Some Identities involving Products of Gamma Functions: a Case Study in Experimental Mathematics

Richard P. Brent

CARMA and ANU

27 October 2015

Joint work with
Christian Krattenthaler (Vienna)
and Ole Warnaar (Univ. of Qld.)

Presented at a CARMA seminar, Newcastle, 27 Oct 2015.

Abstract

We consider identities satisfied by discrete analogues of Mehta-like integrals. The integrals are related to Selberg's integral and the Macdonald conjectures. Our discrete analogues have the form
$S_{\alpha,\beta,\delta}(r,n) := \sum_{k_1,\ldots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} |k_i^\alpha - k_j^\alpha|^\beta \prod_{j=1}^r |k_j|^\delta \binom{2n}{n+k_j}$.
In the ten cases that we consider, it is possible to express $S_{\alpha,\beta,\delta}(r,n)$ as a product of Gamma functions and simple functions such as powers of two. For example, if $1 \le r \le n$, then

$$S_{2,2,3}(r,n) = \prod_{j=1}^{r} \frac{(2n)!\, j!^2}{(n-j)!^2} \cdot$$

The emphasis of the talk is on how such identities can be obtained (or ruled out), with a high degree of certainty, using numerical computation. We outline the ideas behind some of our proofs, which involve q-series identities and arguments based on non-intersecting lattice paths.

Some recent history

This talk is about expressing certain sums as products.

For an application of the probabilistic method to Hadamard's maximal determinant problem, Brent and Osborn (2013) considered the sum

$$S(2,n) := \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} |k^2 - \ell^2| \binom{2n}{n+k} \binom{2n}{n+\ell}$$

and showed that

$$S(2,n) = 2n^2 \binom{2n}{n}^2.$$

We can assume that $k, \ell \in [-n, n]$ as otherwise the product of binomial coefficients vanishes. Thus, there are $O(n^2)$ nonzero terms in the sum.

Some recent history

Ohtsuka conjectured, and Prodinger proved, an analogous triple-sum identity:

$$S(3,n) := \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\Delta(k^2,\ell^2,m^2)| \binom{2n}{n+k}\binom{2n}{n+\ell}\binom{2n}{n+m}$$
$$= 3n^3(n-1)\binom{2n}{n}^2 2^{2n-1},$$

where $\Delta(x,y,z) := (y-x)(z-y)(z-x)$ and $n \geq 2$.

Warnaar suggested generalising this and similar results to r-fold sums. At a certain point we consulted Christian Krattenthaler. This resulted in a collaboration between Brent, Krattenthaler and Warnaar. I will summarise some of the results obtained by this collaboration.

Notation and definitions

The Vandermonde determinant is

$$\Delta(k_1,\ldots,k_r) := \det(k_i^{j-1})_{1\le i,j\le r} = \prod_{1\le i<j\le r}(k_j - k_i).$$

Suppose $\alpha \ge 0$ and $k = (k_1,\ldots,k_r) \in \mathbb{R}^r$. A useful notation is

$$\Delta(k^\alpha) := \Delta(k_1^\alpha,\ldots,k_r^\alpha) = \prod_{1\le i<j\le r}(k_j^\alpha - k_i^\alpha).$$

We consider centered binomial sums involving the absolute value of a generalised Vandermonde, specifically

$$S_{\alpha,\beta,\delta}(r,n) := \sum_{k\in\mathbb{Z}^r} |\Delta(k^\alpha)|^\beta \prod_{j=1}^{r} |k_j|^\delta \binom{2n}{n+k_j}.$$

Here $\alpha, \beta, \delta, r, n$ are non-negative integer parameters.

Our aim is to express these sums as products.

Examples

The two-fold sum $S(2,n)$ mentioned previously is

$$S_{2,1,0}(2,n) := \sum_{k_1} \sum_{k_2} |\Delta(k^2)| \prod_{j=1}^{2} \binom{2n}{n+k_j}.$$

The three-fold sum $S(3,n)$ mentioned previously is

$$S_{2,1,0}(3,n) := \sum_{k_1} \sum_{k_2} \sum_{k_3} |\Delta(k^2)| \prod_{j=1}^{3} \binom{2n}{n+k_j}.$$

We saw that these sums can be expressed as products of factorials and powers of two.

Examples

The r-fold generalisation can also be expressed as a product:

$$S_{2,1,0}(r,n) := \sum_{k_1,\ldots,k_r \in \mathbb{Z}} |\Delta(k^2)| \prod_{j=1}^{r} \binom{2n}{n+k_j}$$

$$= \prod_{j=0}^{r-1} \frac{(2n)!}{(n-j)!} \frac{\Gamma(\frac{j+3}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-j+\frac{3}{2})}{\Gamma(n-\frac{j}{2}+\frac{3}{2})} \frac{\Gamma(\frac{j+1}{2})}{\Gamma(n-\frac{j-1}{2})}.$$

We assume that $n \geq r-1$, as otherwise the sum vanishes.

Thus, a sum with $O(n^r)$ nonzero terms has been expressed as a product of $O(r)$ Gamma functions (or reciprocals of Gamma functions). We call such a product a *Gamma product*. Polynomial powers of constants, e.g. 2^{n-r}, are allowed in such products.

Recall that $n! = \Gamma(n+1)$. Thus factorials are special cases of Gamma functions. We use both notations.

Justification

Why are these binomial sums interesting?
Haven't binomial sums been "solved" by Wilf and Zeilberger?

- Wilf and Zeilberger's method only applies in low dimensions (r).
- No "simple" proof (e.g. by induction on the dimension r) of any of our identities is known.
- Interesting generalisations and/or q-analogues exist, and in some cases are required for the proofs.
- The identities are discrete analogues of important integrals due to Selberg, Bombieri, Macdonald, Mehta, Dyson et al.
- The identities can be interpreted as giving expectations associated with certain random walks in r dimensions.

Mehta's integral

Mehta's r-fold integral is

$$F_r(\gamma) := \int_{\mathbb{R}^r} |\Delta(x)|^{2\gamma} \, \mathrm{d}\psi(x),$$

where $\psi(x)$ is the r-dimensional Gaussian measure, i.e.

$$\mathrm{d}\psi(x) := \frac{\exp(-\frac{1}{2}||x||_2^2)}{(2\pi)^{r/2}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_r \, .$$

Mehta and Dyson evaluated $F_r(\gamma)$ for the cases $\gamma \in \{1/2, 1, 2\}$, and conjectured the general result

$$F_r(\gamma) = \prod_{j=1}^{r} \frac{\Gamma(1 + j\gamma)}{\Gamma(1 + \gamma)} \, .$$

This was later proved by Bombieri and Selberg, using Selberg's integral (see the survey by Forrester and Warnaar).

Discrete approximation

The finite sum

$$S_{1,2\gamma,0}(r,n) = \sum_{-n\le k_j\le n} |\Delta(k)|^{2\gamma} \prod_{j=1}^{r} \binom{2n}{n+k_j}$$

is a (scaled) discrete approximation to $F_r(\gamma)$. Using

$$\binom{2n}{n+k} \sim \frac{2^{2n}}{\sqrt{n\pi}} e^{-k^2/n}$$

as $n \to \infty$ with $k = o(n^{2/3})$, we see that

$$\lim_{n\to\infty} \frac{S_{1,2\gamma,0}(r,n)}{2^{2rn}(n/2)^{\gamma r(r-1)/2}} = F_r(\gamma).$$

To make this rigorous we can use the method of tail-exchange (see for example Graham, Knuth and Patashnik).

Macdonald-Mehta integrals

More generally, we can define a *Macdonald-Mehta integral*

$$\mathcal{F}_{\alpha,\beta,\delta}(r) := \int_{R^r} |\Delta(x^\alpha)|^\beta \prod_{j=1}^{r} |x_j|^\delta \, \mathrm{d}\psi(x),$$

where $\alpha, \beta = 2\gamma$, and δ are non-negative real parameters.

Then $S_{\alpha,\beta,\delta}(r,n)$ is a (scaled) discrete approximation to $\mathcal{F}_{\alpha,\beta,\delta}(r)$, in the sense that

$$\mathcal{F}_{\alpha,\beta,\delta}(r) = \lim_{n\to\infty} \frac{S_{\alpha,\beta,\delta}(r,n)}{2^{2rn}\,(n/2)^{\alpha\beta r(r-1)/4+\delta r/2}}\,.$$

The Macdonald-Mehta integrals arise in Macdonald's (ex-)conjecture related to root systems of finite reflection groups.

When does a Gamma product (probably) exist?

Given a finite sum $f(r,n)$, how can we determine if a Gamma product for the sum is likely to exist?

Observe that, if $f(r,n)$ has a Gamma product, then all the prime factors of $f(r,n)$ are "small". More precisely, they are $O(n)$ as $n \to \infty$ with r fixed.

Example – Gamma product exists

If $f(r,n) := S_{2,1,0}(r,n)$, we find experimentally that the prime factors of $f(r,n)$ are bounded by $2n$. Here is the output of a small Magma program checking prime factors for $r = 3, n \leq 20$. The program finds the largest prime factor p of $S_{2,1,0}(3,n)$, and prints p and p/n.

```
alpha 2 beta 1 delta 0 r 3
n 2 max p 3 p/n 1.500
n 3 max p 5 p/n 1.667
n 4 max p 7 p/n 1.750
n 5 max p 7 p/n 1.400
n 6 max p 11 p/n 1.833
n 7 max p 13 p/n 1.857
n 8 max p 13 p/n 1.625
n 9 max p 17 p/n 1.889
n 10 max p 19 p/n 1.900
n 11 max p 19 p/n 1.727
n 12 max p 23 p/n 1.917
n 13 max p 23 p/n 1.769
n 14 max p 23 p/n 1.643
n 15 max p 29 p/n 1.933
n 16 max p 31 p/n 1.938
n 17 max p 31 p/n 1.823
n 18 max p 31 p/n 1.722
n 19 max p 37 p/n 1.947
n 20 max p 37 p/n 1.850
Max p/n 1.947
```

It appears that p is a (weakly) monotonic increasing function of n, and it is reasonable to conjecture that $p \leq 2n$.

Example continued

Other positive values of r give similar results. Thus, it is plausible that a Gamma product exists. In fact, it does:

$$f(r,n) = \prod_{j=0}^{r-1} \frac{(2n)!}{(n-j)!} \frac{\Gamma(\frac{j+3}{2})}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-j+\frac{3}{2})}{\Gamma(n-\frac{j}{2}+\frac{3}{2})} \frac{\Gamma(\frac{j+1}{2})}{\Gamma(n-\frac{j-1}{2})}$$

for $n \geq r-1$ (otherwise the sum vanishes).

From the product it is easy to see that $p \leq 2n$, confirming what we found experimentally.

Example – Gamma product does not exist

We make a small change and set $\delta = 3$. Here is the output:

```
alpha 2 beta 1 delta 3 r 3
n 3 max p 5 p/n 1.667
n 4 max p 7 p/n 1.750
n 5 max p 11 p/n 2.200
n 6 max p 19 p/n 3.167
n 7 max p 29 p/n 4.143
n 8 max p 41 p/n 5.125
n 9 max p 17 p/n 1.889
n 10 max p 71 p/n 7.100
n 11 max p 89 p/n 8.091
n 12 max p 109 p/n 9.083
n 13 max p 131 p/n 10.08
n 14 max p 31 p/n 2.214
n 15 max p 181 p/n 12.07
n 16 max p 31 p/n 1.938
n 17 max p 239 p/n 14.06
n 18 max p 271 p/n 15.06
n 19 max p 61 p/n 3.210
n 20 max p 37 p/n 1.850
Max p/n 15.06
```

Now p is no longer a monotonic increasing function of n, and p/n can be large. Thus, a Gamma product is unlikely to exist.

Ten cases

By checking prime factors, we determined that Gamma products for $S_{\alpha,\beta,\delta}(r,n)$ are likely to exist in the ten cases

$$\alpha,\beta \in \{1,2\}, \;\; 0 \le \delta \le 2\alpha+\beta-3,$$

and unlikely to exist in any other cases where α,β are positive integers.

α	β	δ
1	1	0
1	2	0, 1
2	1	0, 1, 2
2	2	0, 1, 2, 3

The ten cases naturally fall into three families indicated by the colour-coding. Why just ten cases? The methods used to prove the ten cases may provide some clues.

The three families

For $\alpha = 2$ we have seven cases given by

$$S_{2,2\gamma,\delta}(r,n) = \prod_{j=0}^{r-1} \frac{(2n)!}{\Gamma(n-j+\chi)} \frac{\Gamma(1+j\gamma+\gamma)}{\Gamma(1+\gamma)}$$

$$\times \prod_{j=0}^{r-1} \frac{\Gamma(n-j-\gamma+\chi+1)}{\Gamma(n-j\gamma-\gamma+\chi+1)} \frac{\Gamma(j\gamma+\frac{\delta+1}{2})}{\Gamma(n-j\gamma-\frac{\delta-3}{2}-\chi)},$$

where $\chi := \chi[\delta = 0] = \max(0, 1-\delta)$.

For example, we already gave the case $S_{2,1,0}(r,n)$ which corresponds to $\gamma = \frac{1}{2}$, $\delta = 0$.

The three families continued

For $\alpha = 1, \delta = 0$ we have two cases given by

$$S_{1,2\gamma,0}(r, n/2) = \prod_{j=1}^{r} \frac{2^{n-2\gamma(j-1)}\, n!}{(n-j+1)!} \frac{(n-j+\gamma+1)!}{(n-(j-2)\gamma)!} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} .$$

The one remaining case is

$$S_{1,2,1}(r, n) = r! \prod_{j=1}^{\lceil r/2 \rceil} \frac{(2n)!\,(j-1)!^2}{(n-j)!\,(n-j+1)!} \prod_{j=1}^{\lfloor r/2 \rfloor} \frac{(2n)!\,(j-1)!\,j!}{(n-j)!^2} .$$

This is the only case where the product involves “floor” and/or “ceiling” functions.

How do we find these formulas?

If all prime factors appear to be $O(n)$ and we suspect the existence of a Gamma product, how can we find it explicitly?

Define the "logarithmic difference" operator

$$Q : f(r,n) \mapsto \frac{f(r,n)}{f(r-1,n)} \cdot$$

Q is analogous to the backward difference operator but uses division instead of subtraction.

In our applications $f(r,n)$ is defined for $r \geq 0$, with $f(0,n) = 1$. Thus $Qf(r,n)$ is defined for $r \geq 1$, and $Qf(1,n) = f(1,n)$.

An easy example

Let's start with the "easy" example

$$f(r,n) := S_{2,2,3}(r,n) = \prod_{j=1}^{r} \frac{(2n)!\, j!^2}{(n-j)!^2}$$

of the abstract. Applying Q once we obtain (for $n \geq r \geq 1$)

$$g(r,n) := Qf(r,n) = \frac{(2n)!\, r!^2}{(n-r)!^2}\,.$$

Applying Q again gives (for $n \geq r \geq 2$)

$$h(r,n) := Q\,g(r,n) = Q^2 f(r,n) = r^2(n-r+1)^2.$$

If we fix $r \geq 2$ and compute some values of $h(r,n)$ it is easy to fit a quadratic in n. By inspection, this quadratic is $c(r)(n-r+1)^2$. Then it's easy to vary r and guess that $c(r) = r^2$.

Easy example continued

We have guessed that

$$h(r,n) = Q\,g(r,n) = r^2(n-r+1)^2$$

for $n \ge r \ge 2$. From the definition of the operator Q, we have

$$g(r,n) = g(1,n)\prod_{k=2}^{r} h(k,n),$$

and similarly

$$f(r,n) = f(0,n)\prod_{j=1}^{r} g(j,n).$$

Recall that $f(0,n) = 1$ and $g(1,n) = f(1,n)$. Thus, all we need is $f(1,n)$.

Easy example continued

We need

$$f(1,n) = S_{2,2,3}(1,n) = \sum_{k=-n}^{n} |k|^3 \binom{2n}{n+k}.$$

This is a simple binomial sum that can be done by your favorite method. Since we are only trying to guess $f(r,n)$, there is no point in being rigorous about $f(1,n)$. Using the same "logarithmic difference" idea applied to the function of one variable $f(1,n)$, we find that

$$f(1,n) = n^2 \binom{2n}{n}.$$

Thus, working backwards,

$$g(r,n) = n^2 \binom{2n}{n} \prod_{k=2}^{r} k^2 (n-k+1)^2.$$

Easy example continued

We have

$$g(r,n) = n^2 \binom{2n}{n} \prod_{k=2}^{r} k^2 (n-k+1)^2,$$

and this easily simplifies to

$$g(r,n) = \frac{(2n)!\, r!^2}{(n-r)!^2}\,.$$

Thus, working backwards again,

$$f(r,n) = \prod_{j=1}^{r} g(j,n) = \prod_{j=1}^{r} \frac{(2n)!\, j!^2}{(n-j)!^2}$$

which is the desired Gamma product.

A typical example

Unfortunately, the method that we used in the "easy" example does not usually work. To see why, consider the example

$$f(r,n) := S_{2,1,0}(r,n).$$

Here

$$Qf(r,n) = \frac{(2n)!}{(n-r+1)!} \frac{\Gamma(\frac{r}{2}+1)}{\Gamma(\frac{3}{2})} \frac{\Gamma(n-r+\frac{5}{2})}{\Gamma(n-\frac{r}{2}+2)} \frac{\Gamma(\frac{r}{2})}{\Gamma(n-\frac{r}{2}+1)}.$$

Now, $Q^2 f(r,n)$ is not a simple rational function of r and n, because of the Γ factors involving the argument $r/2$.

Typical example continued

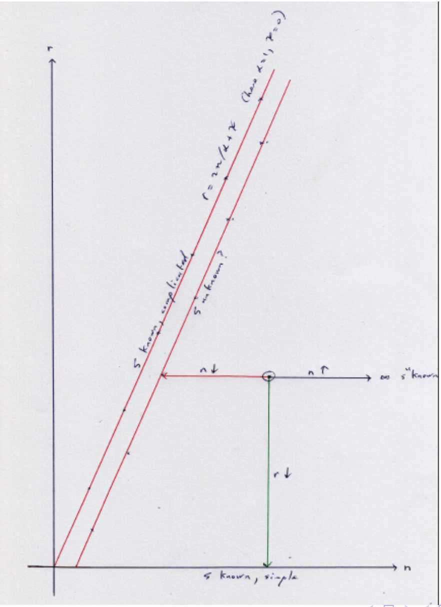
One possible solution is to take a second (logarithmic) difference with respect to n rather than r. This gives a nice rational function

$$\frac{Qf(r,n)}{Qf(r,n-1)} = \frac{4n(n-\frac{1}{2})(n-r+\frac{3}{2})}{(n-r+1)(n-\frac{r}{2})(n-\frac{r}{2}+1)}\,.$$

However, it is no longer easy to work backwards and find the desired Gamma product. The difficulty is that we need to know the $f(r,n)$ on the diagonal $r = n+1$ (more generally, $r = 2n/\alpha + \chi[\delta = 0]$) or asymptotically as $n \to \infty$. See the picture on the next slide.

Both approaches are possible, but lead to complicated results which are difficult to simplify to give the desired Gamma product.

Recursions involving n and r



Typical example continued

Another idea is to consider the “cross ratio”

$$\frac{f(r,n)}{f(r-1,n)}\frac{f(r-3,n)}{f(r-2,n)} = \frac{(\frac{r}{2}-1)_2(n-\frac{r}{2}+1)_2(n-r+2)_2}{(n-r+\frac{3}{2})_2},$$

where the subscripts denote ascending factorials.

This works (in fact it was our original method) but it is difficult to go back from the cross ratio to the desired Gamma product. A straightforward approach leads to a product with several occurrences of $\lceil r/2 \rceil$ and $\lfloor r/2 \rfloor$, but this dependence on the parity of r is only apparent, not real (the final product is an analytic function of r and can be written without floor or ceiling functions).

More typical example continued

Conceptually simpler is to define $f(s,n) := S_{2,1,0}(2s,n)$ and take all differences with respect to s. This works, but only gives a product for $S_{2,1,0}(r,n)$ when r is even. We need to repeat the process with $f(s,n) := S_{2,1,0}(2s+1,n)$ to get a product for $S_{2,1,0}(r,n)$ when r is odd.

The products obtained are over an index ranging from 1 to $s = \lfloor r/2 \rfloor$. By judicious simplification, using the well-known "product formula"

$$\Gamma(x)\Gamma(x+\tfrac{1}{2}) = 2^{1-2x}\Gamma(2x)\Gamma(\tfrac{1}{2})$$

where necessary, the products can be converted to products over an index ranging from 1 to r. With luck they turn out to be equivalent, so we only need one formula, independent of the parity of r.

This happens in 9 of the 10 cases – the exception is $S_{1,2,1}(r,n)$.

The exception

In the exceptional case we get two Gamma products, one for odd r and one for even r. Using the “floor” and “ceiling” functions, they may be written in a unified way as:

$$S_{1,2,1}(r,n) = r! \prod_{j=1}^{\lceil r/2 \rceil} \frac{(2n)!\,(j-1)!^2}{(n-j)!\,(n-j+1)!} \prod_{j=1}^{\lfloor r/2 \rfloor} \frac{(2n)!\,(j-1)!\,j!}{(n-j)!^2}\,.$$

Some generalisations and analogues

We mention some generalisations and analogues. They are of independent interest, and some of the generalisations are necessary for the known proofs of the primary identities.

- A-generalisations.
- K-generalisations.
- q-analogues.
- Sums over $\mathbb{Z} + \frac{1}{2}$.
- Alternating sums.

These classes are not mutually exclusive. We can have Kq-generalisations, alternating sums over $\mathbb{Z} + \frac{1}{2}$, etc.

The next few slides give examples of these generalisations/analogues.

A-generalisations

If $\alpha = \beta = 2$, $0 \le \delta \le 3$, we have identities with an additional real (or complex) variable x:

$$\sum_{-n \le k_1, \ldots, k_r \le n} \Delta(k^2)^2 \prod_{j=1}^{r} |k_j|^\delta \binom{2n}{n+k_j} (x)_{n-k_j} (x)_{n+k_j}$$
$$= \prod_{j=1}^{r} \frac{j!\,(j+\delta')!\,(2n)!\,(x)_{j-1}\,(x)_{n+j-\chi}\,(x+j+r+\delta')_{n-r+\chi}}{(n-j+\chi)!\,(n-j-\chi-\delta')!}\,,$$

valid for $n \ge r - \chi$ and all $x \in \mathbb{R}$. Here $\chi := \chi[\delta = 0]$ and $\delta' := (\delta - 3)/2$. The factorials $z!$ are be to interpreted as $\Gamma(z+1)$ if $z \notin \mathbb{Z}$ (this occurs if δ is even, so $\delta' \notin \mathbb{Z}$).

Similar identities exist if $\alpha = 1, \beta = 2$, $0 \le \delta \le 1$.

Multiplying both sides by x^{-2n} and letting $x \to \infty$, we recover the corresponding primary identities.

K-generalisations

If $\beta = 2$ we have identities with an extra integer variable m symmetric with n. Consider the case $(\alpha, \beta, \delta) = (1, 2, 0)$.
For $m, n, r \in \mathbb{Z}$, $m, n \geq r/2 > 0$, we have

$$\sum_{k \in \mathbb{Z}^r} \Delta(k)^2 \prod_{j=1}^{r} \binom{2m}{m+k_j} \binom{2n}{n+k_j} = \prod_{j=1}^{r} \frac{j!\,(2m)!\,(2n)!\,(2m+2n-2r+j+1)!}{(2m+j-r)!\,(2n+j-r)!\,(m+n+j-r)!^2}.$$

Dividing each side by $\binom{2m}{m}^r$, taking the limit as $m \to \infty$, and simplifying, we obtain the primary identity

$$S_{1,2,0}(r,n) := \sum_{k \in \mathbb{Z}^r} \Delta(k)^2 \prod_{j=1}^{r} \binom{2n}{n+k_j} = \prod_{j=1}^{r} \frac{2^{2n-2j+2}\, j!\,(2n)!}{(2n+j-r)!}.$$

A-generalisations and K-generalisations

The A-generalisations and K-generalisations appear to be different, since the former involves an additional real parameter and the latter involves an additional (symmetric) integer parameter. However, they appear to exist for the same set of parameters (α, β, δ). This suggests a hidden connection.

In fact, if we make the change of variables $m \mapsto -(n+x)$ in any one of the K-generalisations, and rewrite the resulting identity using ascending factorials, we obtain the corresponding A-generalisation, except that $x = -(m+n)$ is now a (non-positive) integer.

Consider fixed r, n and $m \geq n$. When suitable scaled, each side of the identity is a polynomial function of degree $O(nr)$ in $x = -(m+n)$. Since the identity holds for infinitely many x (e.g. all integer $x \leq -2n$) it is true for all real (or complex) x.

Thus, the A-generalisations and K-generalisations are essentially equivalent.

q-functions

Assume that $0 < q < 1$ and m, n are integers such that $0 \le m \le n$. Then the q-shifted factorial, q-binomial coefficient, q-gamma function and q-factorial are defined by:

$$(a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \quad (a;q)_\infty := \prod_{k=1}^{\infty}(1-aq^{k-1}),$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q^{n-m+1};q)_m}{(q;q)_m},$$

$$\Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty},$$

$$[n]_q! := \Gamma_q(n+1) = \prod_{k=1}^{n}\frac{1-q^k}{1-q}.$$

Exercise. Verify that the last three definitions give sensible results as $q \to 1$, e.g. $\lim_{q\to 1}\Gamma_q(x) = \Gamma(x)$.

Quick introduction to Schur functions

For $x = (x_1, \ldots, x_n)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ a partition of length at most n, the Schur function $s_\lambda(x)$ may be defined by

$$s_\lambda(x) := \frac{\det_{1\le i,j\le n}\left(x_i^{\lambda_j+n-j}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j}\right)}.$$

The Schur functions form a basis for the ring Λ_n of symmetric functions in n variables $x_1, \ldots, x_n$. They occur in the representation theory of the symmetric group S_n and of the general linear group $\mathrm{GL}_n(\mathbb{C})$.

Specialisation of Schur functions

The *principal specialisation* of the Schur function $s_\lambda(x)$ arises from substituting $x_j = q^{j-1}$, $1 \le j \le n$.

For example, if λ is a partition of length at most n and largest part at most r, which we write as $\lambda \subseteq (r^n)$, then

$$s_\lambda(1, q, \ldots, q^{n-1}) =$$
$$q^{\sum_{i\ge1}(i-1)\lambda_i} \prod_{i=1}^{r} \begin{bmatrix} n+r-1 \\ \lambda'_i + r - i \end{bmatrix}_q \begin{bmatrix} n+r-1 \\ r-i \end{bmatrix}_q^{-1} \prod_{1\le i<j\le r} \frac{1-q^{\lambda'_i-\lambda'_j+j-i}}{1-q^{j-i}}.$$

Here λ' is the conjugate partition of λ (obtained by reflecting the Young diagram of λ in the main diagonal).

Another useful specialisation (the *odd specialisation*) is $x_j = q^{j-1/2}$, i.e. consider $s_\lambda(q^{1/2}, q^{3/2}, \ldots, q^{n-1/2})$.

q-analogues of the primary identities

Seven of the ten primary identities have q-analogues, in the sense that there are identities involving q which give the corresponding primary identity in the limit as $q \to 1$.

We generally have to divide both sides by a suitable power of $(1-q)$ before taking the limit, in order to ensure that the limit is finite.

The three exceptions are cases $(1,2,1), (2,1,0), (2,1,2)$, where we do not know any q-analogue (they may exist – we can not easily rule them out).

Example

Let $0 < q < 1$, r a positive integer and n an integer or half-integer such that $n \geq (r-1)/2$. Then we have a q-analogue of $S_{1,1,0}(r,n)$:

$$\sum_{k_1,\ldots,k_r=-n}^{n} \prod_{1\le i<j\le r} \left|1-q^{k_i-k_j}\right| \prod_{j=1}^{r} q^{(k_j+n-r+j)^2/2} \begin{bmatrix} 2n \\ n+k_j \end{bmatrix}_q$$

$$= (1-q)^{\binom{r}{2}} \frac{r!}{[r]_{q^{1/2}}!} \prod_{j=1}^{r} (-q^{1/2};q^{1/2})_j \, (-q^{j/2+1};q)_{2n-r}$$

$$\times \prod_{j=1}^{r} \frac{\Gamma_q(1+\frac{1}{2}j)}{\Gamma_q(\frac{3}{2})} \frac{\Gamma_q(2n+1)\,\Gamma_q(2n-j+\frac{5}{2})}{\Gamma_q(2n-j+2)\,\Gamma_q(2n-\frac{1}{2}j+2)}.$$

Dividing both sides by $(1-q)^{\binom{r}{2}}$ and taking the limit as $q \to 1$ yields the primary identity

$$S_{1,1,0}(r,n) = 2^{2rn-\binom{r}{2}} \prod_{j=1}^{r} \frac{\Gamma(1+\frac{1}{2}j)}{\Gamma(\frac{3}{2})} \frac{\Gamma(2n+1)\,\Gamma(2n-j+\frac{5}{2})}{\Gamma(2n-j+2)\,\Gamma(2n-\frac{1}{2}j+2)}.$$

Sketch of proof (1)

How to prove such a q-analogue? Denote the sum on the left by $f_{r,n}$. It can be verified that the summand

$$\prod_{1\le i<j\le r} \left|1 - q^{k_i-k_j}\right| \prod_{j=1}^{r} q^{(k_j+n-r+j)^2/2} \begin{bmatrix} 2n \\ n+k_j \end{bmatrix}_q$$

of $f_{r,n}$ is a symmetric function which vanishes unless the k_i are pairwise distinct. Anti-symmetrisation yields

$$f_{r,n} = r! \sum_{n\ge k_1>\cdots>k_r\ge -n} \prod_{1\le i<j\le r} \left(1 - q^{k_i-k_j}\right) \prod_{j=1}^{r} q^{(k_j+n-r+j)^2/2} \begin{bmatrix} 2n \\ n+k_j \end{bmatrix}_q .$$

Sketch of proof (2)

We replace $n \mapsto (n+r-1)/2$ and write $f_{r,(n+r-1)/2}$ as a sum over partitions $\lambda \subseteq (r^n)$. Using some well-known results on Schur functions, this gives

$$f_{(n+r-1)/2,r} = r! \prod_{1\le i<j\le r} \left(1-q^{j-i}\right) \prod_{j=1}^{r} \begin{bmatrix} n+r-1 \\ r-j \end{bmatrix}_q \times \sum_{\lambda\subseteq(r^n)} s_\lambda\left(q^{1/2}, q^{3/2}, \ldots, q^{n-1/2}\right).$$

MacMahon's formula for the generating function of symmetric plane partitions that fit in a box of size $n \times n \times r$, proved by Andrews and Macdonald, gives

$$\sum_{\lambda\subseteq(r^n)} s_\lambda\left(q^{1/2}, q^{3/2}, \ldots, q^{n-1/2}\right) = \prod_{j=1}^{n} \frac{1-q^{j+(r-1)/2}}{1-q^{j-1/2}} \prod_{1\le i<j\le n} \frac{1-q^{r+i+j-1}}{1-q^{i+j-1}}.$$

Sketch of proof (3)

Simplifying the q-products and replacing $n \mapsto 2n - r + 1$ gives

$$f_{r,n} = r!\,\frac{(q^{(r+1)/2};q)_{2n-r+1}}{(q^{1/2};q)_{2n-r+1}}\prod_{j=1}^{r}\frac{(q;q)_{2n}\,(q;q)_{j-1}\,(q^j;q^2)_{2n-r+1}}{(q;q)_{2n-j+1}^2}\,.$$

The result now follows, using the definition of the q-gamma function.

Kq-generalisations

Some of the K-generalisations have q-analogues, which we call Kq-generalisations. For example, if $\alpha = \beta = 2$, $1 \le \delta \le 3$, then

$$\sum_{k_1,\ldots,k_r} \left(q^{2\binom{r+1}{3}+(\delta'+1)\binom{r+1}{2}} \prod_{1\le i<j\le r} [k_j - k_i]_q^2 \, [k_i + k_j]_q^2 \right.$$
$$\left. \times \prod_{j=1}^{r} q^{k_j^2 - (2j+\delta')k_j} \left(\frac{1+q^{k_j}}{2}\right) \left|[k_j]_q^{\delta}\right| \begin{bmatrix} 2n \\ n+k_j \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_j \end{bmatrix}_q \right)$$
$$= \prod_{j=1}^{r} \frac{[2n]_q!}{[n-j]_q!\,[n-j-\delta']_q!} \frac{[2m]_q!}{[m-j]_q!\,[m-j-\delta']_q!}$$
$$\times \prod_{j=1}^{r} j\,[j-1]_q!\,[j+\delta']_q! \frac{[m+n-j-r-\delta']_q!}{[m+n-j+1]_q!},$$

where $\delta' := (\delta-3)/2$, and we interpret $[x]_q!$ as $\Gamma_q(x+1)$ if $x \notin \mathbb{Z}$.

Sums over $Z + \frac{1}{2}$

For most of the primary identities there is a corresponding identity where the parameter n is a half-integer, and we sum over half-integers k_j rather than integers. Thus, the binomial coefficients $\binom{2n}{n+k_j}$ are well-defined.

In the half-integer cases $(1, 2, 1)$ and $(2, 2, 3)$ there is "almost" a Gamma product (actually a sum of $O(r)$ Gamma products). For example, in case $(1, 2, 1)$ we have: for all positive half-integers n and positive integers $r \le 2n + 1$,

$$\sum_{k_1,\ldots,k_r \in \mathbb{Z}+\frac{1}{2}} \Delta(k)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i}$$

$$= F \times r! \prod_{i=1}^{\lceil r/2 \rceil} \frac{(2n)!\,(i-1)!^2}{(n-i+\frac{1}{2})!^2} \prod_{i=1}^{\lfloor r/2 \rfloor} \frac{(2n)!\,i!^2}{(n-i+\frac{1}{2})!^2},$$

$$F = \sum_{s=0}^{\lfloor r/2 \rfloor} \frac{(n-s+1)_s}{(-16)^{\lfloor r/2 \rfloor - s}\, s!} \binom{2\lfloor r/2 \rfloor - 2s}{\lfloor r/2 \rfloor - s}^2.$$

Alternating sums

We can modify the definition of $S_{\alpha,\beta,\delta}(r,n)$ by inserting a sign that gives a "chess-board" pattern in $\mathbb{Z}^r$:

$$\widehat{S}_{\alpha,\beta,\delta}(r,n) := \sum_{k_1,\ldots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} |\Delta(k^\alpha)|^\beta \prod_{j=1}^{r} (-1)^{k_j} |k_j|^\delta \binom{2n}{n+k_j}.$$

The sums $\widehat{S}_{\alpha,\beta,\delta}(r,n)$ are expressible as Gamma products in the usual ten cases, even though they are not discrete analogues of Macdonald-Mehta integrals. This has been proved in five cases, and is conjectured in the other five cases.

Example – alternating case $(1, 2, 1)$

$$\widehat{S}_{1,2,1}(r,n) \stackrel{?}{=} (-1)^{\binom{r+1}{2}} \prod_{j=1}^{r} \frac{2^{4n-2r}\,(2n)!\,j!^2\,\Gamma(n-j+\frac{3}{2})\,\Gamma(n-j+\frac{1}{2})}{(2n-j)!\,(2n-j+1)!\,\Gamma(\frac{1}{2})^2}\,.$$

The $\Gamma(\cdots)$ factors in the numerator can be negative if $n < r$.
More precisely, $\operatorname{sgn}\,\Gamma(k+\frac{1}{2}) = (-1)^{\min(k,0)}$ for $k \in \mathbb{Z}$.

This leads to an interesting pattern of signs in the region $r \le 2n$:

$$\operatorname{sgn}\,\widehat{S}_{1,2,1}(r,n) \stackrel{?}{=} \begin{cases} (-1)^{r(r-1)/2+\min(r,n)} & \text{if } r \le 2n; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(-1)^{r(r-1)/2}$ is periodic in r with period 4.

Asymptotics

The degree of cancellation in the alternating sum $\widehat{S}_{1,2,1}(r,n)$ can be quantified by comparison with the corresponding sum $S_{1,2,1}(r,n)$ which has no negative terms. We find that

$$\frac{\widehat{S}_{1,2,1}(r,n)}{S_{1,2,1}(r,n)} \stackrel{?}{=} O_r(n^{-r(r+1)/2})$$

as $n \to \infty$ with r fixed.

Proofs

We have proved all of the primary identities and various generalisations. Our proofs fall into several categories.

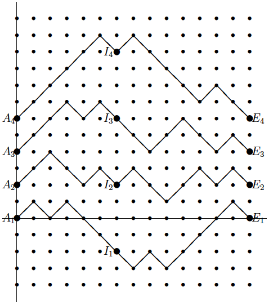
- Proofs via enumeration of non-intersecting lattice paths.
- Proofs via determinantal formulas that do not seem to have a natural interpretation in terms of lattice paths, but follow from elliptic hypergeometric transformation formulas.
- Proofs via Okada-type formulas for the multiplication of Schur functions indexed by partitions of rectangular shape.
- Other proofs involving Schur functions.

I will outline how we use non-intersecting lattice paths. The other categories of proof involve a moderate amount of technical background and there is not enough time to cover it today (though I already sketched one proof involving Schur functions).

Proofs via non-intersecting lattice paths

We thank Helmut Prodinger for suggesting the use of the Lindström-Gessel-Viennot theorem. Our method is similar to Michael Schlosser's non-intersecting path derivation of elliptic hypergeometric series.

We start with non-intersecting lattice paths at points $A_i = (0, 2(i-1))$, $i = 1, 2, \ldots, r$, which end at points $E_i = (m+n, 2(i-1))$, $i = 1, 2, \ldots, r$. The paths must cross the line $x = m$ at some intermediary points $I_i = (m, k_i)$, $i = 1, 2, \ldots, r$.



Non-intersecting lattice paths continued

By a result of Lindström-Gessel-Viennot, the number of non-intersecting lattice paths between the A_i's and the I_i's is given by the determinant

$$\det_{1\le i,j\le r}\left(\binom{m}{j-1+\frac{1}{2}(m-k_i)}\right) = \frac{\prod_{1\le i<j\le r}(\frac{1}{2}(k_j-k_i))}{\prod_{i=1}^{r}(\frac{1}{2}(m-k_i)+r-1)!}\,\frac{\prod_{i=1}^{r}(m+i-1)!}{\prod_{i=1}^{r}(\frac{1}{2}(m+k_i))!}\,.$$

The number of non-intersecting lattice paths between the I_i's and the E_i's is given by the same formula with m replaced by n, and the number of non-intersecting lattice paths between the A_i's and the E_i's is also given by the same formula if we replace m by $m+n$ and take $k_i = 2(i-1)$, $i = 1, 2, \ldots, r$.

Clearly, the last number equals the product of the two former numbers summed over all possible choices for the k_i's (they must have the same parity as m).

Non-intersecting lattice paths continued

After shifting the k_i's appropriately, we deduce the following K-generalisation of the primary identity for case $(1,2,0)$.

Theorem If $m, n, r \in \mathbb{Z}$ and $m, n \geq r/2 > 0$, then

$$\sum_{k \in \mathbb{Z}^r} \Delta(k)^2 \prod_{j=1}^{r} \binom{2m}{m+k_j} \binom{2n}{n+k_j} = \prod_{j=1}^{r} \frac{j!\,(2m)!\,(2n)!\,(2m+2n-2r+j+1)!}{(2m+j-r)!\,(2n+j-r)!\,(m+n+j-r)!^2}\,.$$

If we divide each side of this identity by $\binom{2m}{m}^r$ and take the limit as $m \to \infty$, we obtain the primary identity

$$S_{1,2,0}(r,n) = \prod_{j=1}^{r} \frac{2^{2n-2j+2}\,(2n)!\,j!}{(2n+1-j)!}\,.$$

Comments on the use of lattice paths

The proof sketch above can be modified, using suitable determinant identities (some of which have interpretations in terms of lattice paths) to prove some other of our primary identities that involve squares of the Vandermonde, i.e. $\beta = 2$.

In some cases we can also prove these identities by a different method. For example, the case $(1, 2, 0)$ can be proved rather easily using a known hypergeometric sum $(x, y \in \mathbb{R})$

$$\sum_{0 \le k_1, \ldots, k_r \le n} \Delta(k)^2 \prod_{i=1}^{r} \binom{n}{k_i} (x)_{k_i} (y)_{n-k_i}$$
$$= \prod_{j=1}^{r} j!(n-j+2)_{j-1}(x)_{j-1}(y)_{j-1}(x+y+j+r-2)_{n-r+1} .$$

Taking $x = y$ here implies the K-generalisation on the previous slide.

Some references

P. J. Forrester and S. O. Warnaar, The importance of the Selberg integral, *Bull. AMS (N.S.)* **45**, 4 (2008), 489–534.

R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, Mass., 1989.

M. Ishikawa and M. Wakayama, Minor summation formula for pfaffians, *Linear and Multilinear Algebra* **39** (1995), 285–305.

I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Univ. Press, 1995.

S. Okada, On the generating functions for certain classes of plane partitions, *J. Combin. Theory Ser. A* **51** (1989), 1–23.

S. Okada, Applications of minor summation formulas to rectangular-shaped representations of classical groups, *J. Algebra* **205** (1998), 337–368.

M. Schlosser, Elliptic enumeration of nonintersecting lattice paths, *J. Combin. Theory Ser. A* **114** (2007), 505–521.