# Primes, the Riemann zeta-function, and sums over zeros

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# Abstract

I will give a brief introduction to the Riemann zeta-function $\zeta(s)$ and its connection with prime numbers. I will mention the famous “explicit formula” that gives an explicit connection between Chebyshev’s prime-counting function $\psi(x)$ and an infinite sum that involves the zeros of $\zeta(s)$. Using the explicit formula, many questions about prime numbers can be reduced to questions about these zeros or sums over the zeros.

In the second half of the talk I will consider sums of the form $\sum \phi(\gamma)$, where $\phi$ is a given function and $\gamma$ ranges over a subset of the ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. I will show how the numerical estimation of such sums can be accelerated, and give some numerical examples.

The second half is joint work with Dave Platt and Tim Trudgian. For a preprint, see `https://arxiv.org/abs/2009.13791`.

# The Basel Problem

The *Basel Problem* is a famous problem that was posed by Pietro Mengoli in 1650 (maybe earlier) and solved about 84 years later by Leonhard Euler (1707–1783).

The problem is to evaluate the series

$$S := \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

in closed form.

Nowadays we could compute $S$ numerically, and use the Inverse Symbolic Calculator to guess the answer (though we would still have to prove it). Euler had no inverse symbolic calculator, but he did compute $S \approx 1.644934$ by hand, after making a clever transformation to get a more rapidly converging series $S = (\log 2)^2 + \sum_{n \geqslant 1} 2^{1-n}/n^2$.

Can you guess the answer now? $\cdots$ Pause for 20 minutes $\cdots$

# Euler's Solution

Euler's solution, announced in 1734 and proved more rigorously in 1741, was

$$S = \frac{\pi^2}{6}.$$

One way to prove this is to take logarithms of both sides in the infinite product

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

and then compare the coefficients of $x^2$. Of course, we have to prove that the infinite product is correct. Euler did not know how to do this.

# Euler's generalisation

At this point we'll define the *zeta-function*, more precisely the *Riemann zeta-function* after Bernhard Riemann (1826–1866), because there are similar functions that are also called "zeta functions".

Most likely Euler considered only integer $s$; later, Chebyshev (1821–1894) considered real $s$, and Riemann considered complex $s$. For the moment we'll assume that $s \in \mathbb{R}$ and $s > 1$.

Consider the definition

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

The series converges for $s > 1$ (by comparison with $1 + \int_1^\infty x^{-s}\, ds$). Thus, we can use the series to define a function $\zeta : (1, \infty) \to \mathbb{R}$.

Observe that $\zeta(2)$ is what we called $S$ above, the solution to the Basel problem. Thus

$$\zeta(2) = \pi^2/6.$$

# The connection with Bernoulli numbers

Euler found that, for any positive even integer $2k$,

$$\zeta(2k) = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!},$$

where $B_{2k}$ is a *Bernoulli number*, which can be defined using the *generating function*

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{t}{e^t - 1}.$$

For example, it is easy to show that $B_2 = 1/6$ and $B_4 = -1/30$, so

$$\zeta(2) = \pi^2/6, \ \zeta(4) = \pi^4/90.$$

# An early computer program

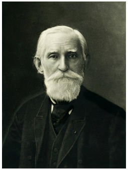
From the generating function, we can easily get a recurrence relation for the Bernoulli numbers. This shows that they are rational numbers, and allows them to be computed easily.

In 1842, Ada Lovelace (1815–1852) outlined a program to compute them on Charles Babbage's *Analytical Engine* (whose construction was never completed).

Thus, the Bernoulli numbers have the distinction of being the subject of the first published (nontrivial) computer program.

# Some people mentioned so far: 18th – 19th century




Euler, Dirichlet, Lovelace, Chebyshev, Riemann

# Positive integer arguments

An *algebraic* number is a zero of some polynomial with rational coefficients, and a number that is not algebraic is called *transcendental*.

We have seen that $\zeta(n)/\pi^n$ is rational for all *even* positive integers $n$. Since $\pi$ is transcendental, this implies that $\zeta(n)$ is transcendental.

Much less is known about $\zeta(n)$ for *odd* positive integers $n$. Infinitely many of them are irrational (Rivoal, 2000).

- Computations suggest (but do not prove) that $\zeta(n)$ and $\zeta(n)/\pi^n$ are irrational for odd $n \geqslant 3$.
- In 1979, Roger Apéry (1916–1994) proved that $\zeta(3)$ is irrational. It is not known if $\zeta(3)$ is transcendental.
- In 2001, Wadim Zudilin (formerly a member of CARMA) proved that *at least one* of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (we do not know which one(s)).

# The connection with primes — Euler's product formula

In 1737, Euler discovered his famous "product formula" that links prime numbers and the zeta function. In modern notation it is, for $s > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

where the infinite product is taken over all *primes* $p$.

If we use $(1 - p^{-s})^{-1} = \sum_{k \geqslant 0} p^{-ks}$, and expand the infinite product, we obtain (exactly once) each possible product of the form $p_1^{-k_1 s} p_2^{-k_2 s} \cdots p_m^{-k_m s}$, where $p_1 < p_2 < \cdots < p_m$ are distinct primes, $m \geqslant 0$, and $k_1, k_2, \ldots, k_m$ are positive exponents. Thus, we get each summand $n^{-s}$ (for positive integer $n$) exactly once.

This is just another way of expressing the theorem that each positive integer has a unique factorisation into products of prime powers! Thus, apart from possible concerns about convergence and reordering terms, we have proved Euler's product formula for $s > 1$.

# Digression — a probability question

What is the probability $P$ that two large integers, chosen at random, are relatively prime?

To make the question rigorous, we could take pairs of integers $a$, $b$, chosen uniformly and independently at random from $\{1, 2, \ldots, N\}$, evaluate the probability of them being relatively prime, and take the limit as $N \to \infty$.

An informal argument is as follows. For each prime $p < N$, the probability that $p$ does *not* divide both $a$ and $b$ is $1 - 1/p^2$ (this is not quite correct for finite $N$, but it holds in the limit as $N \to \infty$). Thus,

$$P = \prod_{p \text{ prime}} (1 - p^{-2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.6079\,.$$

# The Riemann zeta-function in the complex plane

As shown by Riemann and later mathematicians, there are many benefits in considering $\zeta(s)$ as a function of a *complex* variable $s$. From now on, we assume that $s = \sigma + it \in \mathbb{C}$, where $\sigma = \Re(s)$ and $t = \Im(s)$. (This notation mixes Greek and Latin, but it is traditional.)

The definition

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

can still be used, provided $\sigma > 1$, since the series converges absolutely in the halfplane $\{s \in \mathbb{C} : \Re(s) > 1\}$.

The series (1) is an example of a *Dirichlet series*, named after PGL Dirichlet (1805–1859). Such series have the form

$$\sum_{n=1}^{\infty} a_n\, n^{-s},$$

where $a_1, a_2, \ldots$ are constants, and $s$ is a complex variable.

# Complex exponents

If $x \in \mathbb{R}$, $x > 0$, and $s \in \mathbb{C}$, then by definition

$$x^s := \exp(s \log x)$$

where the real branch of the logarithm is taken (we always use natural logarithms). There is no ambiguity in terms like $n^{-s}$.

If $s = \sigma + it$, then $x^s = x^\sigma x^{it}$ and

$$|x^{it}| = |\exp(it \log x)| = |\cos(t \log x) + i \sin(t \log x)| = 1,$$

so $|x^s| = x^\sigma$. This is why the real parts of zeros turn out to be important — often an inequality is governed by the supremum over real parts of zeros of $\zeta(s)$ or some related Dirichlet series.

# Holomorphic, meromorphic, and analytic functions

A *holomorphic* function is a complex-valued function of one (or more) complex variables that is, at every point of its domain, complex differentiable in a neighborhood of the point.

A *meromorphic* function on an open subset $D$ of the complex plane is a function that is holomorphic on all of $D$ except for a set of isolated points, which are poles of the function. The terminology has changed over time. In older literature you might find something like "an analytic function in $D$ except for a set of isolated poles".

In his ground-breaking paper of 1859 (the only paper that he wrote on number theory), Riemann showed that the domain of $\zeta(s)$ could be extended, by analytic continuation, to a meromorphic function on $\mathbb{C}$. In fact, it is a holomorphic function except for having a simple pole at $s = 1$.

# Analytic continuation of $\zeta$

There is a simple way to continue $\zeta(s)$ into the "critical strip", which is the region $\{s \in \mathbb{C} : 0 \leqslant \Re(s) \leqslant 1\}$. We'll see later why this region of the complex plane is called "critical".

The *Dirichlet eta function* (or *alternating zeta function*) is

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2^{1-s})\zeta(s). \tag{2}$$

The Dirichlet series here converges for $\sigma = \Re(s) > 0$, because of the alternating signs. Thus, $\eta(s)$ is a holomorphic function in the half-plane $\sigma > 0$. Note that $\eta(1) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2$.

From (2), $\eta(s)/(1 - 2^{1-s})$ gives the analytic continuation (necessarily unique) of $\zeta(s)$ in the region $\{s \in \mathbb{C} : \sigma > 0, s \neq 1\}$. We have to use l'Hôpital's rule and take the limiting case at points where $2^{1-s} = 1$, i.e. $s = 1 + 2ki\pi/\log(2), k \in \mathbb{Z}$, and exclude the point $s = 1$ where $\zeta(s)$ has a pole.

# Why the location of the zeros of $\zeta(s)$ is important

Define the *prime zeta function* by

$$P(s) := \sum_{p \text{ prime}} p^{-s}$$

for $\Re(s) > 1$, and by analytic continuation for $0 < \Re(s) \leqslant 1$, excluding the singularity at $s = 1$. Taking logarithms on both sides of Euler's product formula, we obtain

$$\log \zeta(s) = \sum_{n>0} \frac{P(ns)}{n},$$

so Möbius inversion gives

$$P(s) = \sum_{n>0} \mu(n) \frac{\log \zeta(ns)}{n} = \log \zeta(s) - \tfrac{1}{2} \log \zeta(2s) - \cdots .$$

Thus, in the right half-plane, $P(s)$ has singularities wherever $\zeta(s)$ has zeros. Information about the distribution of primes can be deduced from this, *if* we know where the zeros of $\zeta(s)$ are (especially their real parts).

# The functional equation

Riemann showed that $\zeta(s)$ satisfies a functional equation which relates its value at $s$ and its value at $1-s$. This equation can be written as

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{3}$$

There are many proofs of (3). Chapter 2 of the book by Titchmarsh book gives seven of them!

There is a more symmetric form of the functional equation. Define

$$\xi(s) := \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

From (3) and properties of the Gamma function,

$$\xi(s) = \xi(1-s). \tag{4}$$

In the other direction, we can prove (4) and deduce (3), as Riemann did.

# Further analytic continuation

Riemann's proof of the symmetric form (4) of the functional equation shows that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-(s+1)/2} + x^{s/2-1}\right)\psi(x)\,dx, \tag{5}$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$ is related to a Jacobi theta function

$$\theta(x) := 2\psi(x) + 1 = \sum_{n=-\infty}^{+\infty} e^{-n^2\pi x},$$

and $\theta(x)$ satisfies a beautiful functional equation

$$\theta(x) = \theta(1/x)/\sqrt{x} \quad \text{(assuming } x > 0\text{)}.$$

The integral in (5) converges for all $s \in \mathbb{C}$, so (5) gives the analytic continuation of $\zeta(s)$ to the whole complex plane, excluding the pole at $s = 1$. By l'Hôpital's rule, $\zeta(0) = -1/2$.

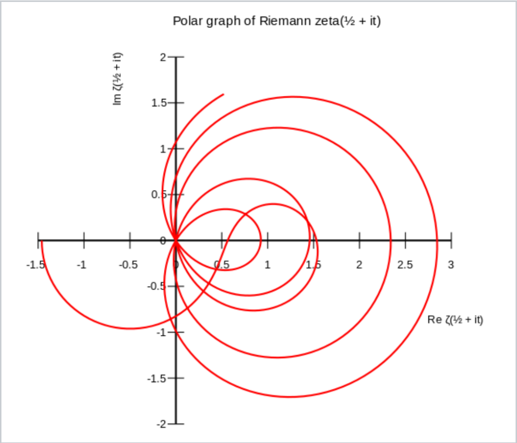
# Zeros and poles of $\zeta(s)$

From the functional equation we see the following.

- $\zeta(s)$ is meromorphic with a single pole at $s = 1$.
- $\zeta(s)$ has real zeros at the negative even integers $-2, -4, -6, \ldots$ (these are called the *trivial zeros*).
- All other zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ are complex and lie in the critical strip $(0 \leqslant \beta \leqslant 1)$ (these are the *nontrivial zeros*), and are either on the *critical line* $\Re(s) = 1/2$, or occur symmetrically about the critical line, i.e. if $\beta + i\gamma$ is a zero, so is $(1 - \beta) + i\gamma$.
- If $\rho = \beta + i\gamma$ is a nontrivial zero, then so is $\overline{\rho} = \beta - i\gamma$ (this follows from $\zeta(\overline{s}) = \overline{\zeta(s)}$). So we usually only need to consider zeros in the upper half-plane $\Im(s) > 0$.

It can also be shown that there are infinitely many nontrivial zeros, and that they are strictly inside the critical strip, i.e. $0 < \beta < 1$. The smallest nontrivial zero $\rho_1$ lies on the critical line $(\beta = 1/2)$, and has imaginary part $\gamma_1 \approx 14.1347$.

# A curious beast: $\zeta(\frac{1}{2}+it)$ for $0 \leqslant t \leqslant 34$




This image shows a plot of the Riemann zeta function along the critical line for real values of $t$ running from 0 to 34. The first five zeros in the critical strip are clearly visible as the place where the spirals pass through the origin.

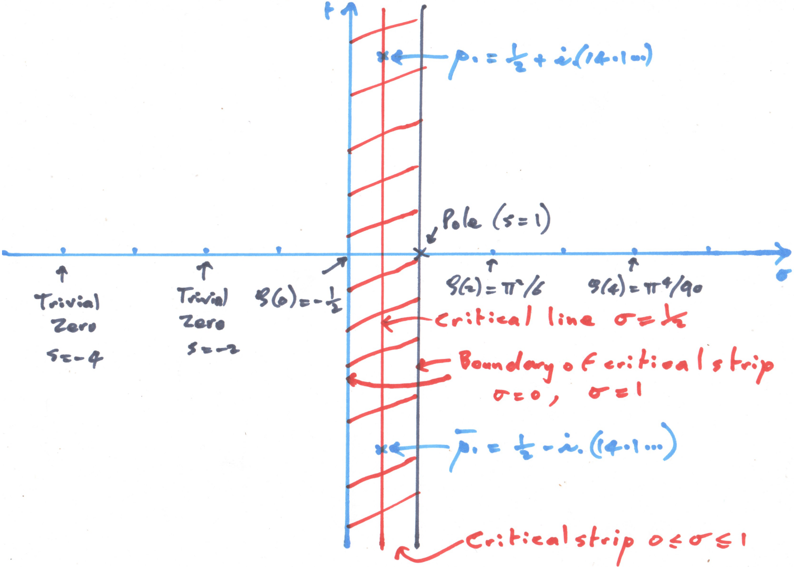
Image from Wikipedia

# Some known properties of the nontrivial zeros

As we already noticed, it is sufficient to consider nontrivial zeros in the upper half-plane. We'll order these in order of increasing imaginary part (usually called the *height*).

- If $N(T)$ is the number of nontrivial zeros up to height $T$, then $N(T) = \frac{T}{2\pi}(\log(\frac{T}{2\pi}) - 1) + O(\log T)$. Thus, the density of zeros of height close to $T$ is approximately $\frac{1}{2\pi}\log(\frac{T}{2\pi})$.
- If $N_0(T)$ is the number of nontrivial zeros *on the critical line* up to height $T$, then, for all sufficiently large $T$, $N_0(T) > N(T)/3$.
- The first $1.23 \times 10^{13}$ nontrivial zeros are simple and lie on the critical line.
- There are many other results about the zeros, such as *zero-density* and *zero-free region* theorems, but I won't say any more about them today.

# Some known zeros/poles of $\zeta(s)$




Features of $\zeta(s)$ close to the origin (not to scale)

# The Riemann Hypothesis (RH)

The *Riemann Hypothesis* is the statement that *all* the non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$.

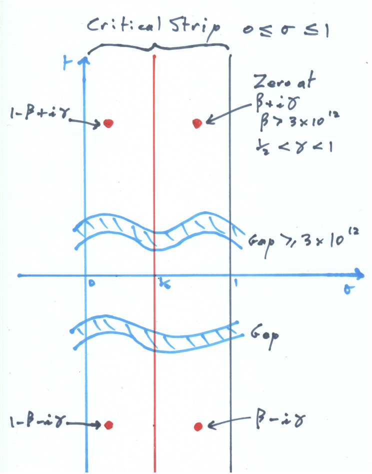
Riemann mentioned this and wrote that it was “very probable”, but he did not claim to have proved it.

RH is one of the famous collection of 23 “Hilbert problems” announced by David Hilbert in 1900, and one of the few that remains unresolved.

It is also one of the 7 “Millenium Problems” announced 100 years later. Only one (the Poincaré conjecture) has been solved so far. There is a $US1,000,000 prize for the solution of any of these problems, but (in my opinion) having a large prize may be detrimental to collaborative research by making people secretive about their work.

Grigory Perelman solved the Poincaré conjecture in 2003, but declined the prize money!

# A hypothetical exception to RH




A hypothetical exception to RH: four symmetrically placed zeros $(\rho, \overline{\rho}, 1-\rho, \overline{1-\rho})$ off the critical line. Not to scale.
With 1-inch units, the gap is at least half the distance from the earth to the sun!

# The prime counting function

The simplest prime counting function is $\pi(x)$, defined to be the number of primes $\leqslant x$.

The *prime number theorem* says that

$$\pi(x) \sim \frac{x}{\log x},$$

i.e. $\lim_{x\to\infty} \pi(x)/(x/\log x) = 1$. This was proved independently in 1896 by Jacques Hadamard (1865–1963) and Charles de la Vallée Poussin (1866–1962). Although their proofs were different, they both used the theorem that $\zeta(s)$ has no zeros on the line $\Re(s) = 1$.

In 1899, de la Vallée Poussin proved the more precise result

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(xe^{-a\sqrt{\log x}}\right) \text{ as } x \to \infty,$$

for some constant $a > 0$. The integral is a *logarithmic integral* and is usually denoted by $\mathrm{Li}(x)$.

# 19th – early 20th century

Hadamard, de la Vallée Poussin, Hardy, Littlewood, Titchmarsh
(relevant, but not all mentioned above)

# A weighted prime counting function $\psi(x)$

The *Mangoldt function* $\Lambda(n)$ is defined by

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p; \\ 0 & \text{otherwise.} \end{cases}$$

The (second) Chebyshev function $\psi(x)$ counts the primes and prime powers up to $x$, weighted by the Mangoldt function:

$$\psi(x) := \sum_{1 < n \leqslant x} \Lambda(n).$$

Equivalently,

$$\psi(x) = \log\left(\mathrm{LCM}\{1, 2, \ldots, \lfloor x \rfloor\}\right)$$

The prime number theorem is equivalent to $\psi(x) \sim x$.

It is easier to work with $\psi(x)$ than $\pi(x)$ because this avoids logarithmic integrals.

# More on $\psi(x)$

If we logarithmically differentiate the Euler product formula,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n\geqslant 2} \Lambda(n) n^{-s}$$

in the region $\Re(s) > 1$, where the sum converges.

Now, using Perron's formula, which is a way of extracting information from a Dirichlet series, we obtain

$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, dx + R(x, T),$$

where $\sigma_0 > 1$ and $R(x, T)$ is a remainder term (details can be found in Montgomery and Vaughan, Chs. 5–6).

# The explicit formula

Since $\zeta'(s)/\zeta(s)$ is a meromorphic function with poles (of residue 1) at the zeros $\rho$ of $\zeta(s)$, standard techniques of contour integration now give the so-called "explicit formula" for $\psi(x)$:

$$\psi_0(x) - x = -\lim_{T\to\infty} \sum_{|\Im\rho|<T} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2})$$

where the sum is over non-trivial zeros $\rho$, and $\psi_0(x)$ is the same as $\psi(x)$ except at primes and prime powers. More precisely, $\psi_0(x) = \lim_{\varepsilon\to 0}(\psi(x-\varepsilon) + \psi(x+\varepsilon))/2$.

We observed that $|x^\rho| = x^\sigma$, where $\sigma = \Re(\rho)$. Thus, it is plausible (and can be proved), that if $\alpha$ is a constant satisfying $\Re(\rho) < \alpha$ for all zeros $\rho$ of $\zeta(s)$, then

$$\psi(x) - x = O(x^\alpha).$$

A similar argument gives $\pi(x) - \mathrm{Li}(x) = O(x^\alpha)$.

# Equivalences to RH

The Riemann hypothesis (if true) implies that we can take any $\alpha > \frac{1}{2}$. In fact, we can say more and give necessary and sufficient conditions for RH to hold:

$$\text{RH} \Leftrightarrow \psi(x) - x = O(x^{1/2} \log^2 x).$$

Similarly,

$$\text{RH} \Leftrightarrow \pi(x) - \mathrm{Li}(x) = O(x^{1/2} \log x).$$

There are many other statements that are equivalent to RH (see the Wikipedia page on "Riemann hypothesis"). Unfortunately, none of them seem any easier to prove (or disprove). An example of Lagarias (2002), is:

$$RH \Leftrightarrow (\forall n \geqslant 1) \left( \sum_{d|n} d \leqslant H_n + \exp(H_n) \log(H_n) \right),$$

where $H_n = \sum_{1 \leqslant j \leqslant n} 1/j$ is the $n$-th harmonic number.

# A Turing machine to decide RH?

We could program a Turing machine to check Lagarias's criterion and halt when, and only when, an exception to the inequality

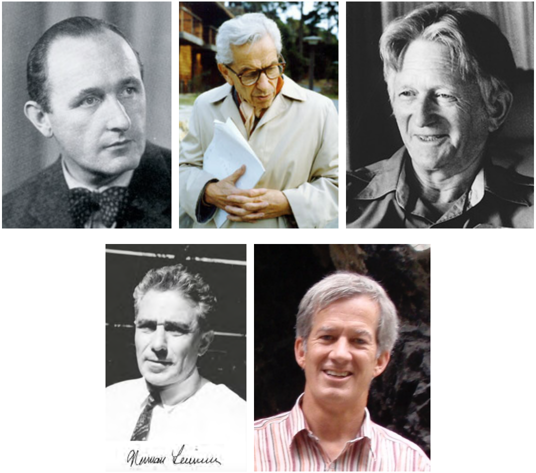
$$\sum_{d|n} d \leqslant H_n + \exp(H_n)\log(H_n)$$

is found. This would disprove RH.

However, if RH is true, then the Turing machine will never halt!

There are other problems like this. For example, you could program a Turing machine to search for integer solutions to Fermat's equation $x^n + y^n = z^n,\ xyz \neq 0,\ n > 2$, and halt when such a solution is found.
Thanks to Wiles and Taylor's proof of "Fermat's Last Theorem", we now know that such a Turing machine will never halt.

# 20th century



Siegel, Erdős, Selberg, Levinson, Conrey

We should not have put Erdős and Selberg next to each other!

We have come to the second half of the talk, which may be intelligible only to “experts”.

Fortunately (if you are not an expert), there is not much time left, so I will have to skip most of what I had hoped to say.

If you are in the “expert” category, you might like to look at my slides for the 2020 AustMS meeting, which are on my website, or at the preprint arXiv:2009.13791.

# Motivation

- In analytic number theory we often encounter sums of the form $\sum \phi(\gamma)$ where $\phi$ is a given function, and the sum is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, perhaps restricted to some finite or semi-infinite interval. We may want to evaluate such sums numerically.
- Examples are $\displaystyle\sum_{\gamma>0} 1/\gamma^2$ and $\sum'_{0<\gamma\leqslant T} 1/\gamma$, where $T$ is large but finite.

The prime mark on the summation sign indicates that a term is given a weight of $\frac{1}{2}$ if it corresponds to an endpoint of the interval.

# Sums over (ordinates of) zeros

Consider sums of the form

$$\sum_{T_1 \leqslant \gamma \leqslant T_2}{}' \phi(\gamma) \,.$$

We can think of this as a Riemann sum approximating

$$\int_{T_1}^{T_2} \phi(t) w(t) \, dt,$$

where $w(t)$ is a weight function that takes into account the non-uniform spacing of the $\gamma$s. The natural weight function is

$$w(t) := L'(t) = \frac{1}{2\pi} \log(t/2\pi),$$

where $L(T) = \frac{T}{2\pi}(\log(\frac{T}{2\pi}) - 1) + \frac{7}{8}$ is a smooth approximation to the zero-counting function $N(T)$. Lehman's lemma bounds the error (the difference between the sum and integral) if we use this weight function.

# Lehman's Lemma

Lemma (Lehman, 1966)

*If* $2\pi e \leqslant T_1 \leqslant T_2$ *and* $\phi : [T_1, T_2] \mapsto [0, \infty)$ *is monotone decreasing on* $[T_1, T_2]$*, and*

$$E(T_1, T_2) := \sum_{T_1 \leqslant \gamma \leqslant T_2}{}' \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi)\, dt,$$

*then*

$$|E(T_1, T_2)| \leqslant A \left( 2\phi(T_1) \log T_1 + \int_{T_1}^{T_2} \frac{\phi(t)}{t}\, dt \right),$$

*where* $A$ *is an absolute constant. We can take* $A = 0.28$*.*

Remark: If $\int_{T_1}^{\infty} \phi(t)/t\, dt < \infty$, we can allow $T_2 \to \infty$.

# An assumption ("Condition A")

Assumption: From now on we assume that $\phi(t)$ is in $C^2[T_0, \infty)$ and for all $t \in [T_0, \infty)$ satisfies

- $\phi(t) \geqslant 0$, (non-negative);
- $\phi'(t) \leqslant 0$, (non-increasing);
- $\phi''(t) \geqslant 0$ (a sort of convexity condition).

These conditions are stronger than those assumed in Lehman's Lemma. However, they are not too restrictive. In most applications $\phi(t)$ is in $C^\infty[T_0, \infty)$, and in this case Lehman's Lemma already assumes the first two conditions. Thus, essentially the only new condition is that $\phi''(t) \geqslant 0$.

With this assumption, we (BPT) can improve on Lehman's lemma.

# Why an improvement is possible

Write

$$N(T) = L(T) + Q(T) = L(T) + S(T) + O(1/T),$$

where

$$S(T) = \pi^{-1} \arg \zeta(\tfrac{1}{2} + iT).$$

The proof of Lehman's lemma depends on the classical result

$$S(T) = O(\log T).$$

Our improvement also uses Littlewood's result

$$S_1(T) := \int_0^T S(t)\, dt = O(\log T),$$

which gives much stronger constraints on the location of the nontrivial zeros of $\zeta(s)$. To take advantage of the bound on $S_1(T)$, we have to assume Condition A.

# Our lemma

Lemma (BPT, 2020)

*If* $2\pi \leqslant T_0 \leqslant T_1 \leqslant T_2$ *and*

$$E(T_1, T_2) := \sideset{}{'}\sum_{T_1 \leqslant \gamma \leqslant T_2} \phi(\gamma) - \frac{1}{2\pi} \int_{T_1}^{T_2} \phi(t) \log(t/2\pi) \, dt \,,$$

*then*

$$E(T_1, T_2) = \phi(T_2)Q(T_2) - \phi(T_1)Q(T_1) + E_2(T_1, T_2),$$

*where*

$$|E_2(T_1, T_2)| \leqslant 2(A_0 + A_1 \log T_1) \, |\phi'(T_1)| + (A_1 + A_2)\phi(T_1)/T_1,$$

*and we may take* $A_0 = 2.067$, $A_1 = 0.059$, $A_2 = 0.007$.

# Example

Consider the convergent sum $c_1 := \sum_{\gamma>0} 1/\gamma^2$.
A first approximation is the sum over $0 < \gamma \leqslant T$
(a finite sum involving $O(T \log T)$ terms).
The error is $\sum_{\gamma>T} 1/\gamma^2 \sim \log(T)/2\pi T$.

We can do better by using Lehman's lemma with $(T_1, T_2) \to (T, \infty)$. This gives

$$\sum_{\gamma \geqslant T}{}' 1/\gamma^2 = \frac{1}{2\pi} \int_T^\infty \frac{\log t/2\pi}{t^2} \, dt + E(T),$$

where

$$|E(T)| \leqslant \frac{0.28\,(0.5 + 2\log T)}{T^2} \, .$$

Lehman's lemma decreases the error bound by a factor of order $T$, from $O(\log(T)/T)$ to $O(\log(T)/T^2)$.

# Example continued

If we use our Lemma (with $T_2 \to \infty$) instead of Lehman's lemma, the error term becomes $E_2(T)$, where

$$|E_2(T)| \leqslant \frac{8.334 + 0.236 \log T}{T^3} .$$

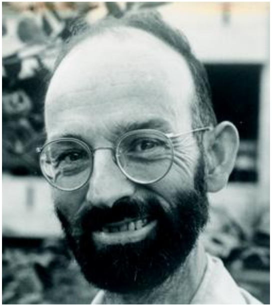
Thus, we get another factor of order $T$, from $O(\log(T)/T^2)$ to $O(\log(T)/T^3)$.

For example, taking $T = 1000$ (corresponding to the first 649 nontrivial zeros), we have the following error bounds.

- Naive truncation of series: $9.7 \times 10^{-4}$.
- Using Lehman's lemma: $4.009 \times 10^{-6}$.
- Using our lemma: $9.965 \times 10^{-9}$.

The improvement over Lehman's lemma is a factor of 400.

# Last photos: 20th – early 21st century



R. Sherman Lehman, Tim Trudgian

# Accurate computation of $c_1$ and its confirmation

Using our theorem with $10^{10}$ zeros, we find that

$$c_1 = 0.0231049931154189707889338104 \pm 5 \cdot 10^{-28}.$$

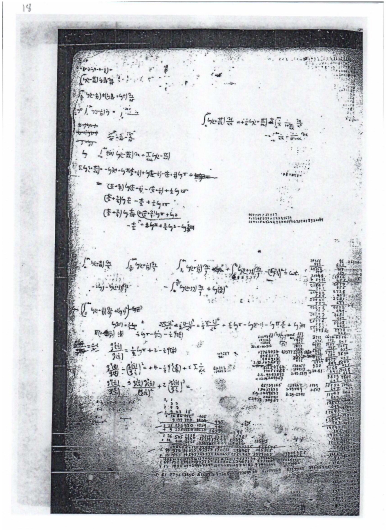
Assuming RH, there is an equivalent expression

$$c_1 = \tfrac{1}{2} d^2 \log \zeta(s)/ds^2|_{s=1/2} + G + \pi^2/8 - 4, \qquad (6)$$

where $G = 1/1^2 - 1/3^2 + 1/5^2 - \cdots \approx 0.916$ is Catalan's constant. This enables us to confirm our result without summing over any zeros of $\zeta(s)$, but assuming RH.

The formula (6) has been proved by Juan Arias de Reyna. In fact, an almost indecipherable sketch of this result may be found in Riemann's Nachlass. We are indebted to Juan Arias de Reyna for information on the identity (6), and for his translation of the relevant page from Riemann's Nachlass.

# A page from Riemann's Nachlass



The page relating to the expression (6) for $c_1$, by courtesy of Juan Arias de Reyna.

# Divergent sums

We can handle divergent sums in much the same way as convergent sums, so long as they don't diverge too fast.

Theorem (BPT, 2020)

*Suppose that* $T_0 \geqslant 2\pi$, *and*

$$\int_{T_0}^{\infty} \frac{\phi(t)}{t}\, dt < \infty.$$

*Then there exists*

$$F(T_0) := \lim_{T\to\infty} \left( \sideset{}{'}\sum_{T_0 \leqslant \gamma \leqslant T} \phi(\gamma) - \frac{1}{2\pi} \int_{T_0}^{T} \phi(t) \log(t/2\pi)\, dt \right).$$

# Example – a "harmonic" series

Corollary (BPT, 2020)

*Let* $G(T) := \sum'_{0<\gamma\leqslant T} 1/\gamma$. *Then there exists*

$$H := \lim_{T\to\infty}\left(G(T) - \frac{\log^2(T/2\pi)}{4\pi}\right).$$

Using $10^{10}$ zeros, we find that

$$H = -0.0171594043070981495 \pm 10^{-18}.$$

The definition of $H$ is analogous to the usual definition of Euler's constant $C$:

$$C := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) = 0.5772\ldots$$

# References


R. Ayoub, Euler and the zeta function,
*Amer. Math. Monthly* 81 (1974), 1067–1086.

R. P. Brent, D. J. Platt, and T. S. Trudgian,
On a harmonic sum over non-trivial zeros of the Riemann zeta-function, *Bull. Austral. Math. Soc.*, online 20 Nov. 2020.
Also https://arxiv.org/abs/2009.05251.

R. P. Brent, D. J. Platt, and T. S. Trudgian,
Accurate estimation of sums over zeros of the Riemann zeta-function, https://arxiv.org/abs/2009.13791.

H. M. Edwards,
*Riemann's Zeta Function*, Academic Press, NY, 1974.

R. S. Lehman, On the difference $\pi(x) - \mathrm{li}(x)$, *Acta Arith.* **11**:397–410, 1966.

H. Montgomery and R. C. Vaughan, *Multiplicative Number Theory, I. Classical Theory*, CUP, Cambridge, 2007.

# References cont.

D. Platt and T. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, arXiv:2004.09765, April 2020.

E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed. (edited by D. R. Heath-Brown), Oxford, 1986.