Lower bounds on maximal determinants via the probabilistic method

Richard P. Brent
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Warren Smith and Judy-anne Osborn

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Abstract

The Hadamard maximal determinant problem is to find the maximal determinant $D(n)$ of a square $\{\pm 1\}$-matrix of given order n. Hadamard proved the upper bound $D(n) \le n^{n/2}$. This talk is concerned with lower bounds on $\mathcal{R}(n) := D(n)/n^{n/2}$.

Define $d := n - h$, where h is the maximal order of a Hadamard matrix no larger than n. Using the probabilistic method, we can show that $\mathcal{R}(n) \ge \kappa_d > 0$, where κ_d depends only on d. Previous lower bounds depend on both d and n. Our bounds are improvements for $d > 1$ and all sufficiently large n.

This talk will outline the main results and methods used to obtain them. For technical details, see the preprint at `http://arxiv.org/abs/1211.3248`.

Introduction – the Hadamard bound and conjecture

- $D(n) :=$ denote the maximum determinant attainable by an $n \times n$ $\{\pm 1\}$-matrix.
- Hadamard proved the upper bound $D(n) \le n^{n/2}$.
- A *Hadamard matrix* is an $n \times n$ ± 1 matrix A with $\det(A) = \pm n^{n/2}$.
- If a Hadamard matrix of order n exists, then $n = 1$, 2, or a multiple of 4. We'll ignore the cases $n \in \{1, 2\}$.
- The *Hadamard conjecture* is that Hadamard matrices exist for every positive multiple of 4.
- This talk is about lower bounds on $D(n)$.

Notation

- $\mathcal{H}$ is the set of all possible orders of Hadamard matrices.
- $\mathcal{R}(n) := D(n)/n^{n/2}$.
  The Hadamard bound is $\mathcal{R}(n) \le 1$.
  We are interested in lower bounds on $\mathcal{R}(n)$.
- $d := n - \max\{h \in \mathcal{H} \mid h \le n\}$.
  In other words, $n = h + d$, $d \ge 0$, and $h \in \mathcal{H}$ is maximal.
  To avoid trivial cases, assume that $n \ge h \ge 4$.
- $f \ll g$ means $f = O(g)$ and $f \gg g$ means $g = O(f)$.

Previous results

For those of you who attended my AustMS talk in Ballarat – the problem is the same, but the results are better!

In all previous papers that we are aware of (including our own), general lower bounds on $\mathcal{R}(n)$ tend to zero as $n \to \infty$, unless $n \in \mathcal{H}$ or $n - 1 \in \mathcal{H}$.

For example, de Launey and Levin (2009) showed that

$$\mathcal{R}(n) \geq \frac{2^{1/2}e}{n}\left(1 + O\left(\frac{1}{n}\right)\right)$$

if $n \equiv 2 \pmod 4$, assuming the Hadamard conjecture.

Under the same assumption, our new result

$$\mathcal{R}(n) > \frac{2}{\pi e} \approx 0.2342$$

is sharper for all $n \geq 18$.

Previous approaches

The most successful previous approaches to obtaining general lower bounds (as opposed to bounds for specific small values of n) used either bordering or minors.

- bordering: choose a Hadamard matrix of order $h < n$, and add a border of $n - h$ rows and columns.
- minors: choose a Hadamard matrix H of order $h > n$, and consider an $n \times n$ submatrix of H.

The best lower bound obtained via bordering or minors is

$$\mathcal{R}(n) \gg n^{-\delta/2} \text{ where } \delta = |n - h|$$

[Koukovinos, Mitrouli and Seberry; de Launey and Levin] with one exception (next slide).

Improved bound for bordering if $\delta = 1$

For $\delta := n - h = 1$, the lower bound can be improved to

$$\mathcal{R}(n) \geq \text{ constant}$$

by using a probabilistic method due to Brown and Spencer (1971), Erdős and Spencer (1974), and Best (1977).

The idea is to add a border of one row and column to a Hadamard matrix in a (semi-)probabilistic manner that gives a large determinant (on average).

The new approach

Our idea is to generalise the bordering method of Best by taking a Hadamard matrix of order $h < n$ and adding a border of $d = n - h$ rows and columns in a (semi-) probabilistic manner.

This enables us to obtain lower bounds of the form

$$\mathcal{R}(n) \geq \kappa_d > 0,$$

where κ_d depends only on d.

For example,

$$\mathcal{R}(n) \geq 0.07\,(0.352)^d.$$

The Schur complement

Let

$$\widetilde{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

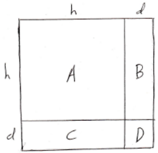
be an $n \times n$ matrix written in block form, where A is $h \times h$, and $n = h + d > h$. The *Schur complement* of A in $\widetilde{A}$ is the $d \times d$ matrix

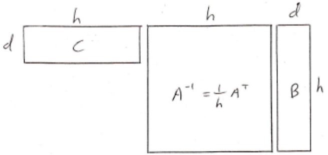
$$D - CA^{-1}B.$$

The Schur complement is relevant to our problem because

$$\det(\widetilde{A}) = \det(A)\det(D - CA^{-1}B).$$

The block matrix $\widetilde{A}$ and Schur complement





Recall that

$$\det(\widetilde{A}) = \det(A)\det(D - CA^{-1}B).$$

Application of the Schur complement

Take A to be an $h \times h$ Hadamard matrix that is a principal submatrix of an $n \times n$ matrix

$$\widetilde{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} .$$

Then $\det(A) = h^{h/2}$ and $A^{-1} = h^{-1} A^T$, so

$$\det(\widetilde{A}) = h^{h/2} \det(D - h^{-1} C A^T B)$$

Thus, the problem is to maximise $|\det(D - h^{-1} C A^T B)|$.

Application of the probabilistic method

Choose the $h \times d$ matrix B uniformly at random from the 2^{hd} possibilities.

We would like to choose C and D (deterministically, but depending on B) to maximise the expected value

$$E(|\det(D - h^{-1}CA^TB)|).$$

We don't know how to do this, but we approximate it by choosing $C = (c_{ij})$,

$$c_{ij} = \mathrm{sgn}(A^TB)_{ji} \text{ for } 1 \le i \le d,\ 1 \le j \le h$$

so that there is no cancellation in the inner products defining the diagonal elements of $C \cdot A^TB$.

In the case $d = 1$ this is the same as Best's choice.

Entries in the Schur complement

Write $F = h^{-1}CA^TB$, so the Schur complement is $D - F$.

The choice of D is not important (at least as $h \to \infty$), so for simplicity we'll ignore D and concentrate on F.

Best, using a counting argument, showed that

$$E(f_{ii}) = 2^{-h}\sum_{k=0}^{h}|h-2k|\binom{h}{k} = \frac{h}{2^h}\binom{h}{h/2} \sim \left(\frac{2h}{\pi}\right)^{1/2}.$$

Also, we can show that, if $i \neq j$, then $E(f_{ij}) = 0$ and $E(f_{ij}^2) = 1$.

Exercise. Show that $|f_{ij}| \leq h^{1/2}$.

Making it rigorous – the off-diagonal elements

We want to approximate the determinant of the Schur complement by the product of its diagonal elements.

One way of showing that the contribution from the off-diagonal elements is (usually) small is to use the Cauchy-Schwarz inequality:

$$E(|f_{ij}f_{k\ell}|) \le \sqrt{E(f_{ij}^2)E(f_{k\ell}^2)} = 1.$$

NB We can not assume that f_{ij} and $f_{k\ell}$ are independent, even if $i \ne j$ and $k \ne \ell$. For example, f_{12} and f_{21} are dependent.

Exercise. Show that f_{ij} depends only on columns i and j of B. Deduce that f_{ij} and $f_{k\ell}$ are independent iff $\{i,j\} \cap \{k,\ell\} = \emptyset$.

Using Cauchy-Schwartz

Consider estimating $E(\det(F))$ for fixed d and large h.

For example, if $d = 3$,

$$\det(F) = \det \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = f_{11}f_{22}f_{33} + \text{ other terms,}$$

and a typical "off-diagonal" term has expectation $O(h^{1/2})$ as

$$|E(f_{12}f_{21}f_{33})| \le E(|f_{12}f_{21}|)\max(|f_{33}|) \le h^{1/2}.$$

Thus, using independence of f_{11}, f_{22} and f_{33},

$$E(\det(F)) = E(f_{11}f_{22}f_{33}) + O_d(h^{1/2}) = \left(\frac{2h}{\pi}\right)^{3/2} + O_d(h^{1/2}).$$

Results

Theorem. If $d \geq 1$, $h \in \mathcal{H}$, $h \geq 4$, $n = h + d$, and

$$h \geq h_0(d) := \left(e(\pi/2)^{d/2}(d-1)! + d\right)^2,$$

then

$$\mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2}.$$

The constant $2/\pi e$ appearing here is nice (though probably not best possible).

We would like to reduce the cutoff $h_0(d)$ which grows faster than exponentially in d. This can be done (see later) using a tail inequality, at the expense of a slightly weaker bound. However, the theorem as it stands is useful for small d.

The case of small d

If $0 \le d \le 3$ then the previous theorem implies (after considering some small cases separately) that

$$\mathcal{R}(n) \ge \left(\frac{2}{\pi e}\right)^{d/2}.$$

$$\left(\frac{2}{\pi e}\right)^{1/2} > 0.4839 \text{ so } \mathcal{R}(n) \ge (0.4839)^d.$$

If the Hadamard conjecture is true, then every positive integer divisible by 4 is a Hadamard order, and we can assume that $0 \le d \le 3$, so the inequality always holds.

Less restrictive result

The following theorem removes the restriction on h at the cost of reducing the constant from $\left(\frac{2}{\pi e}\right)^{1/2} \approx 0.4839$ to $1/3$.

Theorem. If $d \geq 0$, $h \in \mathcal{H}$, and $n = h + d$, then

$$\mathcal{R}(n) > 3^{-(d+3)}.$$

Comparison: the bound of Clements and Lindström (1965) is

$$\mathcal{R}(n) > (3/4)^{n/2}.$$

Our bound is much sharper since $d \ll n^{1/6}$ [Livinskyi 2012]. It is also sharper than the bounds of Koukouvinos, Mitrouli and Seberry (also de Launey and Levin, Brent and Osborn) if $d > 0$ is fixed and $n \to \infty$; all these bounds are at best $\mathcal{R}(n) \gg n^{-1/2}$.

Comments on the proof

The proof uses

- Hoeffding's tail inequality for a sum of bounded independent random variables,
- a new (best possible) lower bound on the determinant of a diagonally dominant matrix, improving on what can be obtained from Gerschgorin's theorem,
- various known constructions for Hadamard matrices,
- results of Livinskyi (2012) on the asymptotic density of Hadamard matrices, and
- a computer-aided analysis of a set of 32 exceptional cases with $n < 60480$.

If you are interested, see our preprint arXiv:1211.3248.

Conjecture

We conjecture that

$$\mathcal{R}(n) \geq \left(\frac{2}{\pi e}\right)^{d/2}.$$

Evidence. The conjecture holds for:

- for $0 \leq d \leq 3$ (implied by the Hadamard conjecture),
- for all $d \geq 0$ if $n \geq n_0(d)$ is sufficiently large,
- for all $n \leq 120$ (in fact $\mathcal{R}(n) > 1/2$ for $n \leq 120$),
- for many larger values of n for which we have computed a lower bound on $\mathcal{R}(n)$ using a probabilistic algorithm based on our construction.

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