# A COMPARATIVE STUDY OF ALGORITHMS FOR COMPUTING CONTINUED FRACTIONS OF ALGEBRAIC NUMBERS (EXTENDED ABSTRACT)

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## 1. Introduction

The obvious way to compute the continued fraction of a real number $\alpha > 1$ is to compute a very accurate numerical approximation of $\alpha$, and then to iterate the well-known truncate-and-invert step which computes the next partial quotient $a = \lfloor \alpha \rfloor$ and the next complete quotient $\alpha' = 1/(\alpha - a)$. This method is called the *basic method.* In the course of this process precision is lost, and one has to take precautions to stop before the partial quotients become incorrect. Lehmer [6] gives a safe stopping-criterion, and a trick to reduce the amount of multi-length arithmetic. Schönhage [12] describes an algorithm for computing the greatest common divisor of $u$ and $v$ and the related continued fraction expansion of $u/v$ in $\mathcal{O}(n \log^2 n \log\log n)$ steps if both $u$ and $v$ do not exceed $2^n$.

A disadvantage of this approach is that if we wish to *extend* the list of partial quotients computed from an initial approximation of $\alpha$, we have to compute a *more accurate* initial approximation of $\alpha$, compute the new complete quotient using this new approximation *and* the partial quotients already computed from the old approximation, and then extend the list of partial quotients using that new complete quotient (we notice that Shiu in [13, p.1312] incorrectly states that *all* the previous calculations have to be repeated, but the partial quotients computed so far don't have to be recomputed).

Bombieri and Van der Poorten [1], and Shiu [13] have recently proposed a remedy for this problem. They give a formula for computing a rational approximation of the next complete convergent from the first $n$ partial quotients. From that complete convergent about $n$ new partial quotients can be computed. So each step gives an approximate doubling of the number of partial quotients. To start the method, a few partial quotients have to be computed with the basic or indirect method. In [1] this method is proposed for algebraic numbers (which are zeros of polynomials) of degree $\geq 3$, whereas Shiu also applies it to more general numbers, namely to transcendental numbers that can be defined as the zero of a function for which the logarithmic derivative at a rational point can be computed with arbitrary precision. This includes numbers like $\pi$, $\log \pi$, and $\log 2$. For each of thirteen different numbers, Shiu computes 10000 partial quotients. Their frequency distributions are compared with the one which almost all numbers should obey, according to the Khintchine–Lévy theory [3, 7]. No significant deviations from this theory are reported. Shiu calls his method the *direct method.*

Curiously, Shiu does not refer to what we would call the *polynomial method* for algebraic numbers [2, 5, 11] of degree $\geq 3$, which computes the partial quotients of $\alpha$ using *only* the coefficients of its defining polynomial. Moreover, Shiu gives neither implementational

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details of his direct method, nor of the indirect method mentioned above (which he applies to four numbers which can not be handled with the direct method). He concludes that his direct method is "superior in the sense that the computing times for a modest number of partial quotients using the indirect and the direct method are similar, whereas it becomes prohibitively long for the basic algorithm".

Since this is not quite a reproducible conclusion, and since the polynomial method is not included in Shiu's study, we felt stimulated to produce a more explicit comparison of the various methods. In addition, we have taken the occasion to compute about 200000 partial denominators of six different algebraic numbers.

A second motivation for this study is the use of the continued fraction expansion of certain algebraic numbers in solution methods for certain Diophantine inequalities. For example, in [10] the system of inequalities

$$|x^3 + x^2y - 2xy^2 - y^3| \le 200 \text{ and } |y| \le 10^{500}$$

(known to have finitely many integral solution pairs $(x, y)$) was solved with the help of the computation of a (modest) number of partial quotients of the continued fraction expansion of one of the real roots of the third degree polynomial $x^3 + x^2 - 2x - 1$.

## 2. Notation and error control

### 2.1 Notation.

Let $\alpha$ be a real number $> 1$. The continued fraction expansion of $\alpha$ is defined by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

where $a_i = \lfloor \alpha_i \rfloor$, $\alpha_{i+1} = 1/(\alpha_i - a_i)$, $i = 0, 1, \ldots$, with $\alpha_0 = \alpha$. The positive integers $a_0$, $a_1, \ldots$ are called the *partial quotients* of $\alpha$ and the real numbers $\alpha_i$ are called the *complete quotients* of $\alpha$. For simplicity, one writes

$$\alpha = [a_0, a_1, a_2, \ldots] = [a_0, a_1, \ldots, a_n, \alpha_{n+1}],$$

where $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$.

If $\alpha$ is rational, say $\alpha = u/v$, then its continued fraction expansion terminates (with some $\alpha_i = 0$) and the basic method is nothing but the Euclidean algorithm for computing the greatest common divisor of $u$ and $v$.

The rational approximation

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}$$

of $\alpha$ is called the $n$-th *convergent* of $\alpha$. The numerators and denominators of these approximations can be computed with the following formulas:

$$\left.\begin{aligned} p_{i+1} &= a_{i+1}p_i + p_{i-1} \\ q_{i+1} &= a_{i+1}q_i + q_{i-1} \end{aligned}\right\} \quad i = 0, 1, \ldots,$$

where $p_0 = a_0$, $q_0 = 1$, $p_{-1} = 1$, and $q_{-1} = 0$. In matrix notation, we have

$$\begin{pmatrix} p_{i+1} & p_i \\ q_{i+1} & q_i \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{i+1} & 1 \\ 1 & 0 \end{pmatrix},$$

which implies, by taking determinants, that

$$p_{i+1}q_i - p_iq_{i+1} = (-1)^i, \quad i = 0, 1, \ldots. \tag{1}$$

**2.2 Error control.**

When we compute the partial quotients $a_0$, $a_1$, ... from a numerical approximation $\overline{\alpha}$ of $\alpha$, we loose precision. The error can be controlled with the help of the following two lemmas. Lemma 1 gives a sufficient condition for $\lfloor\overline{\alpha}\rfloor = \lfloor\alpha\rfloor$ to hold. Lemma 2 gives an upper bound for the relative error in $\overline{\alpha}' = 1/(\overline{\alpha} - \lfloor\overline{\alpha}\rfloor)$ as a function of $\overline{\alpha}$, the relative error in $\overline{\alpha}$, and $\overline{\alpha}'$.

**Lemma 1.** *Let $\overline{\alpha}$ be a numerical (rational) approximation of $\alpha$ with relative error bounded by $\delta$, i.e., $\overline{\alpha} = \alpha(1+\epsilon)$ with $|\epsilon| < \delta$. If $\delta < 1/2$ and*

$$(\lfloor\overline{\alpha}\rfloor + 1)\,\delta < \overline{\alpha} - \lfloor\overline{\alpha}\rfloor < 1 - (\lfloor\overline{\alpha}\rfloor + 1)\,\delta, \tag{2}$$

*then $\lfloor\overline{\alpha}\rfloor = \lfloor\alpha\rfloor$.*

*Proof.* We show that $\lfloor\overline{\alpha}\rfloor < \alpha < \lfloor\overline{\alpha}\rfloor + 1$.

Firstly, since $1 - \delta < 1/(1+\epsilon)$ for $\delta < 1$, we have

$$\overline{\alpha}(1-\delta) < \frac{\overline{\alpha}}{1+\epsilon} = \alpha.$$

Furthermore, we have $\overline{\alpha}\,\delta < (\lfloor\overline{\alpha}\rfloor + 1)\,\delta$ so that, by the left inequality in (2), $\overline{\alpha}\,\delta < \overline{\alpha} - \lfloor\overline{\alpha}\rfloor$, which with the above inequality implies that $\lfloor\overline{\alpha}\rfloor < \overline{\alpha}(1-\delta) < \alpha$.

Secondly, since $1/(1+\epsilon) < 1/(1-\delta)$ for $\delta < \frac{1}{2}$, we have

$$\alpha = \frac{\overline{\alpha}}{1+\epsilon} < \frac{\overline{\alpha}}{1-\delta}.$$

From the right inequality in (2) we have $\overline{\alpha} < (\lfloor\overline{\alpha}\rfloor+1)(1-\delta)$, so that $\overline{\alpha}/(1-\delta) < \lfloor\overline{\alpha}\rfloor+1$. $\square$

**Lemma 2.** *Suppose the conditions of Lemma 1 hold, and let*

$$\alpha' = \frac{1}{\alpha - \lfloor\alpha\rfloor}, \quad \overline{\alpha}' = \frac{1}{\overline{\alpha} - \lfloor\overline{\alpha}\rfloor}.$$

*Then an upper bound for the relative error in $\overline{\alpha}'$ with respect to $\alpha'$ is given by $\overline{\alpha}\,\overline{\alpha}'\delta/(1-\delta)$.*

*Proof.* We have

$$\left|\frac{\overline{\alpha}' - \alpha'}{\alpha'}\right| = \left|\frac{\dfrac{1}{\overline{\alpha} - \lfloor\overline{\alpha}\rfloor} - \dfrac{1}{\alpha - \lfloor\alpha\rfloor}}{\dfrac{1}{\alpha - \lfloor\alpha\rfloor}}\right| = \left|\frac{\alpha - \overline{\alpha}}{\overline{\alpha} - \lfloor\overline{\alpha}\rfloor}\right| =$$

$$= \overline{\alpha}\,\overline{\alpha}'\left|1 - \frac{\alpha}{\overline{\alpha}}\right| = \overline{\alpha}\,\overline{\alpha}'\left|\frac{\epsilon}{1+\epsilon}\right| < \overline{\alpha}\,\overline{\alpha}'\frac{\delta}{1-\delta}.$$

$\square$

An additional way to control the computation is based on the following well-known property of continued fractions which we leave to the reader to prove.

**Lemma 3.** *If $\beta_0/\beta_1$ and $\gamma_0/\gamma_1$ are rational numbers such that*

$$\frac{\beta_0}{\beta_1} < \alpha < \frac{\gamma_0}{\gamma_1},$$

*then as long as the partial quotients of $\beta_0/\beta_1$ and $\gamma_0/\gamma_1$ coincide, they are the partial quotients of $\alpha$. The first time the partial quotients do not coincide, they form a lower and an upper bound for the correct value.*

This suggests Lehmer's method [6] to reduce the amount of multi-precision work. Assuming that we have a very accurate rational approximation $u/v$ of the real number $\alpha > 1$ with very large numbers $u$ and $v$, we can form a suitable lower and upper bound for $u/v$ by just taking the first ten (say) decimal digits of $u$ and $v$: if $\lceil \log_{10} u \rceil = k$, take $u_0 = \lfloor u/10^{k-10} \rfloor$ and $v_0 = \lfloor v/10^{k-10} \rfloor$ and choose[1]

$$\beta_0 = u_0, \quad \beta_1 = v_0 + 1, \quad \gamma_0 = u_0 + 1, \text{ and } \gamma_1 = v_0.$$

Now we compute partial quotients $a_0, a_1, \ldots, a_{i_0}$ of $\gamma_0/\gamma_1$ and hence of $\alpha$ as follows:

$$\left.\begin{array}{l} a_i = \lfloor \gamma_i/\gamma_{i+1} \rfloor \\ \gamma_{i+2} = \gamma_i - a_i\gamma_{i+1}, \quad \beta_{i+2} = \beta_i - a_i\beta_{i+1} \\ \textbf{if } \beta_{i+2} < 0 \vee \beta_{i+2} \geq \beta_{i+1} \textbf{then} \\ \quad i_0 = i - 1, \text{ stop} \\ \textbf{endif} \end{array}\right\} \quad i = 0, 1, \ldots \tag{3}$$

Notice that we do *not* have to compute the partial quotients of $\beta_0/\beta_1$ (contrary to what is suggested in [4, p.328]) since as long as $0 \leq \beta_{i+2} < \beta_{i+1}$, we are sure that $a_i$ is also the correct partial quotient of $\beta_0/\beta_1$. After (3) has stopped, we have to update the fraction $u/v$ with the computed partial quotients $a_0, a_1, \ldots, a_{i_0}$. So with $a_0$ we replace $u/v$ by $v/(u - a_0 v)$. In matrix notation,

$$\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 1 & -a_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

and in general, for $a_0, \ldots, a_{i_0}$ we have

$$\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 1 & -a_{i_0} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a_{i_0-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -a_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The product of the $2 \times 2$ matrices in the right hand side is built up first, and next it is multiplied by the vector $(u \;\; v)^T$, which is the only high-precision computation.

## 3. The basic, polynomial and direct methods

In this section we will describe the three methods which we have considered in this study, namely the basic method, the polynomial method, and the direct method, which is derived from Shiu's direct method.

**3.1 The basic method.** With the notation of Section 2.1, let $\overline{\alpha}_i$ be a rational approximation of $\alpha_i$ with relative error bounded by $\delta_i$. The basic method for computing the continued fraction expansion of $\alpha = \alpha_0$ with safe error control (based on Lemmas 1 and 2) reads as follows.

$$\left.\begin{array}{l} a_i = \lfloor \overline{\alpha}_i \rfloor \\ \textbf{if } (a_i + 1)\,\delta_i < \overline{\alpha}_i - a_i < 1 - (a_i + 1)\,\delta_i \textbf{ then} \\ \quad \overline{\alpha}_{i+1} = 1/(\overline{\alpha}_i - a_i) \\ \quad \delta_{i+1} = \overline{\alpha}_i \overline{\alpha}_{i+1} \delta_i/(1 - \delta_i) \\ \textbf{else} \\ \quad \text{stop} \\ \textbf{endif} \end{array}\right\} \quad i = 0, 1, \ldots \tag{4}$$

[1] If $v_0 = 0$, the first partial quotient of $u/v$ is extremely large, and we have to increase the number of decimal digits in $u_0$ and $v_0$ accordingly.

Since the $\overline{\alpha}_i$ are rational numbers, we can use Lemma 3 and (3) to reduce the amount of multi-precision computations. The numbers $(a_i+1)\,\delta_i$ and $\delta_{i+1}$ are computed in (floating-point) single precision. Since (3) works with low-precision approximations $\gamma_i/\gamma_{i+1}$ and $\beta_i/\beta_{i+1}$ of $\overline{\alpha}_i$, some care has to be taken in the check of the inequalities in (4) and in the computation of $\delta_{i+1}$ from $\delta_i$ in (4). Here we can use that

$$\frac{\beta_{2i}}{\beta_{2i+1}} < \overline{\alpha}_{2i} < \frac{\gamma_{2i}}{\gamma_{2i+1}},$$

and

$$\frac{\gamma_{2i+1}}{\gamma_{2i+2}} < \overline{\alpha}_{2i+1} < \frac{\beta_{2i+1}}{\beta_{2i+2}}.$$

as long as $a_{2i}$ and $a_{2i+1}$ are the correct partial quotients of $\overline{\alpha}_{2i}$ and $\overline{\alpha}_{2i+1}$, respectively. In the full version of this paper, we will give a complete listing of the algorithmic steps in this method.

From the metric theory of continued fractions it is known [9] that if from $d$ decimal digits of $\alpha$ we can compute $p$ partial quotients of its continued fraction, then for almost all $\alpha$

$$\lim_{d\to\infty} \frac{p}{d} = \frac{6\log 2\log 10}{\pi^2} = 0.970....$$

To illustrate this: Lochs [8] has computed 968 partial quotients of $\pi$ from its first 1000 decimals.

A disadvantage of the basic method is that if we have computed as many as possible partial quotients from a given initial approximation of $\alpha$, and if we would like to compute more partial quotients, we have to compute a more accurate initial approximation, use the known partial quotients to compute the last possible complete convergent, and from that extend the list of partial quotients.

**3.2 The polynomial method.** Let $\alpha > 1$ be an an algebraic number of degree $d > 2$ with defining polynomial $f(x)$ (with integral coefficients), so that $f(\alpha) = 0$. Let $f(x)$ have the following three properties:

(i) its leading coefficient is positive;
(ii) it has a unique root $\alpha > 1$ which is simple, i.e., $f'(\alpha) \neq 0$;
(iii) $\alpha$ is irrational.

The polynomial method [5] for computing the continued fraction expansion of $\alpha$ reads as follows. Let $f_0(x) = f(x)$.

$$\left.\begin{array}{l} a_i = \max\{n \in \mathbb{N},\ f_i(n) < 0\} \\ g_i(x) = f_i(x+a_i) \\ f_{i+1}(x) = -x^d g_i(1/x) \end{array}\right\} \quad i = 0, 1, \ldots \tag{5}$$

It is easy to see that $f_1(x)$ has the same three properties as $f_0(x)$ and that the unique root $> 1$ of $f_1(x)$ is given by $1/(\alpha - a_0)$. It follows that the unique root $> 1$ of the polynomial $f_i(x)$ is the $i$-th complete quotient of the continued fraction expansion of $\alpha$, and that this algorithm finds the corresponding partial quotients. The time-consuming work lies in the computation of the coefficients of $f_{i+1}(x)$ from those of $f_i(x)$ (which grow with $i$). The number $a_i$ can be computed quickly as follows. If we write $f_i(x) = c_{id}x^d + c_{i,d-1}x^{d-1} + \ldots$, then the sum of the roots of $f_i(x)$ is given by $s_i = -c_{i,d-1}/c_{id}$. Since, for $i \geq 1$, the $d-1$ roots of $f_i(x)$ which have their real part $< 1$, are all located in the interval $(-1, 0)$, the number $s_i$ approximates $a_i$ with an error not greater than $d-1$; the precise value of $a_i$ is found from $s_i$ by trial and error (with an average of $(d-1)/2$ trials).

Similarly to the basic method, the polynomial method may be accelerated [5] by computing several successive partial quotients (with the basic method) from a low-precision approximation of the real root $> 1$ of $f_n(x)$. Then it is possible to compute $f_{n+m}(x)$ from

$f_n(x)$ and $a_n$, $a_{n+1}$, …, $a_{n+m-1}$ with less computation than is needed to compute all the intermediate polynomials $f_{n+1}(x)$, …, $f_{n+m-1}(x)$. We have not (yet) pursued this.

An advantage of this method is that the computation can always be continued, without any recomputation, provided that we save the exact integral values of the coefficients of the last used polynomial $f_i(x)$. To illustrate the growth of these, for $f(x) = f_0(x) = x^3 - 8x - 10$, the four coefficients of $f_{100}(x)$ are integers of 68 decimal digits each, and the four coefficients of $f_{1000}(x)$ are integers of 570, 571, 570, and 568 decimal digits, respectively.

**3.3 The direct method.** The direct method which we formulate here is based on ideas expressed in [1] and [13], combined with error control facilities described in Section 2. The aim is to compute a very good rational approximation of the complete quotient $\alpha_{n+1}$ when the partial quotients $a_0, a_1, \ldots, a_n$ are known, and from that approximation compute about $n$ partial quotients of $\alpha_{n+1}$. This is done as follows. We have

$$\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}] = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}},$$

from which we find, using (1), that

$$\alpha_{n+1} = \frac{(-1)^{n-1}}{q_n(p_n - \alpha q_n)} - \frac{q_{n-1}}{q_n}.$$

Now using the mean value theorem and $f(\alpha) = 0$, we replace the difference $\frac{p_n}{q_n} - \alpha$ by $f(\frac{p_n}{q_n})/f'(\frac{p_n}{q_n})$, and obtain the approximation

$$\alpha_{n+1} \approx \frac{(-1)^{n-1}}{q_n^2} \frac{f'(\frac{p_n}{q_n})}{f(\frac{p_n}{q_n})} - \frac{q_{n-1}}{q_n}. \tag{6}$$

The error in this approximation is approximately

$$\frac{|f''(\alpha)|}{q_n^2 |f'(\alpha)|}.$$

From this rational approximation of $\alpha_{n+1}$ partial quotients $a_{n+1}, a_{n+2}, \ldots, a_{n+m}, \ldots$ can be computed as long as $q_{n+m} < bq_n^2$, for some small $b = b(\alpha) > 0$. The direct method for computing $N$ partial quotients of the continued fraction expansion of $\alpha$ now reads as follows. *Step 1* Use the basic method (4) to compute a *small* number of partial quotients and the corresponding partial convergents of $\alpha$, say up to $a_n$, $p_n$, $q_n$.
*Step 2* (Check) If $p_n q_{n-1} - p_{n-1} q_n \neq (-1)^{n-1}$ then stop.
Compute the next rational approximation $\alpha'$ of $\alpha_{n+1}$ by

$$\alpha' = \frac{(-1)^{n-1}}{q_n^2} \frac{f'(p_n/q_n)}{f(p_n/q_n)} - \frac{q_{n-1}}{q_n}. \tag{7}$$

Let $B = bq_n^2$ for some suitable constant $b = b(\alpha)$.
Compute the next partial quotients $a_{n+1}$, $a_{n+2}, \ldots, a_{n+m}, \ldots$ with the basic method (4) (using Lemma 3 and (3)) as long as $n + m \leq N$ and $q_{n+m} < B$.
*Step 3* Put $n = n + m$; if $n < N$ go back to Step 2.

The number of partial quotients which can be computed in Step 2 is roughly equal to $n$ so that *after* the completion of Step 2, the number of partial quotients computed has roughly been doubled compared with *before* Step 2. Since (7) is very time-consuming, it is worthwhile to choose $n$ in Step 1 such that the last time Step 2 is carried out, it starts with a value of $n$ which is *slightly larger* than $N/2$. Since in the beginning of the method the behaviour of Step 2 may be rather erratic, one should compute so many partial quotients of $\alpha$ in Step 1 that the "stable" behaviour phase of Step 2 (an approximate doubling of the number of partial quotients) is reached. In practice, this works for $n \approx 100$, but this also depends on the sizes of the first partial quotients of the continued fraction of $\alpha$.

## 4. Experiments

We have implemented the three methods described in Section 3 on a SUN workstation, partially in GP/PARI and partially in Magma. The first package is developed by Henri Cohen and his co-workers at Université Bordeaux I, the second comes from John Cannon and his group at the University of Sydney. Initially, we only worked with GP, but at a certain point in the direct method we ran into problems with the stack size, due to the enormous size of the integers involved in this method. Later we learned that these problems can be solved, for example, by programming PARI in Library Mode, but in the meantime we learned about the Magma-package at the University of Sydney and decided to experiment with that. With Magma we did not encounter any stack problems.

In Table 1 we give some timings with Magma and GP for the basic, the polynomial, and the direct methods. Based on these results, we decided to run bigger experiments with our Magma implementation of the direct method.

In Table 2 we give the frequency distributions of the first 200001 partial quotients of the continued fraction of six algebraic numbers, computed with the direct method. For comparison, the last column gives the frequencies of occurrence of partial quotients $j$:

$$\log_2\left(1+\frac{1}{j}\right)-\log_2\left(1+\frac{1}{j+1}\right)$$

from the well-known Gauss-Kusmin Theorem. Let

$$K(\alpha,n)=(a_0a_1\dots a_n)^{1/(n+1)}$$

and

$$L(\alpha,n)=q_n^{1/(n+1)}.$$

Then for almost all $\alpha$

$$\lim_{n\to\infty}K(\alpha,n)=\prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\log k/\log 2}=2.68545...,$$

and

$$\lim_{n\to\infty}L(\alpha,n)=\exp\left(\frac{\pi^2}{12\log 2}\right)=3.27582....$$

The latter result implies that for almost all $\alpha$ the number of decimal digits in $q_n$ is about $n\log_{10}L\approx 0.515n$. Table 2 gives the values of $K(\alpha,200000)$ for the six algebraic numbers which we considered. Table 2 also lists the largest partial quotient $a_n$ found, and the corresponding index $n$. Only in case (A) there is an early occurrence of a large partial quotient ($a_{121}=16467250$), but soon after that, no extremely large partial quotients occur anymore. To illustrate this, Table 3 lists $a_n$ for $n=0,\dots,200$ and for $n=199901,\dots,200000$. The "abnormal" initial behaviour is explained in [14]. Table 4 presents, for some values of $n$, the number of decimal digits in $q_n$ and that number divided by $n$. The values of $n$ in Table 4 are those for which the direct method computed a new rational approximation of $\alpha_n$: it illustrates the approximate doubling of these $n$-values, especially for larger values of $n$. The last column shows good convergence to the value $\pi^2/(12\log 2\log 10)=0.51532....$

## 5. Conclusion

We have compared three different methods (the basic, the polynomial, and the direct method) for computing the continued fraction expansion of algebraic numbers, and observed that the direct method is the most efficient one in terms of CPU-time and memory, at least for our implementations (in GP/PARI and Magma). We have applied the direct method to the computation of 200,001 partial quotients of six different algebraic numbers, and found no apparent deviation from the theory of Khintchine and Lévy, which holds for almost all real numbers.

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TABLE 1 *CPU-time in seconds to compute 10000 partial quotients of the root* $\alpha = 3.31862...$ *of* $f(x) = x^3 - 8x - 10$.

| | basic | polynomial | direct |
|---|---|---|---|
| with Magma | 98 | | 55 |
| with GP | | 192 | 172 |

TABLE 2 *Frequency distribution of the first 200001 partial quotients of the continued fraction of various algebraic numbers.*

| | | |
|---|---|---|
| (A) | $f(x) = x^3 - 8x - 10,$ | $\alpha = 3.31862...$ |
| (B) | $f(x) = x^3 - 2,$ | $\alpha = 1.25992... = 2^{1/3}$ |
| (C) | $f(x) = x^3 - 5,$ | $\alpha = 1.70997... = 5^{1/3}$ |
| (D) | $f(x) = x^4 + 6x^3 + 7x^2 - 6x - 9$ | $\alpha = 1.03224...$ $= (5^{1/2} - 1)/2 + 2^{1/2} - 1$ $= [0, \overline{1}\,] + [0, \overline{2}\,]$ |
| (E) | $f(x) = x^3 + x^2 - 2x - 1$ | $\alpha = 1.24697... = 2\cos(2\pi/7)$ |
| (F) | $f(x) = x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$ | $\alpha = 2.99197... = 2^{1/3} + 3^{1/2}$ |

| digit | (A) | (B) | (C) | (D) | (E) | (F) | “expected” |
|---|---|---|---|---|---|---|---|
| 1 | 82705 | 82862 | 83186 | 82865 | 83159 | 82566 | 83008 |
| 2 | 34277 | 34180 | 33883 | 34538 | 33900 | 34382 | 33985 |
| 3 | 18641 | 18680 | 18570 | 18588 | 18560 | 18616 | 18622 |
| 4 | 11693 | 11795 | 11785 | 11503 | 11835 | 11931 | 11779 |
| 5 | 8192 | 8114 | 8165 | 8114 | 8070 | 8083 | 8128 |
| 6 | 6082 | 5900 | 5864 | 5880 | 5826 | 5916 | 5949 |
| 7 | 4470 | 4443 | 4535 | 4512 | 4519 | 4532 | 4544 |
| 8 | 3470 | 3636 | 3557 | 3540 | 3671 | 3594 | 3584 |
| 9 | 2862 | 2841 | 2975 | 2896 | 2866 | 2911 | 2900 |
| 10 | 2474 | 2424 | 2329 | 2400 | 2428 | 2347 | 2395 |
| >10 and ≤100 | 22156 | 22240 | 22298 | 22309 | 22302 | 22230 | 22264 |
| >100 | 2979 | 2886 | 2854 | 2856 | 2865 | 2893 | 2843 |
| $K$ | 2.6944 | 2.6871 | 2.6832 | 2.6848 | 2.6844 | 2.6919 | 2.68545 |
| $\max(a_n)$ | 16467250 | 320408 | 489859 | 7295890 | 179545 | 1075748 | |
| index $n$ | 121 | 190270 | 21125 | 142839 | 44595 | 52062 | |
| CPU-time in minutes | 109 | 99 | 107 | 196 | 113 | 424 | |

TABLE 3 *Some partial quotients $a_n$ of the real root $\alpha = 3.31861...$ of $x^3 - 8x - 10$ (those $> 10000$ are underlined).*

| $n$ | $a_n$ |
|---|---|
| 0 | 3 |
| 1 – 20 | 3 7 4 2 30 1 8 3 1 1 1 9 2 2 1 3 $\underline{22986}$ 2 1 32 |
| 21 – 40 | 8 2 1 8 55 1 5 2 28 1 5 1 $\underline{1501790}$ 1 2 1 7 6 1 1 |
| 41 – 60 | 5 2 1 6 2 2 1 2 1 1 3 1 3 1 2 4 3 1 $\underline{35657}$ 1 |
| 61 – 80 | 17 2 15 1 1 2 1 1 5 3 2 1 1 7 2 1 7 1 3 25 |
| 81 – 100 | $\underline{49405}$ 1 1 3 1 1 4 1 2 15 1 2 83 1 162 2 1 1 1 2 |
| 101–120 | 2 1 $\underline{53460}$ 1 6 4 3 4 13 5 15 6 1 4 1 4 1 1 2 1 |
| 121–140 | $\underline{16467250}$ 1 3 1 7 2 6 1 95 20 1 2 1 6 1 1 8 1 $\underline{48120}$ 1 |
| 141–160 | 2 17 2 1 2 1 4 2 3 1 2 23 3 2 1 1 1 2 1 27 |
| 161–180 | $\underline{325927}$ 1 60 1 87 1 2 1 5 1 1 1 2 2 2 2 2 17 4 9 |
| 181–200 | 9 1 7 11 1 2 9 1 14 4 6 1 22 11 1 1 1 1 4 1 |
| | |
| 199901–199920 | 1 1 2 1 2 1 3 3 3 4 2 2 1 1 1 1 1 8 11 1 |
| 199921–199940 | 10 1 4 1 1 2 4 2 1 1 3 5 1 6 1 2 7 7 50 1 |
| 199941–199960 | 2 12 1 1 1 3 1 7 1 6 2 2 1 1 2 3 2 6 1 3 |
| 199961–199980 | 1 3 10 1 1 1 3 1 6 2 1 16 1 1 1 1 8 1 1 1 |
| 199981–200000 | 2 5 1 1 2 1 2 2 3 2 1 1 5 2 5 1 1 1 268 1 |

TABLE 4 *Sizes of $q_n$ for the continued fraction expansion of the real root $\alpha = 3.31862...$ of $x^3 - 8x - 10$.*

| n | # dec. digits of $q_n$ | # dec. digits/$n$ |
|---|---|---|
| 301 | 192 | 0.638 |
| 654 | 380 | 0.581 |
| 1347 | 752 | 0.558 |
| 2830 | 1499 | 0.530 |
| 5667 | 2992 | 0.528 |
| 11502 | 5977 | 0.520 |
| 22982 | 11945 | 0.520 |
| 46208 | 23880 | 0.517 |
| 92514 | 47754 | 0.516 |
| | | $0.51532...=\frac{\pi^2}{12\log 2\log 10}$ |

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