# Asymptotics of a Mahler Function

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# Abstract

We give the asymptotic behaviour as $z \to 1^-$ of a Mahler function $F(z)$ that was introduced by Dilcher and Stolarsky and is related to the Stern sequence. $F(z)$ satisfies the recurrence

$$F(z) = (1 + z + z^2)F(z^4) - z^4 F(z^{16}).$$

We also consider the associated function $\mu(z) = F(z)/F(z^4)$ which may be written as a continued fraction. The analysis depends on locating singularities of the Mellin transforms of $\ln F(e^{-t})$ and $\ln \mu(e^{-t})$.

There are applications to questions of transcendence and algebraic independence.

# Motivation – Mahler's method

*One of the first significant contributions of Mahler is an approach, now called "Mahler's method", yielding transcendence and algebraic independence results for the values at algebraic points of a large family of power series satisfying functional equations of a certain type. In the seminal paper [9][1] Mahler established that the Fredholm series* $f(z) = \sum_{k \geq 0} z^{2^k}$*, which satisfies* $f(z^2) = f(z) - z$*, takes transcendental values at any nonzero algebraic point in the open unit disc.*

*J. Borwein, Y. Bugeaud and M. Coons*
*The legacy of Kurt Mahler*
*AustMS Gazette, March 2014, pg. 16.*

[1] K. Mahler, *Math. Ann.* 101 (1929), 342–366.

# The function $F(z)$

Dilcher and Stolarsky [Acta Arithmetica, 2009] introduced a Mahler function $F(z) = 1 + z + \cdots$ satisfying the recurrence

$$F(z) = (1 + z + z^2)F(z^4) - z^4F(z^{16}).$$

$F(z)$ is related to the Stern sequence.

We consider the asymptotic behaviour of $F(z)$ as $z \to 1^-$.

# The Stern sequence

Stern's diatomic sequence (or Stern-Brocot sequence) is defined by

$$
\begin{aligned}
a_0 &= 0, \\
a_1 &= 1, \\
a_{2n} &= a_n \text{ for } n > 0, \\
a_{2n+1} &= a_n + a_{n+1} \text{ for } n > 0.
\end{aligned}
$$

This sequence has many interesting properties (see the OEIS entry A002487). For example, $a_n/a_{n+1}$ runs through all the reduced nonnegative rationals exactly once.

# Some properties of $F(z)$

Dilcher and Stolarsky (2009) defined $F(z)$ using a polynomial analogue of the Stern sequence, and deduced the recurrence

$$F(z) = (1 + z + z^2)F(z^4) - z^4 F(z^{16}). \tag{1}$$

However, for our purposes it is simpler to *define* $F(z)$ by the recurrence (1) and the auxiliary condition $F(z) = 1 + O(z)$ as $z \to 0$.

Using Mahler's method, Adamczewski (2010) proved that $F(q)$ is transcendental for every algebraic $q$, $0 < |q| < 1$.

Independently, Michael Coons (2010) proved that $F(z)$ is a transcendental function, along with results on transcendence at algebraic arguments.

# The auxiliary function $\mu(z)$

We are interested in the behaviour of $F(z)$ for $z \in [0,1)$, and in particular the asymptotic behaviour of $F(z)$ as $z \to 1^-$.

It is useful to define an auxiliary function $\mu : [0,1) \mapsto \mathbb{R}$ by

$$\mu(z) = \frac{F(z)}{F(z^4)} \,. \tag{2}$$

From the recurrence for $F(z)$ and (2), $\mu(z)$ satisfies the recurrence

$$\mu(z) = 1 + z + z^2 - \frac{z^4}{\mu(z^4)} \,. \tag{3}$$

Our strategy is to analyse the asymptotic behaviour of $\mu(z)$ and then deduce the corresponding behaviour of $F(z)$.

# $\mu(z)$ as a continued fraction

Observe that $\mu(z)$ may be written as a continued fraction

$$\begin{aligned}\mu(z) &= (1+z+z^2) - z^4/\mu(z^4)\\ &= (1+z+z^2) - \frac{z^4}{(1+z^4+z^{2\cdot 4}) - z^{4^2}/\mu(z^{4^2})} = \cdots\end{aligned}$$

Since $\mu(z) = F(z)/F(z^4)$, we have an explicit expression for $F(z)$ as an infinite product:

$$F(z) = \prod_{k=0}^{\infty} \mu\left(z^{4^k}\right). \qquad (4)$$

In this sense we have an explicit solution for $F(z)$ as an infinite product of continued fractions.

# Some properties of $F(z)$ as an analytic function

## Lemma

*The Maclaurin series*

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

*has coefficients* $f_n \in \{0, 1\}$*. Also,* $F(z)$ *is strictly monotonic increasing and unbounded for* $z \in [0, 1)$*, and can not be analytically continued past the unit circle.*

From the functional equation for $F(z)$ it follows that $F(z)$ has a singularity at $z = \exp(2\pi i/2^k)$ for all non-negative integers $k$. Thus, there is a dense set of singularities on the unit circle, which is a natural boundary.

# Properties of $\mu(z)$

## Lemma

*If* $\mu_1 := \lim_{x \to 1-} \mu(x)$ *and* $\mu'_1 := \lim_{x \to 1-} \mu'(x)$*, then*

$$\mu_1 = \frac{3+\sqrt{5}}{2} = \rho^2 \approx 2.618 \tag{5}$$

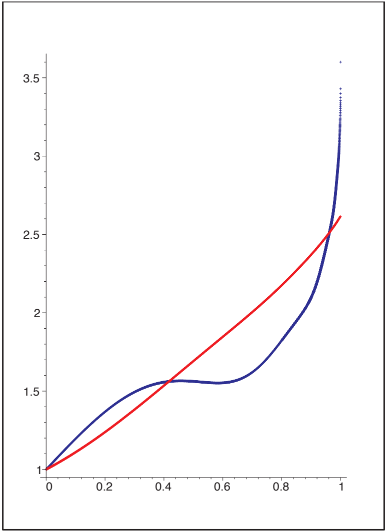
*and*

$$\mu'_1 = \frac{21+8\sqrt{5}}{11} \approx 3.535. \tag{6}$$

## Sketch of proof.

Let $Q(x)$ be the larger root of $Q(x) = 1 + x + x^2 - x^4/Q(x)$. Show that $\mu(x) < Q(x)$ for all $x \in (0,1)$. Hint – use induction on $x = x_0^{4^{-n}}$, where $x_0$ is sufficiently small. □

# $\mu(x)$ and $\mu'(x)$ for $x \in [0, 1)$



# What can we say about $\mu''(x)$?

It appears from the graph of $\mu'(x)$ that $\mu''(x)$ is unbounded as $x \to 1^-$, and this is indeed true. We have the following result, where the constant $2\lg(\rho)$ is best possible.[2]

## Lemma

*Let $\alpha \le 2\lg(\rho) \approx 1.388$. Then, for $t \in (0,1)$ we have*

$$\mu''(e^{-t}) = O(t^{\alpha-2}) \tag{7}$$

*and*

$$\mu(e^{-t}) = \mu_1 - t\mu_1' + O(t^{\alpha}). \tag{8}$$

[2] We write $\lg(x)$ for $\log_2(x)$.

# Why the exponent $\alpha \approx 1.388$?

Differentiating the recurrence for $\mu(z)$ twice, we obtain

$$\mu''(e^{-t}) = A(t) + B(t)\mu''(e^{-4t}),$$

where $A(t)$ is bounded, and

$$B(t) = 16e^{-10t}/\mu(e^{-4t})^2 = 16/\mu_1^2 + O(t).$$

The exponent $\alpha$ is chosen so that $16/\mu_1^2 \le 4^{2-\alpha}$, since this inequality is necessary (and sufficient) for the inductive proof to go through.

Since $\mu_1 = \rho^2$, we have to choose $\alpha \le 2\lg(\rho) \approx 1.388$.

# Mellin transforms

Our strategy is to deduce the asymptotic behaviour of $\mu(z)$ and $F(z)$ as $z \to 1^-$ from certain Mellin transforms.

Specifically, define

$$\mathcal{F}(s) := \int_0^\infty \ln(F(e^{-t}))\, t^{s-1}\, \mathrm{d}t$$

and

$$\mathcal{M}(s) := \int_0^\infty \ln(\mu(e^{-t}))\, t^{s-1}\, \mathrm{d}t.$$

The integrals converge in the half-plane $\Re(s) > 0$. For $\Re(s) \leq 0$ we define $\mathcal{F}(s)$ and $\mathcal{M}(s)$ by analytic continuation (if possible).

# Properties of the Mellin transforms

Since

$$\ln \mu(e^{-t}) = \ln F(e^{-t}) - \ln F(e^{-4t}),$$

we see that

$$\mathcal{M}(s) = (1 - 4^{-s})\mathcal{F}(s).$$

We can deduce the behaviour of $\ln F(e^{-t})$ for small positive $t$ from knowledge of the singularities of $\mathcal{F}(s)$.

Since $\mathcal{F}(s) = (1 - 4^{-s})^{-1}\mathcal{M}(s)$, it is sufficient to determine the singularities of $\mathcal{M}(s)$ and (easy) those of $(1 - 4^{-s})^{-1}$.

First we use the Lemmas above to extend the domain of definition of $\mathcal{M}(s)$ into the left half-plane.

# Analytic continuation of $\mathcal{M}(s)$

Define

$$\widetilde{\mu}(t) := \ln(\mu(e^{-t})) - \ln(\mu_1)e^{-\lambda t},$$

where

$$\lambda := \frac{\mu_1'}{\mu_1 \ln \mu_1} \approx 1.403\,.$$

Since $\lambda \ge 1$, $\widetilde{\mu}(t) = O(e^{-t})$ as $t \to +\infty$.
Also, from the Lemmas above, as $t \to 0^+$ we have

$$\widetilde{\mu}(t) = (\lambda \ln \mu_1 - \mu_1'/\mu_1)t + O(t^\alpha).$$

Our choice of $\lambda$ makes the coefficient of $t$ vanish, so $\widetilde{\mu}(t) = O(t^\alpha)$.

# Analytic continuation of $\mathcal{M}(s)$

Let

$$\widetilde{\mathcal{M}}(s) := \int_0^\infty \widetilde{\mu}(t) t^{s-1}\,\mathrm{d}t.$$

Since $\widetilde{\mu}(t) = O(t^\alpha)$, the integral converges for $\Re(s) > -\alpha$. Now

$$\mathcal{M}(s) = \widetilde{\mathcal{M}}(s) + \ln(\mu_1)\lambda^{-s}\Gamma(s)$$

gives the analytic continuation of $\mathcal{M}(s)$ into the half-plane

$$\mathcal{H} := \{s \in \mathbb{C} : \Re(s) > -2\lg(\rho)\}.$$

In $\mathcal{H}$, the only singularities of $\mathcal{M}(s)$ occur at the singularities of $\Gamma(s)$, i.e. at $s \in \{0, -1\}$.

# Singularities of $\mathcal{F}(s)$ in $\mathcal{H}$

The Mellin transform $\mathcal{F}(s) = (1 - 4^{-s})^{-1}\mathcal{M}(s)$ has three types of singularities in $\mathcal{H}$.

(a) A double pole at $s = 0$, since $\Gamma(s)$ has a pole there, and the denominator $1 - 4^{-s}$ vanishes at $s = 0$.

(b) Poles at $s = ik\pi/\ln(2)$ for $k \in \mathbb{Z}\backslash\{0\}$, since the denominator $1 - 4^{-s}$ vanishes at these points.

(c) A pole at $s = -1$, since $\Gamma(s)$ has a pole there.

# Asymptotics of $\ln F(e^{-t})$

**Theorem**

For arbitrary $\varepsilon > 0$ and small positive $t$,

$$\ln F(e^{-t}) = -\lg(\rho)\ln(t) + c_0 + \sum_{k=1}^{\infty} a_k(t) + c_1 t + O(t^{2\lg(\rho)-\varepsilon}),$$

where $c_0 \approx 0.1216$ and $c_1 \approx 0.4501$ are constants, and

$$a_k(t) = \frac{1}{\ln 2} \Re \left( \mathcal{M}\left(\frac{ik\pi}{\ln 2}\right) \exp(-ik\pi \lg(t)) \right).$$

Note. It is easy to see that $a_k(4t) = a_k(t)$, so the $a_k(t)$ are periodic in $\log(t)$.

# The oscillatory terms $a_k(t)$

We can write

$$a_k(t) = A_k \cos(k\pi \lg(t)) + B_k \sin(k\pi \lg(t)).$$

Define

$$C_k := \sqrt{A_k^2 + B_k^2} = \max_{t>0} |a_k(t)| = \frac{|\mathcal{M}(ik\pi / \ln 2)|}{\ln 2}\,.$$

Numerically, we find
$C_1 \approx 2.1 \times 10^{-3}$, $C_2 \approx 2.2 \times 10^{-6}$, $C_3 \approx 2.8 \times 10^{-9}$,
$C_4 \approx 3.3 \times 10^{-12}$, ...

The constants $C_k$ appear to decrease exponentially fast as $k \to \infty$.

# Sketch proof of the theorem

Consider the singularity of type (a).

Define $L(s) := \mathcal{M}(s)/\Gamma(s)$. Then

$$L(0) = 2\ln\rho,\ \ L'(0) = \widetilde{\mathcal{M}}(0) - 2\ln(\lambda)\ln(\rho) \approx 0.06.$$

Near the double pole at $s = 0$,

$$\mathcal{F}(s) = \frac{L(0)}{2\ln 2}\, s^{-2} + c_0\, s^{-1} + O(1),$$

where

$$c_0 = \frac{(\ln 2 - \gamma)L(0) + L'(0)}{2\ln 2}\,.$$

Standard arguments applied to the inverse Mellin transform now give the first two terms $(-\lg(\rho)\ln(t) + c_0)$.

# Sketch proof continued

Now consider the singularities of type (b).

These are simple poles at $s = ik\pi/\ln 2$ for $k \in \mathbb{Z}\backslash\{0\}$.

From the pole at $ik\pi/\ln 2$ we get a term

$$T_k(t) := \frac{1}{\ln 4}\mathcal{M}\left(\frac{ik\pi}{\ln 2}\right)\exp(-ik\pi\lg(t)).$$

Combining the terms $T_k(t)$ and $T_{-k}(t)$ for $k \geq 1$, the imaginary parts cancel and we are left with the oscillatory term $a_k(t)$.

# Sketch proof continued

Now consider the singularity of type (c).

At $s = -1$, $\mathcal{F}(s)$ has a pole with residue

$$c_1 = \frac{\lambda \ln \mu_1}{3} = \frac{\mu_1'}{3\mu_1} = \frac{23 + 3\sqrt{5}}{66}.$$

This accounts for the term $c_1 t$.

Finally, the error term $O(t^{2\lg(\rho)-\varepsilon})$ allows for the fact that we have only considered the singularities of $\mathcal{F}(s)$ in $\mathcal{H}$.

There could be (in fact are) other singularities in the half-plane

$$\{s \in \mathbb{C} : \Re(s) \le -2\lg(\rho)\}.$$

# A corollary

All that we actually need for the applications is the following.

**Corollary**

For $z \in [0, 1)$,

$$F(z) = \frac{C(z)}{(1 - z)^{\lg \rho}} ,$$

where $C(z)$ is a positive oscillatory term, bounded away from zero and infinity.

**Remark**

We find numerically that $C(z) \in [1.11, 1.14]$ for all $z \in [1/2, 1)$.

# A conjecture

We conjecture that $\mathcal{M}(s)$ and $\mathcal{F}(s) = (1 - 4^{-s})^{-1}\mathcal{M}(s)$ have poles at $s = -2\lg(\rho) + ik\pi/\ln(2)$ for $k \in \mathbb{Z}$.

This would account for numerical evidence that the error $e_1(t)$ in the linear approximation to $\mu(e^{-t})$ is of order $t^{2\lg(\rho)}$ but does not tend to a limit as $t \to 0^+$, instead it has small oscillations that are periodic in $k = -\lg t$.

| $k$ | $t = 2^{-k}$ | $\mu(e^{-t})$ | $e_1(t)$ | $e_1(t)/t^{2\lg\rho}$ |
|---|---|---|---|---|
| 20 | 9.5367e-7 | 2.6180306 | 1.1708e-8 | 2.6790 |
| 21 | 4.7684e-7 | 2.6180323 | 4.4999e-9 | 2.6958 |
| 22 | 2.3842e-7 | 2.6180331 | 1.7079e-9 | 2.6787 |
| 23 | 1.1921e-7 | 2.6180336 | 6.5648e-10 | 2.6956 |
| 24 | 5.9605e-8 | 2.6180338 | 2.4917e-10 | 2.6786 |

Approximation of $\mu(e^{-t})$ for $t = 2^{-k}$, $20 \le k \le 24$, $e_1(t) = \mu(e^{-t}) - (\mu_1 - t\mu_1')$.

# References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