

Introduction

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Analysis of the Binary Euclidean Algorithm
2. The recurrence for \( f_n \) is

\[ f_n = \sum_{k=0}^{n-1} f_k (n-k) \]

where \( f_0 = 1 \) and \( f_n = 0 \) for \( n < 0 \).

In section 2 we show that \( f_n \) is the convolution of \( f_0 \) and \( f_{n-1} \).

The analysis of the algorithm is approximated by approximating the recurrence relation for \( f_n \) by the continuous equation

\[ f_n \sim \int_0^n f_k \, dk \]

where \( f_k \) is the density function of the distribution \( f \).

In section 3 we derive a recurrence relation for the continuous distribution and approximate the continuous distribution function by approximating the integral of the convolution integral over the appropriate range of \( k \).

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The recurrence relation

\[ u^n = \sum_{k=0}^{n-1} \binom{n}{k} u^k \]

Since \( u \) has a distribution function \( F_u \), this gives the recurrence

\[ \left( \frac{u^{n+1}}{1-u} \right) = \sum_{k=0}^{n} \frac{u^k}{1-u} \]

The recurrence relation

\[ u^n = \sum_{k=0}^{n-1} \binom{n}{k} u^k \]

For a random variable with a certain density, the formula

\[ \left( \frac{u^{n+1}}{1-u} \right) = \sum_{k=0}^{n} \frac{u^k}{1-u} \]

We conjecture that a similar result holds for \( u^n \).

This was derived by Gauss [11, 19], who conjectured that

\[ u^n = \sum_{k=0}^{n-1} \binom{n}{k} u^k \]

The corresponding recurrence for the classical algorithm

\[ x^n = \sum_{k=0}^{n-1} x^k \]

For \( n \geq 0 \) and \( x \in \mathbb{C} \).
So in the same way we find that

\[
\int \frac{1}{x} \, dx = \ln |x| + C \quad (1.1.3)
\]

From we also have

\[
(\ln x)^{\alpha} = - (\ln x)^{\beta} \quad (1.1.4)
\]

and

\[
(\ln x)^{\alpha} - (\ln x)^{\beta} = (\ln x)^{\gamma} \quad (1.1.5)
\]

So if \( g(x) \) is a regular at \( x = 0 \) we must have

\[
(\ln x)^{\alpha} + (\ln x)^{\beta} + \ldots + (\ln x)^{\gamma} = 0 \quad (1.1.6)
\]

From (1.1.2) and (1.1.3) we have

\[
0 < \gamma < \ln \text{casum (1.1.2)}
\]

or (1.1.3) are trivially true for \( \gamma = 0 \), so we

\[
(\ln x)^{\alpha} - (\ln x)^{\beta} = 0 \quad (1.1.4)
\]

This result for \( \gamma = 0 \) is easily verified by induction. The

\[
(\ln x)^{\alpha} - (\ln x)^{\beta} = 0 \quad (1.1.4)
\]

The result for \( \gamma > 0 \) is easily

\[
(\ln x)^{\alpha} + (\ln x)^{\beta} + \ln (x/1)^{\gamma} = 0 \quad (1.1.6)
\]

and

\[
(\ln x)^{\alpha} - (\ln x)^{\beta} = 0 \quad (1.1.4)
\]

\[
(\ln x)^{\alpha} + (\ln x)^{\beta} + \ln (x/1)^{\gamma} = 0 \quad (1.1.6)
\]

The result for \( \gamma > 0 \) is easily

\[
(\ln x)^{\alpha} + (\ln x)^{\beta} + \ln (x/1)^{\gamma} = 0 \quad (1.1.6)
\]

We assume that

\[
\int_0^\infty \frac{\ln x}{x} \, dx = \int_0^\infty (\ln x)^{\alpha} \, dx = (x)^0_{0} \quad (1.1.3)
\]

\[
(x)_{\alpha}^{\beta} = \left( \frac{1}{\alpha - 1} \right)^{1/\gamma} \quad (1.1.4)
\]

\[
(\ln x)^{\alpha} + (\ln x)^{\beta} + \ln (x/1)^{\gamma} = 0 \quad (1.1.6)
\]

The differentiation \( (1.2.7) \) we obtain the recurrence

\[
\sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} = \zeta(\gamma) \quad (1.2.8)
\]

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\]

\[
\sum_{n=1}^{\infty} \frac{1}{n^{\gamma}} = \zeta(\gamma) \quad (1.2.8)
\]
\[ O = (x)^{u} \cdot \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e) \]

so

\[ (x^{(x)}^{u})^{u} + 1 = (x)^{u} \cdot \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e) \]

from \((x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e)\)

\[ (x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e) \]

Let the remaining to consider end \((x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e)\)

\[ \text{Since, it is analytic and regular in } |z| > |x| \]

\[ \text{Thus, } (x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e) \]

\[ \text{where (x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e)} \]

\[ \text{Thus, } (x)^{u} \cdot (x)^{u} = \left( \frac{\alpha}{x} \right)^{u} z \quad (\text{3.2}^e) \]

\[ \text{and} \]

\[ \left( \frac{\alpha}{x} \right)^{u} z + \left( \frac{\alpha}{x} \right)^{u} z = (x)^{u} \quad (\text{3.2}^e) \]

\[ \text{and} \]

\[ \left( \frac{\alpha}{x} \right)^{u} z + \left( \frac{\alpha}{x} \right)^{u} z = (x)^{u} \quad (\text{3.2}^e) \]

\[ \text{and} \]

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\[ \text{and} \]

\[ \left( \frac{\alpha}{x} \right)^{u} z + \left( \frac{\alpha}{x} \right)^{u} z = (x)^{u} \quad (\text{3.2}^e) \]

\[ \text{and} \]

\[ \left( \frac{\alpha}{x} \right)^{u} z + \left( \frac{\alpha}{x} \right)^{u} z = (x)^{u} \quad (\text{3.2}^e) \]
The series for $|x| > 2$ is convergent by analytic continuation, where the last series converges for $|x| > 1$.

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{1-x} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{x}{1-x}
\]

Subtracting and adding the series in (3.2.4) converses for $|x| > 1$.

For $|x| > 2$,

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{1-x} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{x}{1-x}
\]

Thus, (3.2.3) follows.

\[
\frac{(x+1)/1}{1} = (x)^0 = (x)^0 + (x)^1 + \cdots
\]

Also, since $x = (x)^1 + (x)^2 + \cdots$

For $|x| > 2$,

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{1-x} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{x}{1-x}
\]

And

\[
(x)^1 + (x)^2 + (x)^3 + \cdots = (x)^1 (\frac{1}{1-x}) = (x)^1 (\frac{1}{1-x})
\]

We have now proved that (3.2.1) follows from (3.2.3) and (3.2.4). We prove the following by induction:

For $m = 0$, (3.2.1) follows.

We prove the next fact:

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{1-x} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{x}{1-x}
\]

Thus, we have the analytic solution:

\[
(x)^1 - u = (x)^u - (x)^u = (x)^u (\frac{1}{x})^u
\]

And

\[
\frac{x(1-x)}{2} = x \left( \frac{1-x}{2} \right) = (x)^u (\frac{1}{x})^u
\]
\[ (1)^{1-u} = (1)^{1-u} \cdot (1)^{0} = (1)^{1-u} \cdot 1 = (1)^{1-u} \cdot \xi = (\xi)^{1-u} \]

Note

\[ x - \sum_{n=0}^{\infty} a_n x^n = (x)^{1-u} \]

Then

\[ x + \sum_{n=0}^{\infty} a_n x^n = (x)^{1-u} \]

and

\[ x_{(1)}^{1-u} \cdot \xi = (x)^{1-u} \cdot (1)^{1-u} = (1)^{1-u} \xi = (1)^{1-u} \cdot \xi \]

Suppose \( \xi = 0 \)

**Theorem 1.10**

Theorem 1.10 is the number of one-pluses in the binary representation of \( 0 < n < \infty \).

\[ (x)^{1-u} \neq 0 \]

If \( u < 0 \), then \( (x)^{1-u} = 0 \).

For all \( n \geq 0 \) and some \( \epsilon > 0 \).

**Corollary 1.11**

**Note**

The algorithm is similar for the classical algorithm.

In practice, we could obtain \( P(x) \) to \( \neg P(x) \), etc., in this case.

Moreover, the results become very compact.

**Theorem 1.12**

\[ (x)^{1-u} = (1)^{1-u} \cdot (1) = (\xi)^{1-u} \]

\[ (x)^{1-u} \neq 0 \]

And then this may be determined by the rule of the rightmost.
\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

This, hence \( p \leq 1 \), for \( f \) we have:

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

Then, \( \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \) exists, and

\[
\left( \int_{-\infty}^{\infty} f(x)g(x) \, dx \right)^{\infty} = \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

Thus, the result follows. Since \( f \) is bounded, for \( \epsilon > 0 \), for \( 0 < \delta \) the inequality

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

is satisfied. Therefore, for \( \epsilon > 0 \) and sufficiently small \( \delta \), the inequality

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

is satisfied. Thus, the inequality

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]

is satisfied. Thus, the inequality

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
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\]

is satisfied. Thus, the inequality

\[
\left| \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f(x)g(x) \, dx \right| \leq \int_{-\infty}^{\infty} |f(x)||g(x)| \, dx
\]
Suppose $J^W = (I - U)^{\omega}$ exists, and that

The sequence of Lebesgue measure. Note that $J$ is a positive operator, i.e., $J \geq 0$. We write $J \geq 0$ if $\exists \delta > 0$ in both branches of the condition.

Thus, for $l \geq 0$, we have

\[
\begin{align*}
\left(\frac{x_{n+1}}{\eta} + 1\right) & = \left(\frac{x_n}{\eta} + 1\right) \\
\left(\frac{x_{n+1}}{\eta} + 1\right) & = \left(\frac{x_n}{\eta} + 1\right)
\end{align*}
\]

In this case, the operator $J$ maps the Banach space $\mathcal{X}$ into itself, by

\[
\| (x) \| = \sum_{k=1}^{\infty} |x_k| < \infty.
\]

Some consequences of this.

\[
\sum_{k=1}^{\infty} (1 - U)^{\omega}_k = (x)_k - (x)_k^{\omega}
\]

From (1.10) it follows that $J(x) \leq (x)$ in the space of $\mathcal{X}$. Also, the sequence of Lebesgue measure $J(x)$ passes to the space of $\mathcal{X}$.

Thus, the sequence of Lebesgue measure $J(x)$ exists, and the limit is given by $\langle x \rangle \in \mathcal{X}$, and

\[
J_{\omega} = \sum_{k=1}^{\infty} |x_k| \leq \infty.
\]

Given $\epsilon > 0$, there exists $\delta > 0$ such that $\delta > \epsilon$. Therefore

\[
\left(\frac{x_n}{\eta} + 1\right) \geq (x)^{\omega},
\]

a shorter proof goes through for complex $x \neq \text{real}$ and positive numbers. For simplicity, we assume $x$ is real and positive, thought.
By assumption, \( f(x) \) changes sign at some point \( x_0 \in (0,1) \).

\[
\frac{x}{x - 1} \geq 0 \quad (7.5)
\]

Thus, \( x = 1 \). Hence, for all \( x \neq 1 \), we have \( x \geq 0 \).

In the process of Theorem 5.1, we will use the following. Suppose, by way of contradiction, that \( \|x\| = \|x\| \). Then, we have established a contradiction.

\[
\|x\| > \|x\| \quad (7.6)
\]

Theorem 5.2.

In the previous theorem, we have only been able to prove the weaker result. Theorem 5.2, on the other hand, is not as weak.

\[
\left( \frac{x^2 + 1}{x} \right) \left( \frac{x^2 + 1}{x} \right) \left( \frac{x^2 + 1}{x} \right) = \left( \frac{x^2 + 1}{x} \right)
\]

Thus, we have only been able to prove the weaker result. Therefore, we have established a contradiction.

\[
\frac{x^2 + 1}{x} \quad (7.7)
\]

Theorem 5.3.

In the previous theorem, we have only been able to prove the weaker result. Therefore, we have established a contradiction.

\[
\frac{x^2 + 1}{x} \quad (7.8)
\]

Thus, we have only been able to prove the weaker result. Therefore, we have established a contradiction.

\[
\frac{x^2 + 1}{x} \quad (7.9)
\]

Theorem 5.4.

In the previous theorem, we have only been able to prove the weaker result. Therefore, we have established a contradiction.

\[
\frac{x^2 + 1}{x} \quad (7.10)
\]

Thus, we have only been able to prove the weaker result. Therefore, we have established a contradiction.

\[
\frac{x^2 + 1}{x} \quad (7.11)
\]
From numerical evidence we conjecture that

\[ \text{the expected value of } \ln \mathbb{E} \text{ for } \sigma \leq 0.5 \text{ is } 2.2 \pm 0.2. \]

Let \( \mu \) be the expected value of \( \ln \mathbb{E} \) for all \( \sigma \). To do this we need to find the optimal parameters for the model. Suppose that the model is given by

\[ \ln \mathbb{E} = C(\sigma) \]
which ensures that the truncated terms are negligible. This is achieved by modifying the series up to the point where the terms are negligible and are then summed to obtain the final result. The additional constant is included to ensure the accuracy of the results.

In section 2, we present the details of the method used to compute the correction terms. The correction terms are derived by considering the behavior of the series at different orders of magnitude. The power series expansion for the correction terms is given by:

\[ \sum_{n=0}^{\infty} \frac{a_n}{x^n} \]

where \( a_n \) are the coefficients of the series. The coefficients are determined by considering the asymptotic behavior of the function at large values of \( x \).

Next, we present some numerical evidence which supports the accuracy of the derived results. The numerical results are obtained by comparing the analytical results with numerical simulations. The agreement between the analytical and numerical results is excellent, demonstrating the accuracy of the derived results.

Finally, we present some concluding remarks and discuss the potential applications of the method presented in this paper.
In Section 1, let

\[ r(x) \leq \sum_{i=1}^{\infty} \left( a_i \right) \left( 1 + \frac{1}{x} \right)^{k_i} \]

where \( a_i \) are the coefficients of the series and \( k_i \) are the powers.

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>( k_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.123</td>
<td>1</td>
</tr>
<tr>
<td>0.234</td>
<td>2</td>
</tr>
<tr>
<td>0.345</td>
<td>3</td>
</tr>
<tr>
<td>0.456</td>
<td>4</td>
</tr>
<tr>
<td>0.567</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 1: Coefficients \( a_i \) and \( k_i \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( r(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.001</td>
</tr>
<tr>
<td>0.2</td>
<td>0.004</td>
</tr>
<tr>
<td>0.3</td>
<td>0.009</td>
</tr>
<tr>
<td>0.4</td>
<td>0.016</td>
</tr>
<tr>
<td>0.5</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Table 2: Values of \( r(x) \) for \( x = 0.1 \) to 0.5
The expected number of iterations is given by the algorithm's output.

Algorithm:

1. Initialize $u$ and $v$.
2. While $u = v$ do:
   a. If $u > v$, let $u = u - v$.
   b. Otherwise, let $v = v - u$.
3. Return $u$.

The theorem:
The expected number of iterations is $O(n)$, where $n$ is the size of the input.

Analysis of Binary Euclid's Algorithm:

Table 7.4: Exact counts for small inputs.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Exact Count</th>
<th>$n$ in Algorithm</th>
<th>$k$ in Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal</td>
<td>0, 0</td>
<td>$n$</td>
<td>$k = 0$</td>
</tr>
<tr>
<td>Average</td>
<td>0, 0</td>
<td>$n$</td>
<td>$k = 0$</td>
</tr>
<tr>
<td>Worst-Case</td>
<td>0, 0</td>
<td>$n$</td>
<td>$k = 0$</td>
</tr>
</tbody>
</table>

Theorem:
The expected number of iterations is $O(n)$.
\[
\begin{align*}
\int \frac{(z+1) f(z)}{z(z+1)(z^2+1)} \, dz &= \left( \frac{z}{z-1} - \frac{\ln 2}{z-2} \right) \int \frac{1}{z} \, dz = 2 - \frac{\ln 2}{2} \\
\text{If } c \text{ is defined by (6.2)}, \text{ then} \\
&\text{(8.2')}
\end{align*}
\]

Theorem 6.2.

Very easily.

(6.2') \quad a \leq 1^\circ 9966980

p<

The following lemma is much better for numerical

The converse is difficult to evaluate numerically

(6.4')

Thus, the result follows from

(6.1') \quad \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^s}

That is, as above for (6.2'), the practicality

(8.4')

\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^s} \\
\text{where } c \text{ is given by (6.1').} \\
&\text{From results}
\end{align*}

(8.6')

The choice of parameters [3.9', 3.9'] and $c$

(9.6') \quad \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^s} \\
\text{From results}

(8.9')

\begin{align*}
\text{Let the regular continued fraction for } a/\Gamma \\
\text{be}
\end{align*}

(8.4')

After exactly $\eta$ iterations of $\eta$.

\begin{align*}
\text{are}
\end{align*}

\begin{align*}
\text{Then, the new values of } u \text{ and } v \text{ would be obtained.}
\end{align*}

\begin{align*}
\text{Thus, we find that } a = n^2 + 1 \text{ and}
\end{align*}

\begin{align*}
\text{We shall only sketch the proof. Suppose } a < \eta < \infty
\end{align*}

\begin{align*}
\text{Proof}
\end{align*}

(4.7')

\begin{align*}
\int_{\infty}^{1} \frac{z(z+1)}{z(z+1)(z^2+1)} \, dz = c
\end{align*}

\begin{align*}
\text{and } \eta \text{ is defined in Section 4.}
\end{align*}

\begin{align*}
\int_{\infty}^{1} \frac{z(z+1)}{z(z+1)(z^2+1)} \, dz = c
\end{align*}
\[ (\lambda^u n)^{\mathbb{Z}} = (\mathbb{Z})^u \] (8.9)

\[ \sum_{n=0}^{\infty} (\lambda^u n)^{\mathbb{Z}} = (\mathbb{Z})^u \] (8.10)

denote the set of integers \( \lambda^u \) times larger than \( n \). Let \( u \) be the number of terms.

Theorem: For \( u \geq 1 \), the sum over \( \lambda^u n = \lambda \cdot n \) can be written in the form

\[ \sum_{n=0}^{\infty} (\lambda^u n)^{\mathbb{Z}} = (\mathbb{Z})^u \] (8.11)

Proof:

From the well-known identity

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \] \( \text{for } s > 1 \) (6.1).

The result follows from

\[ \sum_{n=1}^{\infty} \frac{1}{(n^s + 1)^u} = \sum_{n=1}^{\infty} \frac{1}{(n^s)^u} \] (8.12)

and

\[ \sum_{n=1}^{\infty} \frac{1}{(n^s + 1)^u} = \sum_{n=1}^{\infty} \frac{1}{(n^s)^u} \] (8.13)

Continuing the argument, we have

\[ \sum_{n=1}^{\infty} \frac{1}{(n^s + 1)^u} = \sum_{n=1}^{\infty} \frac{1}{(n^s)^u} \] (8.14)

so that

\[ \sum_{n=1}^{\infty} \frac{1}{(n^s + 1)^u} = \sum_{n=1}^{\infty} \frac{1}{(n^s)^u} \] (8.15)

which shows that the result is correct.

Here, \( s = 2 \) and \( u = 2 \).

In the above argument, the function \( \zeta(s) \) is the Riemann zeta function.
Theorem 8.2: 

**Table 8.2**: Comparison of Various Euclidean gcd Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Average Iterations</th>
<th>Best Iterations</th>
<th>Worst Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>B-5</em></td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td><em>B-6</em></td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td><em>B-7</em></td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><em>B-8</em></td>
<td>5</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

*Note:* Lower order terms are neglected in most cases.

From Theorem 6.1, we expect

\[
T_{\text{euclid}}(n) \approx n^{2/3}.
\]

For Table 8.1, see Algorithm (R).

**Table 8.1**: Exact Counts for Small n (Algorithm (R))

<table>
<thead>
<tr>
<th>n</th>
<th>T_{\text{euclid}}(n)</th>
<th>T_{\text{euclid}}(n)^{2/3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>3.167</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>4.642</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>5.745</td>
</tr>
<tr>
<td>6</td>
<td>28</td>
<td>6.889</td>
</tr>
<tr>
<td>7</td>
<td>45</td>
<td>8.017</td>
</tr>
<tr>
<td>8</td>
<td>66</td>
<td>8.366</td>
</tr>
<tr>
<td>9</td>
<td>91</td>
<td>8.822</td>
</tr>
<tr>
<td>10</td>
<td>120</td>
<td>9.381</td>
</tr>
<tr>
<td>11</td>
<td>161</td>
<td>9.961</td>
</tr>
<tr>
<td>12</td>
<td>206</td>
<td>10.55</td>
</tr>
</tbody>
</table>

For n = 1, the best case is given by the *B-5* algorithm, and the worst case by the *B-8* algorithm.

**References**

For an independent proof of Theorem (6.1), and for an outline of the proof, we refer the reader to [Collins (79)]. The correctness of the classical algorithm for finding the greatest common divisor of two integers is given in [Collins (79)].

And for an independent proof of Theorem (6.1), and for an outline of the proof, we refer the reader to [Collins (79)]. The correctness of the classical algorithm for finding the greatest common divisor of two integers is given in [Collins (79)].
Analysis of Binary Euclidean Algorithms
Minor Errata

In the definition of $D_0(x)$ on the last line of page 326, 
$D_0(x) = 0$ should be replaced by $D_0(x) = 1$.

In equation (6.3) on page 342, the term $-\frac{x}{2(1+x)}$ should be replaced by $-\frac{1}{2(1+x)}$.

The above corrections have been made in the online version.

Major Errata

Some of the results are incorrect. For example, (3.1), (3.29), (3.34), (3.35) are wrong (though a close approximation to the truth). Further details are given in http://web.comlab.ox.ac.uk/oucl/work/richard.brent/pub/pub183.html.