Bounds on minors of binary matrices

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joint work with
Richard P. Brent

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Eg.

$$H_{12} = \begin{pmatrix}
+ & + & + & + & - & - & - & + & + & + & - & - \\
+ & + & + & - & + & - & + & - & + & - & + & - \\
+ & + & + & - & - & + & + & + & - & - & - & + \\
+ & - & - & + & - & - & + & - & - & - & - & - \\
- & + & - & - & + & - & - & + & - & - & - & - \\
- & - & + & - & - & + & - & - & + & - & - & - \\
- & + & + & + & - & - & - & - & - & - & + & + \\
+ & - & + & - & + & - & - & - & - & + & - & + \\
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+ & - & - & - & - & - & - & + & + & - & + & + \\
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What do we know of the minors of Hadamard matrices?

Outline

- de Launey and Levin, on minors of Hadamard matrices
- Main result: our generalisation
- our (simpler) proof
- Corollary 1
- Corollary 2

de Launey and Levin

Theorem. [de Launey and Levin (2009), Proposition 2]

Let A be a Hadamard matrix of order n. Then for M chosen uniformly at random from square order m submatrices of A,

$$E(\det(M)^2) = \frac{n^m}{\binom{n}{m}}$$

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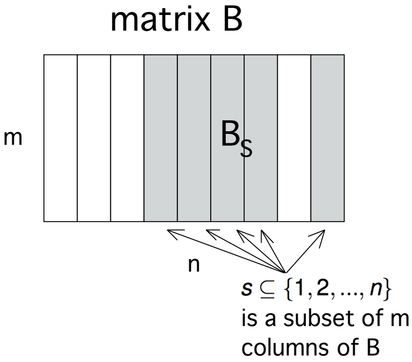
Answers: Yes to both.

We'll find useful: the Cauchy-Binet formula (1812)



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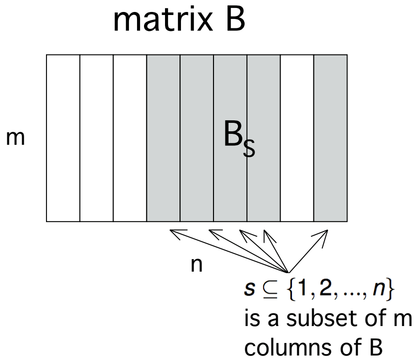

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Proof and history: Muir, 1906; or Brualdi & Schneider, 1983.

Main result

Theorem 1. Let A be a square $\{\pm 1\}$ matrix of order $n \geq m \geq 1$. Then for M be chosen uniformly at random from square order m submatrices of A,

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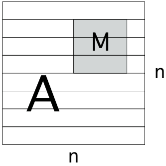

$$E(\det(M)^2) = \frac{\sum_{|\text{rows}|=m} \sum_{|\text{columns}|=m} \det(M)^2}{\binom{n}{m}^2}$$

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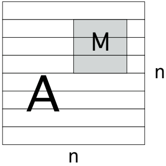
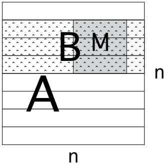


$$= \frac{\sum_{|\text{rows}|=m} \det(BB^T)}{\binom{n}{m}^2} \text{ by Cauchy-Binet}$$



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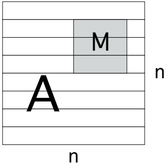

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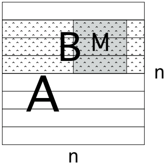

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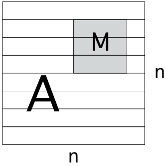

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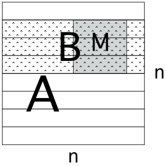

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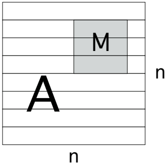

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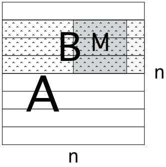

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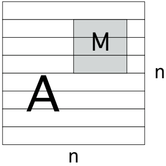

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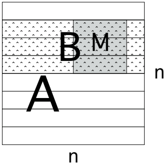

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Q.E.D.

Turán and Corollary 1

Theorem (Turán)

Let M be chosen uniformly at random from the set of square $\{\pm 1\}$ matrices of order m. Then

$$E(\det(M)^2) = m!$$

Corollary (1, to our Main Theorem, or de Launey & Levin)

Let H be a Hadamard matrix of order $n \geq m > 1$. Let M be chosen uniformly at random from the set of square order m submatrices of H. Then

$$E(\det(M)^2) > m!$$

Proof of Corollary 1

Let H be Hadamard, M be a square submatrix of order $m > 1$ chosen uniformly at random. Then by Theorem 1,

$$E(\det(M)^2) = \frac{n^m}{\binom{n}{m}}$$

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$$\begin{aligned} E(\det(M)^2) &= \frac{n^m}{\binom{n}{m}} \\ &= \frac{n^m m!}{n(n-1)(n-2)\ldots(n-(m-1))} \end{aligned}$$

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Example:

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+ & - & - & + & - & - & + & - & - & - & - & - \\
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+ & - & + & - & + & - & - & - & - & + & - & + \\
+ & + & - & - & - & + & - & - & - & + & + & - \\
+ & - & - & - & - & - & - & + & + & - & + & + \\
- & + & - & - & - & - & + & - & + & + & - & + \\
- & - & + & - & - & - & + & + & - & + & + & -
\end{pmatrix}$$

Let M be a submatrix of order 2 chosen uniformly at random in H_{12}. Then

$$E(\det(M)^2) = 2.181818... > 2$$

Corollary 2

Definition

$Z(m, A) = |$zero minors of order m of $\{\pm 1\}$ matrix $A|$

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Corollary (2, to our Main Theorem)

Let A be a square $\{\pm 1\}$ *matrix of order* $n \geq m > 1$*. Then*

$$Z(m, A) \geq \binom{n}{m}^2 - 4\binom{n}{m}\left(\frac{n}{4}\right)^m$$

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Proof - see the paper

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Proof - see the paper

Example: $Z(2, H_{12}) = 1980$, from 4356 submatrices of order 2.

Explicitly for H_{12}

m	$\left\lvert \frac{\text{minor}}{2^{m-1}} \right\rvert^{\text{frequency}}$
1:	1^{144}
2:	0^{1980}, 1^{2376}
3:	0^{24640}, 1^{23760}
4:	0^{109890}, 1^{126720}, 2^{8415}
5:	0^{205920}, 1^{318384}, 2^{95040}, 3^{7920}
6:	0^{150480}, 1^{348480}, 2^{239184}, 3^{76032}, 4^{31680}, 5^{7920}
7:	0^{205920}, 3^{318384}, 6^{95040}, 9^{7920}
8:	0^{109890}, 9^{126720}, 18^{8415}
9:	0^{24640}, 27^{23760}
10:	0^{1980}, 81^{2376}
11:	243^{144}
12:	1458^{1}

$$H_{12} = \begin{pmatrix} + & + & + & + & - & - & - & + & + & + & - & - \\ + & + & + & - & + & - & + & - & + & - & + & - \\ + & + & + & - & - & + & + & + & - & - & - & + \\ + & - & - & + & - & - & + & - & - & - & - & - \\ - & + & - & - & + & - & - & + & - & - & - & - \\ - & - & + & - & - & + & - & - & + & - & - & - \\ - & + & + & + & - & - & - & - & - & - & + & + \\ + & - & + & - & + & - & - & - & - & + & - & + \\ + & + & - & - & - & + & - & - & - & + & + & - \\ + & - & - & - & - & - & - & + & + & - & + & + \\ - & + & - & - & - & - & + & - & + & + & - & + \\ - & - & + & - & - & - & + & + & - & + & + & - \end{pmatrix}$$

NB: North-south symmetry follows from Szöllősi, 2010.

References

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