^1_{K^{\mathrm{g}}} \subseteq \mathrm{fil}'_{n-1}\Omega^1_{K^{\mathrm{g}}}.$$

Then, by 5.9 and 5.12, we have

$$\phi_m(\chi|_{A^{\mathrm{g}}}) \equiv \phi_n(\chi)|_{A^{\mathrm{g}}} \not\equiv 0 \mod \mathrm{fil}'_{n-1}\Omega^1_{K^{\mathrm{g}}}.$$

Therefore, $\chi \notin \mathrm{fil}'_{n-1}\Omega^1_{K^{\mathrm{g}}}$, and so 5.11 implies $\mathrm{ar}_{\mathrm{K}}(\chi) = n$.

Now let χ be an arbitrary class in $H^1(K)$. If χ is tame, the result follows from 2.4. If χ is wild, then write $\chi = \chi' + \chi''$, where χ' is in $H^1(K, \mathbf{Q}_p/\mathbf{Z}_p)$ and χ'' is tame. Because χ'' is tame, χ and χ' have the the same refined Swan conductor and, hence, the same Kato-Artin conductor. Similarly, $\chi'|_{A^{\mathrm{g}}}$ is wild (again by 2.4), and so $\chi|_{A^{\mathrm{g}}}$ and $\chi'|_{A^{\mathrm{g}}}$ have the same Kato-Artin conductor. The work above then implies $\mathrm{ar}_{\mathrm{K}}(\chi') = \mathrm{ar}_{\mathrm{K}}(\chi'|_{A^{\mathrm{g}}})$, and this completes the proof.

6. Some lemmas

The purpose of this section is to prove some lemmas needed in the proof of theorem B in mixed characteristic. All the results in this section are, however, still valid in equal characteristic. Let π be a uniformizer of A.

6.1. Let $U^i K_2^{\mathrm{M}}(K)$ (for $i \geqslant 1$) denote the subgroup of $K_2^{\mathrm{M}}(K)$ (see 4.1) generated by the set $\{U_A^i, K^*\}$. This filtration satisfies [3, 4.1]

$$\{U_A^i, U_A^j\} \subseteq U^{i+j} K_2^{\mathrm{M}}(K). \tag{6.1.1}$$

6.2. Lemma. Let x and y be non-zero elements of $\mathfrak{p}_A$. Then

$$\{1+x, 1+y\} \equiv \{1+xy, -y\} \mod U^{v_A(xy)+1} K_2^{\mathrm{M}}(K).$$

Proof. We have

$$\begin{aligned}\{1+x, 1+y\} &= \{-y(1+x), 1+y\}\\ &= -\{-y(1+x), 1+xy(1+y)^{-1}\}\\ &\equiv -\{-y(1+x), 1+xy\} \mod U^{v_A(xy)+1} K_2^{\mathrm{M}}(K)\\ &\equiv \{1+xy, -y\} \mod U^{v_A(xy)+1} K_2^{\mathrm{M}}(K).\end{aligned}$$

6.3. Lemma. Suppose A is residually perfect. Let $x \neq 0$ be an element of $\mathfrak{p}_A$, and let $z \in A^*$ be such that the element $z' = z - s_A(\bar{z})$ is non-zero. Then

$$\{1+x, z\} \equiv v_A(z')\{1+xz's_A(\bar{z}^{-1}), \pi\} \mod U^{v_A(xz')+1} K_2^{\mathrm{M}}(K) + DK_2^{\mathrm{M}}(K),$$

where $DK_2^{\mathrm{M}}(K)$ is the infinitely p-divisible subgroup of $K_2^{\mathrm{M}}(K)$.

Proof. The defining property of the lift $s_A(\bar{z})$ of $\bar{z}$ is that it is infinitely p-divisible in K^*. Therefore, it suffices to assume $\bar{z} = 1$. By 6.2, we have

$$\begin{aligned}\{1+x, 1+z'\} \equiv{}& v_A(z')\{1+xz', \pi\}\\ &+\{1+xz', -z'/\pi^{v_A(z')}\} \mod U^{v_A(xz')+1} K_2^{\mathrm{M}}(K).\end{aligned}$$

Because $-z'/\pi^{v_A(z')}$ is in $A^* = s_A(k^*)U_A^1$, we also have

$$\{1+xz', -z'/\pi^{v_A(z')}\} \in U^{v_A(xz')+1} K_2^{\mathrm{M}}(K) + DK_2^{\mathrm{M}}(K).$$

6.4. Proposition. If F is a field of characteristic p, the p^∞-torsion subgroup of $H^q(F)$ is p-divisible.

Proof. By the surjectivity of the higher Artin-Schreier maps ξ_s and the natural maps $\Omega^\bullet_{W_s(F)} \to W_s\Omega^\bullet_F$ and $W_{s+1}(F) \to W_s(F)$.

6.5. Lemma. Let $\chi \in H^1(K)$ be a class such that $\mathrm{ar}_{\mathrm{K}}(\chi) = \mathrm{sw}_{\mathrm{K}}(\chi) \neq 0$. Then, in the notation of 4.3, the element $\{\chi|_{\tilde{A}}, 1+\pi^{\mathrm{sw}_{\mathrm{K}}(\chi)-1}T\}$ is in $\mathrm{fil}_1 H_p^2(\tilde{K}) + H_{p^2}^2(\tilde{k})$ but is not in $\mathrm{fil}_0 H^2(\tilde{K})$.

Proof. Write $n = \mathrm{sw_K}(\chi)$. Then by 4.9, we have $n > 1$. Let $\psi = \{\chi|_{\tilde{A}}, 1 + \pi^{n-1}T\}$. Since $n > 1$, we have

$$(1 + \pi^{n-1}T)^{p^2} \in U_{\tilde{A}}^{n+1} \ \text{ and } \ (1 + \pi^{n-1}T)^p \in U_{\tilde{A}}^{n}.$$

Applying (6.1.1), we get $\psi \in \mathrm{fil}_1 H^2_{p^2}(\tilde{K})$ and $p\psi \in \mathrm{fil}_0 H^2_p(\tilde{K})$.

Let A' be a henselian extension of A whose residue field k' is perfect and has the property that k is separably closed in it. (Take $A' = A^{\mathrm{g}}$, for example.) Because $\mathrm{ar_K}(\chi) = \mathrm{sw_K}(\chi)$ and because k' is perfect, 4.10 implies $\mathrm{sw_K}(\chi|_{A'}) \leqslant n-1$. Therefore, $p\psi|_{\tilde{A}'}$ is zero. Since k is separably closed in k', the field $\tilde{k} = k(T)$ is separably closed in $\tilde{k}' = k'(T)$. The natural map $H^1_p(\tilde{k}) \to H^1_p(\tilde{k}')$ is therefore an injection. So, by chasing diagram (4.13.1) applied to the extension $\tilde{A}'/\tilde{A}$, we conclude $p\psi \in H^2_p(\tilde{k})$. By 6.4, there is a class $\psi'' \in H^2_{p^2}(\tilde{k})$ such that $p\psi'' = p\psi$. Then, $\psi - \psi''$ is in $\mathrm{fil}_1 H^2_p(\tilde{K})$.

On the other hand, putting $\eta = \kappa_n(\chi)$, we have

$$\begin{aligned} \{\psi, 1-\pi\} &= \{\chi|_{\tilde{A}}, 1 + \pi^{n-1}T, 1-\pi\} & \\ &= \{\chi|_{\tilde{A}}, 1 - \pi^{n}T, \pi\} & \text{by 6.2} \\ &= \{\lambda_{\tilde{A}}(-\eta\pi^n T), \pi\} & \text{by 4.7.} \end{aligned}$$

But, by assumption, η is in $\mathfrak{p}_A^{-n} \otimes \Omega^1_k$, and so $\lambda_{\tilde{A}}(-\eta\pi^n T)$ equals $\xi_1(-\eta\pi^n T)$, which is non-zero [17, 3.8]. Because of this, $\{\psi, 1-\pi\}$ is non-zero (4.4) and, hence, ψ is not in $\mathrm{fil}_0 H^2(\tilde{K})$.

7. A diagram

Assume A is of mixed characteristic. Let $k_2(K)$ denote $K_2^{\mathrm{M}}(K)/pK_2^{\mathrm{M}}(K)$, and let $U^\bullet k_2(K)$ denote the image of the filtration $U^\bullet K_2^{\mathrm{M}}(K)$.

The primary purpose of this section is to prove a certain diagram (7.6) commutes. This will allow us to understand in terms of symbols how some classes in $H^2_p(K)$ change when pulled back to A^{g}. The relation between symbols and cohomology classes is provided by a theorem from Bloch-Kato [3, 5.12] and Kato [17, 4.1(6)]:

7.1. Theorem. If A contains the group μ_p of all p-th roots of unity, then h_K (of 4.1) induces isomorphisms

$$k_2(K) \otimes \check{\mu}_p \xrightarrow{\sim} H^2_p(K) \quad \text{and} \quad U^{\hat{e}-n} k_2(K) \otimes \check{\mu}_p \xrightarrow{\sim} \mathrm{fil}_n H^2_p(K),$$

where $\hat{e} = e_{A/\mathbf{Z}_p} p(p-1)^{-1}$, $0 \leqslant n < \hat{e}$, and $\check{\mu}_p = \mathrm{Hom}(\mu_p, \mathbf{Z}/p)$.

7.2. Fix an extension A_μ of A that is generically generated by a primitive p-th root of unity. We will assume throughout this section that the extension A_μ/A is residually trivial. Let A' be a henselian residually perfect extension of A of ramification index one. Because we will use this only when $A' = A^{\mathrm{g}}$, the reader is free to assume it (even though it does not simplify anything).

Also fix the following notation: A'_μ is $A' \otimes_A A_\mu$ (a residually perfect henselian discrete valuation ring); K', K_μ, K'_μ are the fraction fields of A', A_μ, A'_μ, and k' is the residue field of A'; Γ is $\mathrm{Gal}(K_\mu/K)$, and m is the degree of K_μ/K; μ_p is the group of all p-th roots of unity in K_μ, and $\check{\mu}_p$ is $\mathrm{Hom}(\mu_p, \mathbf{Z}/p)$; ζ is some non-trivial element of μ_p, and $\check{\zeta}$ is the element in $\check{\mu}_p$ such that $\check{\zeta}(\zeta) = 1$. (None of the constructions below will depend on the choice of ζ.)

7.3. Proposition. There is a map $\gamma_{A'/A}$ that makes the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{fil}_m H^2_p(K_\mu) & \twoheadrightarrow & \mathrm{gr}_m H^2_p(K_\mu) \\
\downarrow & & \vdots \\
H^2_p(K'_\mu) & & \vdots \\
\| & & \vdots\ \gamma_{A'/A} \\
\mathrm{fil}_0 H^2_p(K'_\mu) & & \vdots \\
\downarrow & & \downarrow \\
H^1_p(k') & \twoheadrightarrow & \mathrm{coker}\left[H^1_p(k) \to H^1_p(k')\right].
\end{array}
$$

(Note that $\mathrm{fil}_0 H^2_p(K'_\mu) = H^2_p(K'_\mu)$ since A'_μ is residually perfect [17, 6.1].)

Proof. Let χ be a class in $\mathrm{fil}_{m-1} H^2_p(K_\mu)$, and put $\chi' = m^{-1}\sum_{\sigma\in\Gamma}\sigma(\chi)$. It is clear that χ' lies in $\mathrm{fil}_{m-1} H^2_p(K_\mu)^\Gamma$, which by 4.12, agrees with $\mathrm{fil}_0 H^2_p(K)$. On the other hand, 4.14 implies Γ acts trivially on $\mathrm{fil}_0 H^2_p(K'_\mu) = H^2_p(K'_\mu)$, and so χ and χ' have the same image in $H^2_p(K'_\mu)$. The image of χ in $H^1_p(k')$ is therefore contained in the image of $H^1(k)$.

7.4. Proposition. The composite map

$$A \to \mathfrak{p}_{A'}/(\mathfrak{p}^2_{A'} + \mathfrak{p}_A) = (k'/k) \otimes_A \mathfrak{p}_A \longrightarrow (k'/k^{p^{-1}}) \otimes_A \mathfrak{p}_A,$$

where the leftmost map sends x to the class of $x - s_{A'}(\bar{x})$, is a derivation that vanishes on $\mathfrak{p}_A$.

Proof. It is immediate that it vanishes on $\mathfrak{p}_A$ and a short computation shows it satisfies the Leibniz rule. To see it is additive, it suffices to show that for all $x, y \in k$, we have

$$s_{A'}(x+y) \equiv s_{A'}(x) + s_{A'}(y) \mod \mathfrak{p}_{A'}^2 + s_{A'}(k^{p^{-1}})\mathfrak{p}_A.$$

Let W be the ring of Witt vectors [13, 0.1] with entries in $k^{p^{-\infty}}$. By the definition of addition in W, there is an element $z \in k$ such that

$$(x, 0, \dots) + (y, 0, \dots) = (x+y, z, \dots), \quad \text{i.e.,}$$

$$s_W(x) + s_W(y) \equiv s_W(x+y) + ps_W(z^{p^{-1}}) \mod p^2.$$

Because there is a map $W \to A'$ that is compatible with multiplicative sections, the congruence above holds.

If p does not generate $\mathfrak{p}_A$, even the map $A \to (k'/k) \otimes_A \mathfrak{p}_A$ is a derivation (in both equal and mixed characteristic).

7.5. Construction. *Morphisms* $\partial_{A'/A}$, ν, *and* θ

Let $\partial_{A'/A} : \mathfrak{p}_A^{-1} \otimes_A \Omega_k^1 \longrightarrow k'/k^{p^{-1}}$ denote the k-linear homomorphism induced by the derivation in 7.4.

Put

$$\hat{e} = e_{A_\mu/\mathbf{Z}_p} p(p-1)^{-1} = v_{A_\mu}\big(p(\zeta - 1)\big),$$

and let

$$\nu : \mathrm{gr}^{\hat{e}-m} k_2(K_\mu) \otimes \check{\mu}_p \longrightarrow \mathfrak{p}_A^{-1} \otimes \Omega_k^1$$

denote the map determined by

$$\{1 - p(\zeta - 1)x, y\} \otimes \check{\zeta} \mapsto mx \otimes \mathrm{dlog}(\bar{y}),$$

where $x \in \mathfrak{p}_A^{-1} A_\mu$, $y \in A_\mu$. Bloch and Kato show it is (well-defined and) an isomorphism [3, 4.3, 5.2]. Because $U_{A_\mu}^{\hat{e}+1} \subseteq (K_\mu^*)^p$, we have $U^{\hat{e}} k_2(K'_\mu) = \mathrm{gr}^{\hat{e}} k_2(K'_\mu)$. We can therefore define a map

$$\theta : k' \longrightarrow U^{\hat{e}} k_2(K'_\mu) \otimes \check{\mu}_p$$

by $x \mapsto \{1 - p(\zeta - 1)x, \pi_\mu\}$, where π_μ is a uniformizer of A_μ. (The map is independent of the choice.)

7.6. Proposition. The following diagram commutes:

$$
\begin{tikzcd}
& & k'/k^{p^{-1}} \arrow[rrr, two heads, "\xi_1"] & & & \operatorname{coker}\left[H^1_p(k^{p^{-1}}) \to H^1_p(k')\right] \\
& & & & & \operatorname{coker}\left[H^1_p(k) \to H^1_p(k')\right] \arrow[u, "\simeq"'] \\
\mathfrak{p}_A^{-1} \otimes_A \Omega^1_k \arrow[uurr, "\partial_{A'/A}"] & & & & & \\
& & k' \arrow[uuu, two heads] \arrow[rrr, "\xi_1"] \arrow[ddl, two heads, "\theta"'] & & & H^1_p(k') \arrow[uu, two heads] \\
\operatorname{gr}^{\hat{e}-m} k_2(K_\mu) \otimes \check{\mu}_p \arrow[uu, "\nu", "\simeq"'] \arrow[rrr, "\simeq", "h_{K_\mu}"'] & & & \operatorname{gr}_m H^2_p(K_\mu) \arrow[uuurr, "\gamma_{A'/A}"] & & \\
& U^{\hat{e}} k_2(K'_\mu) \otimes \check{\mu}_p \arrow[rrr, "\simeq", "h_{K'_\mu}"'] & & & \operatorname{fil}_0 H^2_p(K'_\mu) \arrow[uur, "\simeq"] & \\
U^{\hat{e}-m} k_2(K_\mu) \otimes \check{\mu}_p \arrow[uu, two heads] \arrow[ur] \arrow[rrr, "\simeq", "h_{K_\mu}"'] & & & \operatorname{fil}_m H^2_p(K_\mu). \arrow[uu, two heads] \arrow[ur] & &
\end{tikzcd}
$$

Proof. The commutativity of the rear lower face follows from the splittings of 4.4 and the usual compatibility between Artin-Schreier theory and Kummer theory. It is clear the other three faces for which it makes sense to ask the question commute. Therefore, it only remains to check that the perimeter commutes.

Because ν is an isomorphism, it is enough to consider elements

$$x = \nu^{-1}(\pi^{-1} \otimes dz) = \{1 - p(\zeta - 1)\pi^{-1}z, z\} \otimes \check{\zeta} \in U^{\hat{e}-m}k_2(K_\mu) \otimes \check{\mu}_p.$$

where π is a uniformizer of A and $z \in A^*$. Write $z|_{A'} = s_{A'}(\bar{z}) + \pi y$, where $y \in A'$. Then, letting $[x]$ denote the graded class of x, we have

$$\partial_{A'/A} \circ \nu([x]) = \bar{y} \bmod k^{p^{-1}}.$$

On the other hand, by 6.3, we have

$$x|_{K'_\mu} = \{1 - p(\zeta - 1)yzs(\bar{z})^{-1}, \pi_\mu\} \otimes \check{\zeta} = \theta(\bar{y}).$$

Since the front lower and rear faces commute, the proof is complete.

8. The proof in mixed characteristic

Assume in this section that A is of mixed characteristic, and let A_μ be an extension of A that is generically generated by a primitive p-th root of unity. (The extension A_μ/A is no longer assumed to be residually trivial.)

8.1. Proposition. The composite map

$$\mathfrak{p}_A^{-1} \otimes_A \Omega_k^1 \xrightarrow{\partial_{A^{\mathrm{g}}/A}} k^{\mathrm{g}}/k^{p^{-1}} \xrightarrow{\xi_1} \operatorname{coker}\left[H_p^1(k^{p^{-1}}) \to H_p^1(k^{\mathrm{g}})\right]$$

is injective.

(Compare with 5.12.)

Proof. Let T be a p-basis for k. Then dT is a basis for Ω_k^1. Let η be an element of the kernel of $\xi_1 \circ \partial_{A^{\mathrm{g}}/A}$ and write

$$\eta = \pi^{-1} \otimes \sum_{t \in T} a_t dt, \ a_t \in k$$

where π is a uniformizer of A and a_t is zero for all but finitely many $t \in T$. By the construction (1.7) of the map $A \to A^{\mathrm{g}}$, we have

$$\partial_{A^{\mathrm{g}}/A}(\eta) = \sum_{t \in T} a_t u_{t,1} \bmod k^{p^{-1}}.$$

Because the image of this in

$$\operatorname{coker}\left[H_p^1(k^{p^{-1}}) \to H_p^1(k^{\mathrm{g}})\right]$$

is assumed to be zero, there are elements $x \in k^{\mathrm{g}}$ and $y \in k$ such that

$$x^p - x = y + \sum_{t \in T} a_t u_{t,1}.$$

Suppose for a contradiction that there is an element $s \in T$ such that $a_s \neq 0$. Put

$$F = k(u_{t,1} \mid t \neq s)^{p^{-\infty}}.$$

Now, $F(u_{s,1})$ is separably closed in $\mathrm{Fr}(R) = k^{\mathrm{g}}$, and therefore we have $x \in F(u_{s,1})$ and $x^p - x = y' + a_s u_{s,1}$, where $y' = y + \sum_{t \neq s} a_t u_{t,1} \in F$. But valuation considerations in the completion $F((u_{s,1}^{-1}))$ show this is impossible.

8.2. Corollary. If A_μ/A is residually trivial, the map $\gamma_{A^{\mathrm{g}}/A}$ is injective.

Proof. By 7.6 and 8.1.

8.3. Proposition. Let $s \geqslant 1$ be an integer. Then the following sequences are exact:

$$0 \longrightarrow H_{p^s}^2(k) \longrightarrow \mathrm{fil}_0 H_{p^s}^2(K) \longrightarrow H_{p^s}^2(K^{\mathrm{g}}), \tag{8.3.1}$$

$$0 \longrightarrow H_p^2(k) \longrightarrow \mathrm{fil}_1 H_p^2(K) \longrightarrow H_p^2(K^{\mathrm{g}}). \tag{8.3.2}$$

Proof. Because k^{g} is perfect, we have $H^2_{p^s}(k^{\mathrm{g}}) = 0$ (by, say, higher Artin-Schreier theory [17, 1.3]). Now, 2.4 implies the map $H^1_{p^s}(k) \to H^1_{p^s}(k^{\mathrm{g}})$ is an injection. The exactness of (8.3.1) then follows from 4.13.

Let A_0 be the maximal unramified subextension of A_μ/A. Applying 8.2 to the extension A_μ/A_0, we conclude that the map $\gamma_{A_0^{\mathrm{g}}/A_0}$ is injective, and therefore the kernel of the map

$$\mathrm{fil}_m H^2_p(K_\mu) \longrightarrow H^2_p(K_0^{\mathrm{g}} \otimes_{K_0} K_\mu)$$

is contained in $\mathrm{fil}_{m-1} H^2_p(K_\mu)$. By 4.11, we have

$$\mathrm{fil}_{m-1} H^2_p(K_\mu) \cap H^2_p(K) = \mathrm{fil}_0 H^2_p(K).$$

Because there is (2.1) a map $K^{\mathrm{g}} \to K_0^{\mathrm{g}}$, we then have

$$\begin{aligned}\ker\left[\mathrm{fil}_1 H^2_p(K) \to H^2_p(K^{\mathrm{g}})\right] &\subseteq \ker\left[\mathrm{fil}_m H^2_p(K_\mu) \to H^2_p(K_0^{\mathrm{g}} \otimes_{K_0} K_\mu)\right] \cap H^2_p(K)\\ &\subseteq \ker\left[\mathrm{fil}_0 H^2_p(K) \to H^2_p(K^{\mathrm{g}})\right],\\ &= H^2_p(k),\end{aligned}$$

which proves the exactness of (8.3.2).

We can finally prove theorem B in mixed characteristic.

Proof. Let $\pi \in A$ be a uniformizer. Put $n = \mathrm{sw}_{\mathrm{K}}(\chi)$. If $n = 0$, then χ is tame, and the result follows immediately from 2.4. Now assume $n > 0$.

By definition, $\mathrm{ar}_{\mathrm{K}}(\chi)$ is either n or $n+1$. If it is $n+1$, then $\kappa_n(\chi) \notin \mathfrak{p}_A^{-n} \otimes_A \Omega^1_k$. Therefore, 4.10 implies $\kappa_n(\chi|_{A^{\mathrm{g}}}) \notin \mathfrak{p}_{A^{\mathrm{g}}}^{-n} \otimes_A \omega^1_{k^{\mathrm{g}}}$, and so $\mathrm{ar}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) = n+1$, as desired.

Now consider the case $\mathrm{ar}_{\mathrm{K}}(\chi) = n$. Because A^{g} is residually perfect, $\mathrm{ar}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) = n$ if and only if $\mathrm{sw}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) = n-1$. By 4.10, we know $\mathrm{sw}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}})$ is at most $n-1$. Put $\psi = \{\chi|_{\tilde{A}}, 1+\pi^{n-1}T\} \in H^2(\tilde{K})$. Then $\mathrm{sw}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}}) \geqslant n-1$ if and only if $\psi|_{\widetilde{A^{\mathrm{g}}}}$ is not zero.

Now we will construct a map $\widetilde{A^{\mathrm{g}}} \longrightarrow \tilde{A}^{\mathrm{g}}$. (See figure 1.) Because $\tilde{A}/A$ is residually separable, there is (2.1) an injection $A^{\mathrm{g}} \to \tilde{A}^{\mathrm{g}}$. Sending $T \mapsto T$ yields another injection $A^{\mathrm{g}}[T] \longrightarrow \tilde{A}^{\mathrm{g}}$. By the universal properties of localization and henselization [23, VIII], this naturally induces a map $\widetilde{A^{\mathrm{g}}} \longrightarrow \tilde{A}^{\mathrm{g}}$. It is therefore enough to show $\psi|_{\tilde{A}^{\mathrm{g}}} \neq 0$.

By 6.5, we can write $\psi = \psi' + \psi''$, where ψ' is in $\mathrm{fil}_1 H^2_p(\tilde{K}) - \mathrm{fil}_0 H^2_p(\tilde{K})$ and ψ'' is in $H^2_{p^2}(\tilde{k})$. We then have $\psi|_{\tilde{A}^{\mathrm{g}}} = \psi'|_{\tilde{A}^{\mathrm{g}}} \neq 0$ by 8.3.

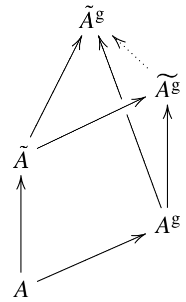

Fig. 1.

9. Comparison with Kato's conductor

Let Λ be a field whose characteristic is not p, and fix an injection from the torsion subgroup of Λ^* to $\mathbf{Q}/\mathbf{Z}$. In each of the two results below, B is a finite extension of A that is generically Galois with group G and χ is the class in $H^1(K, \mathbf{Q}/\mathbf{Z})$ corresponding to a homomorphism $\rho : G \to \Lambda^*$.

9.1. Proposition. If B/A is residually separable, we have

$$\mathrm{ar}_{\mathrm{K}}(\chi) = \mathrm{ar}_{\mathrm{n}}(\rho)$$

Proof. If ρ is tame, the result is clear. Now assume ρ is wild and let $n = \mathrm{sw}_{\mathrm{K}}(\chi)$. Then Kato [17, 6.8][15, 3.6(1),3.16] shows $\kappa_n(\chi)$ is not in $\mathfrak{p}_A^{-n} \otimes \Omega_k^1$ and that n agrees with the naive Swan conductor [17, 6.7.1]. This in turn agrees [28, 2.1] with

$$e_{B/A}^{-1} \sum_{i \geqslant 1} |G_i| \operatorname{codim} \Lambda^{G_i}. \tag{9.1.1}$$

Because $\mathrm{ar}_{\mathrm{K}}(\chi)$ is one more than $\mathrm{sw}_{\mathrm{K}}(\chi)$ and because the number $\mathrm{ar}_{\mathrm{n}}(\rho) = \mathrm{ar}_{\mathrm{n}}^B(\rho)$ is one more than (9.1.1), the equality of conductors follows.

9.2. Corollary. $\mathrm{ar}(\rho) = \mathrm{ar}_{\mathrm{K}}(\chi)$.

Proof. Since A^{g} is residually perfect, 3.2 implies $\mathrm{ar}(\rho)$ agrees with $\mathrm{ar}_{\mathrm{n}}^B(\rho|_{A^{\mathrm{g}}})$ and, by the previous proposition, with $\mathrm{ar}_{\mathrm{K}}(\chi|_{A^{\mathrm{g}}})$. Applying theorem B completes the proof.

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