A Conjectured Integer Sequence Arising From the Exponential Integral

Richard P. Brent
Australian National University
Canberra, ACT 2601, Australia
ei@rpbrent.com

M. L. Glasser
Clarkson University
Potsdam, NY 136-99-5820, USA
lglasser@clarkson.edu

Anthony J. Guttmann
School of Mathematics and Statistics
The University of Melbourne
Vic. 3010, Australia
tony.guttmann@unimelb.edu.au

Abstract

Let $f_0(z) = \exp(z/(1-z))$, $f_1(z) = \exp(1/(1-z))E_1(1/(1-z))$, where $E_1(x) = \int_x^\infty e^{-t}t^{-1}\,dt$. Let $a_n = [z^n]f_0(z)$ and $b_n = [z^n]f_1(z)$ be the corresponding Maclaurin series coefficients. We show that a_n and b_n may be expressed in terms of confluent hypergeometric functions.

We consider the asymptotic behaviour of the sequences (a_n) and (b_n) as $n \to \infty$, showing that they are closely related, and proving a conjecture of Bruno Salvy regarding (b_n).

Let $\rho_n = a_n b_n$, so $\sum \rho_n z^n = (f_0 \odot f_1)(z)$ is a Hadamard product. We obtain an asymptotic expansion $2n^{3/2}\rho_n \sim -\sum d_k n^{-k}$ as $n \to \infty$, where the $d_k \in \mathbb{Q}$, $d_0 = 1$. We conjecture that $2^{6k}d_k \in \mathbb{Z}$. This has been verified for $k \leqslant 1000$.

1 Introduction

We consider two analytic functions,

$$f_0(z) := e^{z/(1-z)} = e^{-1}\, e^{1/(1-z)}$$

and

$$f_1(z) := e^x E_1(x), \text{ where } x := 1/(1-z) \text{ and } E_1(x) := e^x \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

These functions are regular in the open disk $D = \{z \in \mathbb{C} : |z| < 1\}$. We write their Maclaurin coefficients as $a_n := [z^n]f_0(z)$ and $b_n = [z^n]f_1(z)$. Thus, in the disk D, $f_0(z) = \sum_{n\geqslant 0} a_n z^n$ and $f_1(z) = \sum_{n\geqslant 0} b_n z^n$.

The functions $f_0(z)$ and $f_1(z)$ satisfy the same third-order linear differential equation with polynomial coefficients. Thus, the sequences (a_n) and (b_n) are D-finite and satisfy the same recurrence relation (for sufficiently large n).

There are several entries in the OEIS related to the rational sequence $(a_n)_{n\geqslant 0}$. The numerators are OEIS A067764, and the denominators are OEIS A067653. The integers $n!a_n$ are given by OEIS A000262 and, with alternating signs, by OEIS A293125. The numbers $(b_n)_{n\geqslant 0}$ are unlikely to be rational.[1]

The numbers a_n and b_n may be expressed in terms of confluent hypergeometric functions. If $M(a, b, z) = {}_1F_1(a; b; z)$ and $U(a, b, z)$ are standard solutions of Kummer's differential equation, then Lemmas 1–2 show that $a_n = e^{-1}M(n + 1, 2, 1)$ and $b_n = -\Gamma(n)U(n, 0, 1)$.

We are interested in the asymptotics of a_n and b_n for large n. Perron [11] showed that

$$a_n \sim \frac{e^{2\sqrt{n}}}{2n^{3/4}\sqrt{\pi e}}\,.$$

Salvy[2] conjectured that b_n is of order $e^{-2\sqrt{n}}n^{-3/4}$. We have verified this conjecture. In fact,

$$b_n \sim -\frac{\sqrt{\pi e}}{n^{3/4}e^{2\sqrt{n}}}\,.$$

[1]In particular, $b_0 = G$, where $G := eE_1(1) \approx 0.596$ is the Euler-Gompertz constant, whose decimal digits are given by OEIS A073003. We have $b_n = a_nG - a'_n$, where $a'_n \in \mathbb{Q}$ and a'_n satisfies essentially the same recurrence as a_n, but with different initial conditions. Clearly $b_n \in \mathbb{Q}$ if and only if $G \in \mathbb{Q}$. All that is known is that at least one of γ and G is irrational [1, 12].

[2]Bruno Salvy, email to A. J. Guttmann et al., May 28, 2018.

A function of the form $f(n) = \exp(\alpha n^{\beta+o(1)})$ for $\alpha \neq 0$, $\beta \in (0,1)$, is called a *stretched exponential* in the physics/statistics literature (the term *sub-exponential* is used in complexity theory). Thus, a_n and b_n are stretched exponentials, with $\alpha = \pm 2$ and $\beta = 1/2$. The motivation for this paper was a query about the existence of simple functions whose Maclaurin coefficients are stretched exponentials with $\alpha < 0$ and $\beta \in \mathbb{Q}$. Here we consider the case $\beta = 1/2$; other cases are more complicated and will be considered in a separate paper.

Theorem 5 gives complete asymptotic expansions of a_n and b_n. These may be written as

$$a_n = \frac{F(n^{1/2})}{2n^{3/4}\sqrt{\pi e}} \text{ and } b_n = -\frac{\sqrt{\pi e}}{n^{3/4}}\, F(-n^{1/2}),$$

where $F(x) \sim e^{2x} \sum_{k \geqslant 0} c_k x^{-k}$, for certain constants $c_k \in \mathbb{Q}$, $c_0 = 1$. The c_k may be computed using Theorem 5 or Lemma 7.

The *Hadamard product* $f_0 \odot f_1$ of f_0 and f_1 is the analytic function defined for $z \in D$ by

$$(f_0 \odot f_1)(z) = \sum_{n \geqslant 0} a_n b_n z^n.$$

The asymptotic expansions of a_n and b_n imply an asymptotic expansion for $\rho_n := a_n b_n$ of the form

$$\rho_n \sim -\frac{1}{2n^{3/2}} \sum_{k \geqslant 0} d_k n^{-k},$$

where $d_k \in \mathbb{Q}$, $d_0 = 1$ (see Corollary 10).

A *dyadic rational* is a rational number of the form p/q, where q is a power of two. Let $Q_2 := \{j/2^k : j, k \in \mathbb{Z}\}$ denote the set of dyadic rationals.

We conjecture, from numerical evidence for $k \leqslant 1000$, that $d_k \in Q_2$. More precisely, defining $r_k := 2^{6k} d_k$, Conjecture 11 is that $r_k \in \mathbb{Z}$. Remark 12 gives numerical evidence for a slightly stronger conjecture. In Theorem 18 we prove the weaker (but still nontrivial) result that $k! r_k \in \mathbb{Z}$.

In Remark 14 we mention an analogous (easily proved) result for modified Bessel functions, where the product $I_\nu(x) K_\nu(x)$ for fixed $\nu \in \mathbb{Z}$ has an asymptotic expansion whose coefficients are in $\mathbb{Q}_2$.

Some comments on notation: $f(x) \sim \sum_{k \geqslant 0} f_k x^{-k}$ means that the sum on the right is an asymptotic series for $f(x)$ in the sense of Poincaré. Thus, for any fixed $m > 0$, $f(x) = \sum_{k=0}^{m-1} f_k x^{-k} + O(x^{-m})$ as $x \to \infty$. The letters j, k, m, n always denote integers (except for n in Remark 4). The notation $(x)_n$ for $n \geqslant 0$ denotes the *ascending factorial* or *Pochhammer symbol*, defined by $(x)_n := x(x+1)\cdots(x+n-1)$.

2 The Maclaurin coefficients a_n and b_n

In this section we obtain recurrence relations and closed-form expressions for a_n and b_n. The connection with confluent hypergeometric (Kummer) functions is discussed in §3, and asymptotics are considered in §4.

The function $f_0(z)$ is the exponential generating function counting several combinatorial objects, such as the number of "sets of lists", i.e., the number of partitions of $\{1, 2, \ldots, n\}$ into ordered subsets, see Wallner [18, §5.3].

Observe that $f_0(z)$ satisfies the differential equation

$$(1-z)^2 f_0'(z) - f_0(z) = 0, \tag{1}$$

and from this it is easy to see that the a_n satisfy a three-term recurrence

$$na_n - (2n-1)a_{n-1} + (n-2)a_{n-2} = 0 \quad \text{for} \quad n \geqslant 2. \tag{2}$$

The initial conditions are $a_0 = a_1 = 1$. Thus

$$(a_n)_{n \geqslant 0} = (1, 1, 3/2, 13/6, 73/24, 167/40, \ldots).$$

The recurrence (2) holds for $n \geqslant 0$ provided that we define $a_n = 0$ for $n < 0$. A closed-form expression, valid for $n \geqslant 1$ (but not for $n = 0$), is

$$a_n = \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1}.$$

The constants a_n may be expressed in terms of the generalised Laguerre polynomials $L_n^{(\alpha)}(x)$, which have a generating function (see [10, 18.12.13])

$$\sum_{n \geqslant 0} z^n L_n^{(\alpha)}(x) = (1-z)^{-(\alpha+1)} e^{-xz/(1-z)}.$$

With $\alpha = x = -1$ we obtain $\sum_{n \geqslant 0} z^n L_n^{(-1)}(-1) = e^{z/(1-z)}$, so $a_n = L_n^{(-1)}(-1)$.

Using the chain rule and the definition of $f_1(z)$ in §1, we see that $f_1(z)$ satisfies the differential equation

$$(1-z)^2 f_1'(z) - f_1(z) = z - 1, \tag{3}$$

which differs from (1) only in the right-hand side $z-1$. Differentiating twice more with respect to z, we see that $f_0(z)$ and $f_1(z)$ both satisfy the same third-order differential equation

$$(1-z)^2 f''' + (4z-5)f'' + 2f' = 0.$$

From (3), the b_n satisfy a recurrence

$$nb_n - (2n-1)b_{n-1} + (n-2)b_{n-2} = \begin{cases} 1, & \text{if } n = 2; \\ 0, & \text{if } n \geqslant 3. \end{cases} \tag{4}$$

This is essentially (i.e., for $n \geqslant 3$) the same recurrence as (2), but the initial conditions $b_0 = G$, $b_1 = G - 1$ are different. Here $G := eE_1(1) \approx 0.596$ is the Euler-Gompertz constant [9, §2.5].

We remark that computation of the b_n using the recurrence (4) in the forward direction is numerically unstable. A stable method of computation is to use an adaptation of Miller's algorithm, originally used to compute Bessel functions. See Gautschi [7, §3] and Temme [14, §4].

As noted in §1, the b_n may be expressed as $a_n G - a'_n$, where a_n is as above, and a'_n satisfies essentially the same recurrence with different initial conditions. In fact,

$$na'_n - (2n-1)a'_{n-1} + (n-2)a'_{n-2} = \begin{cases} -1, & \text{if } n = 2; \\ 0, & \text{if } n \geqslant 3. \end{cases}$$

The initial conditions are $a'_0 = 0$, $a'_1 = 1$. Thus

$$(a'_n)_{n\geqslant 0} = (0, 1, 1, 4/3, 11/6, 5/2, 121/36, \ldots).$$

Since $b_n \to 0$ as $n \to \infty$ (see (15) below), the sequence $(a'_n/a_n)_{n\geqslant 1}$ is a convergent sequence of rational approximations to G. The sequence of approximants is $(1, 2/3, 8/13, 44/73, 100/167, \ldots)$.

Bala [2] gives the continued fraction

$$1 - G = 1/(3 - 2/(5 - 6/(7 - \cdots - n(n+1)/(2n+3) - \cdots))),$$

with convergents $1/3, 5/13, 28/73, 201/501, \ldots$. The corresponding convergents to G are $2/3, 8/13, 45/73, 100/167, \ldots$. We see that the n-th convergent is just a'_{n+1}/a_{n+1}. This explains the formula

$$n!a'_n = \underline{A000262}(n) - |\underline{A201203}(n-2)| \text{ for } n \geqslant 2.$$

Also, Theorem 5 implies that $G - a'_n/a_n = b_n/a_n \sim -2\pi e^{1-4\sqrt{n}}$ as $n \to \infty$.

3 Connection with hypergeometric functions

The numbers a_n and b_n may be expressed in terms of confluent hypergeometric functions (Kummer functions), for which we refer to [10, §13.2]. If $M(a,b,z)$ and $U(a,b,z)$ are standard solutions $w(a,b,z)$ of Kummer's differential equation $zw'' + (b-z)w' - aw = 0$, then Lemmas 1–2 below express a_n and b_n in terms of $M(n+1,2,1)$ and $U(n,0,1)$.

Kummer [8] considered

$$M(a,b,z) = {}_1F_1(a;b;z) = \sum_{k\geqslant 0} \frac{(a)_k\, z^k}{(b)_k\, k!}\,, \tag{5}$$

which is undefined if b is zero or a negative integer. In the case $a \neq b = 0$, we can use the solution

$$zM(a+1,2,z) = \lim_{b\to 0} \frac{b}{a} M(a,b,z).$$

Tricomi [16] introduced the function $U(a,b,z)$ as a second (minimal) solution of Kummer's differential equation. For our purposes it is convenient to use the integral representation [10, (13.4.4)] (valid for $\Re(a) > 0$, $\Re(z) > 0$)

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}\, t^{a-1}\, (1+t)^{b-a-1}\, dt. \tag{6}$$

We remark that the functions M and U satisfy recurrence relations, known as "connection formulas". For example, we mention [10, (13.3.1) and (13.3.7)], both (essentially) due to Gauss (see Erdélyi [4, §6.4 and §6.6]):

$$(b-a)M(a-1,b,z) + (2a-b+z)M(a,b,z) - aM(a+1,b,z) = 0, \tag{7}$$

$$U(a-1,b,z) + (b-2a-z)U(a,b,z) + a(a-b+1)U(a+1,b,z) = 0. \tag{8}$$

Lemma 1. *If $n \in \mathbb{Z}$, $n \geqslant 1$, and a_n is as above, then*

$$a_n = e^{-1} M(n+1, 2, 1). \tag{9}$$

Proof. If we put $a = n+1$, $b = 2$, and $z = 1$ in the connection formula (7), we see that $\widetilde{a_n} := e^{-1} M(n+1, 2, 1)$ satisfies the same recurrence (2) as a_n. Thus, to show that $a_n = \widetilde{a_n}$ for all $n \geqslant 1$, it is sufficient to show that $a_n = \widetilde{a_n}$ for $n \in \{1, 2\}$. Now

$$\widetilde{a_1} = e^{-1} M(2, 2, 1) = e^{-1} \sum_{k \geqslant 0} \frac{(2)_k}{(2)_k \, k!} = 1 = a_1,$$

and, similarly,

$$\widetilde{a_2} = e^{-1} M(3, 2, 1) = e^{-1} \sum_{k \geqslant 0} \frac{(3)_k}{(2)_k \, k!} = e^{-1} \sum_{k \geqslant 0} \frac{k+2}{2 \, k!} = 3/2 = a_2,$$

so the result follows. □

Lemma 2. *If $n \in \mathbb{Z}$, $n \geqslant 1$, and b_n is as above, then*

$$b_n = -\Gamma(n) \, U(n, 0, 1). \tag{10}$$

Proof. We start with [10, (6.7.1)]:

$$I(a, b) := \int_0^\infty \frac{e^{-at}}{t+b} \, dt = e^{ab} E_1(ab), \quad a, b > 0.$$

Note that, by definition, $b_n = [z^n] I(1, 1/(1-z))$. Setting $a = 1$, $b = 1/(1-z)$, the term $1/(t+b)$ inside the integral can be rearranged as follows:

$$\left(t + \frac{1}{1-z}\right)^{-1} = \frac{1-z}{1+t-tz} = \frac{1}{1+t} - \frac{1}{t(1+t)} \left(\frac{1}{1 - zt/(1+t)} - 1\right),$$

and making the substitution $s = t/(1+t)$ gives

$$I(1, 1/(1-z)) = \int_0^\infty \frac{e^{-t}}{1+t} \, dt - \int_0^1 e^{-s/(1-s)} \left(\frac{z}{1-zs}\right) ds = \sum_{n \geqslant 0} b_n \, z^n.$$

Thus, $b_0 = e E_1(1)$ and, for $n > 0$,

$$b_n = -\int_0^1 e^{-s/(1-s)} \, s^{n-1} \, ds. \tag{11}$$

Writing $e^{-s/(1-s)} = e^{1-1/(1-s)}$ gives, for $n > 0$,

$$b_n = -e \int_0^1 e^{-1/(1-s)}\, s^{n-1}\, ds. \tag{12}$$

Substitute $t = s/(1-s)$ in (6), giving

$$\Gamma(a) U(a, b, z) = e^z \int_0^1 e^{-z/(1-s)}\, s^{a-1}\, (1-s)^{-b}\, ds. \tag{13}$$

Comparison of (12) and (13) now gives $b_n = -\Gamma(n)\, U(n, 0, 1)$. □

Remark 3. We could prove Lemma 2 in the same manner as Lemma 1, using the connection formula (8) instead of (7), and the recurrence (4) instead of (2), but in order to verify the initial conditions we would have to resort to some explicit representation for U, such as the integral representation (6), so the proof would be no simpler.

Remark 4. We can generalise our definitions of a_n and b_n to permit $n \in \mathbb{C}$, using Lemmas 1–2. Such generalizations do not seem particularly useful, so in what follows we continue to assume that $n \in \mathbb{Z}$.

4 Asymptotic expansions of a_n and b_n

Theorem 5 gives the complete asymptotic expansions of a_n and b_n in ascending powers of $n^{-1/2}$. Wright [19] proved the existence of an asymptotic expansion of the form (14) for a_n, but did not state an explicit formula or algorithm for computing the constants c_m occurring in the expansion.

Theorem 5. *For positive integer n, if a_n and b_n are as above, then*

$$a_n \sim \frac{e^{2\sqrt{n}}}{2n^{3/4}\sqrt{\pi e}} \sum_{m \geqslant 0} c_m n^{-m/2} \tag{14}$$

and

$$b_n \sim -\frac{\sqrt{\pi e}}{n^{3/4} e^{2\sqrt{n}}} \sum_{m \geqslant 0} (-1)^m c_m n^{-m/2}, \tag{15}$$

where

$$c_m = (-1)^m \sum_{j=0}^{m} [h^{m-j}] \exp(\mu(h))\, \frac{(m-2j+3/2)_{2j}}{4^j j!} \tag{16}$$

and

$$\mu(h) = h^{-1} - (e^h - 1)^{-1} - \tfrac{1}{2}\,. \tag{17}$$

Remark 6. The function $\mu(h)$ defined by (17) could also be defined using Bernoulli numbers, since

$$\mu(h) = -\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} h^{2k-1} = -\frac{h}{12} + \frac{h^3}{720} - \cdots\,, \tag{18}$$

The function $\exp(\mu(h))$ occurring in (16) has the Maclaurin expansion

$$\exp(\mu(h)) = 1 - \frac{h}{12} + \frac{h^2}{288} + \frac{67h^3}{51840} + \cdots\,. \tag{19}$$

The numerators and denominators of the coefficients $[h^n]\exp(\mu(h))$ do not appear to be in the OEIS.

Proof of Thm. 5*.* We first prove (15). From Lemma 2, $b_n = -\Gamma(n)\,U(n,0,1)$. Temme [15, Sec. 3] gives a general asymptotic result for $U(a,b,z^2)$ as $a \to \infty$. We state Temme's result for the case $(a,b,z) = (n,0,1)$, which is what we need. Let $c'_k := [h^k]\exp(\mu(h))$. (Temme uses c_k, but this conflicts with our notation.) From Temme [15, (3.8)–(3.10)], we have

$$U(n,0,1) \sim \frac{\sqrt{e}}{\Gamma(n)} \sum_{k \geqslant 0} c'_k \Phi_k(n), \tag{20}$$

where

$$\Phi_k(n) = 2n^{-(k+1)/2} K_{k+1}(2n^{1/2}),$$

and K_ν denotes the usual modified Bessel function.

From [10, (10.40.2)], $K_\nu(z)$ has an asymptotic expansion

$$K_\nu(z) \sim e^{-z}\sqrt{\frac{\pi}{2z}} \sum_{j \geqslant 0} \frac{(\nu - j + 1/2)_{2j}}{j!\,(2z)^j}\,. \tag{21}$$

Setting $\nu = k$ and $z = 2n^{1/2}$ in (21), we obtain

$$\Phi_{k-1}(n) = 2n^{-k/2} K_k(2n^{1/2}) \sim \frac{\sqrt{\pi}e^{-2\sqrt{n}}}{n^{1/4}} \sum_{j \geqslant 0} \frac{(k - j + 1/2)_{2j}}{j!\,4^j\, n^{(j+k)/2}}\,.$$

Substituting this expression into (20), and grouping like powers of n, we obtain

$$b_n = -\Gamma(n)\, U(n,0,1) \sim -\frac{\sqrt{\pi e}}{n^{3/4}e^{2\sqrt{n}}} \sum_{m\geqslant 0}\sum_{j=0}^{m} \frac{c'_{m-j}\,(m-2j+3/2)_{2j}}{j!\,4^j\,n^{m/2}}\,.$$

Now, comparison with (15) shows that

$$(-1)^m c_m = \sum_{j=0}^{m} \frac{c'_{m-j}\,(m-2j+3/2)_{2j}}{j!\,4^j}\,,$$

which completes the proof of (15).

The proof of (14) is similar. We use Lemma 1 instead of Lemma 2, and Temme's asymptotic result [15, (3.29)] for $M(a,b,z^2)$ as $a \to \infty$ instead of (20); the modified Bessel function I_ν replaces K_ν. From [10, (10.40.1)], $I_\nu(z)$ has an asymptotic expansion

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{j\geqslant 0} (-1)^j\, \frac{(\nu - j + 1/2)_{2j}}{j!\,(2z)^j}\,, \tag{22}$$

which replaces (21). □

Theorem 5 gives an expression for c_m which (indirectly) involves Bernoulli numbers, in view of (18). Lemma 7 gives a different expression for c_m that is recursive, as the expression for c_m depends on the values of c_k for $k < m$, but has the advantage of avoiding reference to Bernoulli numbers. The idea of the proof is similar to that used in the "method of Frobenius" [6].

Lemma 7. *We have* $c_0 = 1$ *and, for all* $m \geqslant 1$,

$$mc_m = [h^{m+3}] \sum_{j=0}^{m-1} c_j h^j \sum_{s\in\{\pm 1\}} (1+sh^2)^{\frac{1-2j}{4}} \exp\left(\frac{2}{h}\left((1+sh^2)^{\frac{1}{2}} - 1\right)\right). \tag{23}$$

Proof. We substitute the expression (14) for a_n into the recurrence (2), and show that this determines the constants c_m uniquely (subject to $c_0 = 1$). It is convenient to define $h := n^{-1/2}$. Thus $h \to 0$ as $n \to \infty$. Write (2) in the symmetric form

$$(n+1)a_{n+1} + (n-1)a_{n-1} = (2n+1)a_n \;\text{ for }\; n \geqslant 1. \tag{24}$$

Now substitute the expression (14) into (24), divide both sides by $n^{1/4}e^{2\sqrt{n}}$, and replace $\sqrt{n}$ by h^{-1}. As we consider formal Laurent series in h, questions of convergence are irrelevant. We find that

$$\begin{aligned}(1+h^2)^{1/4}\exp(2h^{-1}((1+h^2)^{1/2}-1))\times&\\ (1+c_1h(1+h^2)^{-1/2}+c_2h^2(1+h^2)^{-2/2}+\cdots)&\\ +(1-h^2)^{1/4}\exp(2h^{-1}((1-h^2)^{1/2}-1))\times&\\ (1+c_1h(1-h^2)^{-1/2}+c_2h^2(1-h^2)^{-2/2}+\cdots)&\\ =(2+h^2)(1+c_1h+c_2h^2+\cdots),& \qquad (25)\end{aligned}$$

where each side of (25) is to be regarded as a formal Laurent series in h. In fact, since $2h^{-1}((1\pm h^2)^{1/2}-1) = \pm h + O(h^3)$, there are no negative powers of h, and both sides of (25) are formal power series in h.

A slight reorganisation of (25) gives

$$\begin{aligned}\exp(2h^{-1}((1+h^2)^{1/2}-1))\times&\\ ((1+h^2)^{1/4}+c_1h(1+h^2)^{-1/4}+c_2h^2(1+h^2)^{-3/4}+\cdots)&\\ +\exp(2h^{-1}((1-h^2)^{1/2}-1))\times&\\ ((1-h^2)^{1/4}+c_1h(1-h^2)^{-1/4}+c_2h^2(1-h^2)^{-3/4}+\cdots)&\\ -(2+h^2)(1+c_1h+c_2h^2+\cdots)=0.& \qquad (26)\end{aligned}$$

In (26), the terms on the left-hand-side not involving c_m for $m \geqslant 1$ are $-5h^4/48 + O(h^5)$, as the lower-order terms cancel. The terms involving c_1 are $-c_1h^4 + O(h^5)$, and the terms involving c_m for $m \geqslant 2$ are $O(h^5)$. Thus, to ensure that the left side of (26) is $O(h^5)$, it is necessary and sufficient that $c_1 = -5/48$.

Continuing in this way, suppose that, for some $m \geqslant 1$, we have determined $c_1, \ldots, c_{m-1}$ such that the left side of (26) is $O(h^{m+3})$. A small computation shows that the terms involving c_m are $-mc_mh^{m+3} + O(h^{m+4})$, and the terms involving $c_{m+1}, c_{m+2}, \ldots$ are $O(h^{m+4})$. Thus, for a proof by induction on m, it is sufficient to choose c_m such that

$$mc_m = [h^{m+3}]A(c_1, \ldots, c_{m-1}, 0, 0, \ldots),$$

where $A(c_1, c_2, \ldots)$ is the left side of (26), regarded as a function of the parameters c_1, c_2, etc.

Finally, we note that the expression $(2+h^2)(1+c_1h+\ldots+c_{m-1}h^{m-1})$ that occurs in (26) is a polynomial in h of degree (at most) $m+1$, so may be omitted without changing $[h^{m+3}]A$. $\square$

Remark 8. With small modifications, we could use (15) instead of (14) in the proof of Lemma 7. This would provide an independent proof that the constants c_m occurring in the asymptotic expansions of a_n and b_n are the same, apart from changes of sign.

Remark 9. Computation using (16) and, as a check, (23), gives

$$(c_k)_{k\geqslant 0} = \left(1, -\frac{5}{48}, -\frac{479}{4608}, -\frac{15313}{3317760}, \frac{710401}{127401984}, -\frac{3532731539}{214035333120}, \ldots\right).$$

The numerators and denominators do not appear to be in the OEIS. With the exception of c_0 and c_4, the c_k all appear to be negative. This has been verified numerically for $k \leqslant 1000$.

5 The Hadamard product of f_0 and f_1

Define $\rho_n := a_n b_n$. Thus $\sum_{n=0}^{\infty} \rho_n z^n$ is the Hadamard product $(f_0 \odot f_1)(z)$. From Lemmas 1–2,we have

$$\rho_n = -e^{-1}\Gamma(n)M(n+1, 2, 1)U(n, 0, 1).$$

Using Theorem 5, we can obtain a complete asymptotic expansion for ρ_n in decreasing powers of n. This is given in Corollary 10.

Corollary 10. *We have*

$$\rho_n \sim -\frac{1}{2n^{3/2}} \sum_{k\geqslant 0} d_k n^{-k},$$

where

$$d_k = \sum_{j=0}^{2k} (-1)^j c_j c_{2k-j},$$

and $c_0, \ldots, c_{2k}$ *are as in Lemma 7.*

A computation shows that

$$(d_k)_{k\geqslant 0} = (1, -7/32, 43/2048, -915/65536, \ldots).$$

We observe that the d_k appear to be dyadic rationals More precisely, it appears that $2^{6k} d_k \in \mathbb{Z}$. Define a scaled sequence $(r_k)_{k\geqslant 0}$ by $r_k := 2^{6k} d_k$. Computation gives

$$(r_k)_{k\geqslant 0} = (1, -14, 86, -3660, -1042202, -247948260, -108448540420, \ldots).$$

This leads naturally to the following conjecture.

Conjecture 11. For all $k \geqslant 0$, $r_k \in \mathbb{Z}$.

Remark 12. Conjecture 11 has been verified for all $k \leqslant 1000$. We also showed numerically, for $3 \leqslant k \leqslant 1000$, that $r_k < 0$ and $r_k \equiv \binom{2k}{k} \pmod{32}$.

Remark 13. A problem that is superficially similar to our conjecture was solved by Tulyakov [17]. However, we do not see how to adapt his method to prove our conjecture.

Remark 14. Corollary 10 is reminiscent of the result

$$I_0(x)K_0(x) \sim \frac{1}{2x}\sum_{k\geqslant 0} e_{k,0}\, x^{-2k}$$

in the theory of Bessel functions [3, (1.2)]. The coefficients $e_{k,0}$ are given by

$$e_{k,0} = \frac{(2k)!^3}{2^{6k}k!^4}\,,$$

so $2^{4k}e_{k,0} \in \mathbb{Z}$. The modified Bessel functions $I_0(x)$ and $K_0(x)$ are solutions of the same ordinary differential equation $xy'' + y' - xy = 0$, but $I_0(x)$ increases with x while $K_0(x)$ decreases. This is analogous to the behaviour of a_n, which increases as $n \to \infty$, and $|b_n|$, which decreases as $n \to \infty$.

More generally, from [10, 10.40.6], we have

$$I_\nu(x)K_\nu(x) \sim \frac{1}{2x}\sum_{k\geqslant 0} e_{k,\nu} x^{-2k},$$

where

$$e_{k,\nu} = (-1)^k 2^{-2k}(\nu - k + 1/2)_{2k}\binom{2k}{k},$$

and $2^{4k}e_{k,\nu} \in \mathbb{Z}$ for $\nu \in \mathbb{Z}$.

6 Other expressions for d_n

Since (a_n) and (b_n) are D-finite, it follows that (ρ_n) is D-finite.[3] In fact, ρ_n satisfies the 4-term recurrence

$$\begin{aligned} n^2(n-1)(2n-3)\rho_n = \; & (n-1)(2n-1)(3n^2-5n+1)\rho_{n-1} \\ & - (n-2)(2n-3)(3n^2-5n+1)\rho_{n-2} \\ & + (n-2)(n-3)^2(2n-1)\rho_{n-3} \end{aligned} \tag{27}$$

[3] See Flajolet and Sedgewick [5, Appendix B.4], and Stanley [13, Theorem 2.10], for relevant background on D-finite sequences.

for $n \geqslant 3$, with initial conditions $\rho_0 = G$, $\rho_1 = G - 1$, $\rho_2 = (9G - 6)/4$.

The recurrence (27) can be simplified by defining $\sigma_n := n\rho_n$. Then σ_n satisfies the recurrence

$$n(n-1)(2n-3)\sigma_n = (2n-1)(3n^2-5n+1)\sigma_{n-1} \\ - (2n-3)(3n^2-5n+1)\sigma_{n-2} + (n-2)(n-3)(2n-1)\sigma_{n-3} \quad (28)$$

for $n \geqslant 3$, with initial conditions $\sigma_0 = 0$, $\sigma_1 = G - 1$, $\sigma_2 = 9G/2 - 3$. Also, Corollary 10 gives an asymptotic series for σ_n:

$$\sigma_n \sim -\frac{1}{2n^{1/2}} \sum_{k \geqslant 0} d_k n^{-k}. \quad (29)$$

Using (28), we can give a recursive algorithm for computing the sequence (d_n) (and hence (r_n)) directly, without computing the sequence (c_n).

Lemma 15. *We have $d_0 = 1$ and, for all $k \geqslant 1$,*

$$8kd_k = -[h^{k+2}] \left(\sum_{j=0}^{k-1} d_j h^j \Big(B(h)(1-h)^{-(j+1/2)} \\ + C(h)(1-2h)^{-(j+1/2)} + D(h)(1-3h)^{-(j+1/2)} \Big) \right), \quad (30)$$

where

$$\begin{aligned} B(h) &= -6 + 13h - 7h^2 + h^3 &&= -(2-h)(3-5h+h^2), \\ C(h) &= +6 - 19h + 17h^2 - 3h^3 &&= (2-3h)(3-5h+h^2), \quad \textit{and} \\ D(h) &= -2 + 11h - 17h^2 + 6h^3 &&= -(1-2h)(1-3h)(2-h). \end{aligned}$$

Proof. Define $h := n^{-1}$, so $h \to 0$ as $n \to \infty$. From Corollary 10, there exists an asymptotic series of the form

$$-2\sigma_n \sim \sum_{j \geqslant 0} d_j n^{-j-1/2}$$

as $n \to \infty$. Moreover, $d_0 = 1$. Define $A(h) := (1-h)(2-3h)$ in addition to $B(h)$, $C(h)$ and $D(h)$. Using the recurrence (28) and the elementary identity

$1/(n-m) = h/(1-mh)$ for $m \in \{0,1,2,3\}$, we have

$$\sum_{j\geqslant 0} d_j \Bigg(A(h)h^{j+1/2} + B(h)\left(\frac{h}{1-h}\right)^{j+1/2} + C(h)\left(\frac{h}{1-2h}\right)^{j+1/2} + D(h)\left(\frac{h}{1-3h}\right)^{j+1/2}\Bigg) \sim 0.$$

Now, dividing both sides by $h^{1/2}$, we obtain

$$\sum_{j\geqslant 0} d_j h^j \Big(A(h) + B(h)(1-h)^{-(j+1/2)} + C(h)(1-2h)^{-(j+1/2)} + D(h)(1-3h)^{-(j+1/2)}\Big) \sim 0. \qquad (31)$$

An easy computation shows that

$$\begin{aligned} A(h)+B(h)+C(h)+D(h) &= -4h^2 + O(h^3),\\ B(h)+2C(h)+3D(h) &= 8h + O(h^2), \quad \text{and}\\ B(h)+2^2C(h)+3^2D(h) &= O(h). \end{aligned}$$

Thus, for all $j \geqslant 1$, the terms involving d_j in (31) are $8jh^{j+2} + O(h^{j+3})$. (The "$8j$" arises from $-4 + 8(j+1/2) = 8j$.) This shows that the choice of d_k in (30) is necessary and sufficient to give an asymptotic series of the required form. Finally, we note that $[h^{k+2-j}]A(h) = 0$, since $j \leqslant k-1$ and $\deg(A(h)) = 2$. Thus, a term involving $A(h)$ has been omitted from (30). □

Using Lemma 15, we computed the sequences (d_n) and (r_n) for $n \leqslant 1000$, and verified the values previously computed (more slowly) via Corollary 10.

Since the power series occurring in (30) have a simple form, we can extract the coefficients of the required powers of h to obtain a recurrence for the d_k, as in Corollary 16. This gives a third way to compute the sequence (d_n).

Corollary 16. *We have $d_0 = 1$ and, for all $k \geqslant 1$,*

$$8k\, d_k = \sum_{j=0}^{k-1} \alpha_{j,k}\, d_j.$$

Here the $\alpha_{j,k}$ are defined by

$$\begin{aligned}\alpha_{j,k} = \; & (-1 + 3 \cdot 2^{m-1} - 2 \cdot 3^m)(\tau)_{m-1}/(m-1)! \\ & + (7 - 17 \cdot 2^m + 17 \cdot 3^m)(\tau)_m/m! \\ & + (-13 + 38 \cdot 2^m - 33 \cdot 3^m)(\tau)_{m+1}/(m+1)! \\ & + 6(1 - 4 \cdot 2^m + 3 \cdot 3^m)(\tau)_{m+2}/(m+2)!, \end{aligned} \tag{32}$$

where $m := k - j$ and $\tau := j + 1/2$.

Proof (sketch). To prove Corollary 16, we apply the binomial theorem to the power series in (30), multiply by the polynomials $B(h)$, $C(h)$, and $D(h)$, and extract the coefficient of h^{k+2-j}. □

The following corollary is an easy deduction from Corollary 16, and gives an explicit recurrence for $r_k = 2^{6k} d_k$.

Corollary 17. *We have $r_0 = 1$ and, for all $k \geqslant 1$,*

$$k\, r_k = \sum_{j=0}^{k-1} \beta_{j,k}\, r_j, \quad \textit{where} \;\; \beta_{j,k} = 8^{2k-2j-1}\, \alpha_{j,k}\,.$$

Although we have not proved Conjecture 11, the following result goes part of the way.

Theorem 18. *For all $k \geqslant 0$, we have $k!\, r_k \in \mathbb{Z}$.*

Proof. Let $R_k := k! r_k$. We show that $R_k \in \mathbb{Z}$. From Corollary 17, $R_0 = 1$ and, for $k \geqslant 1$, R_k satisfies the recurrence

$$R_k = \sum_{j=0}^{k-1} \beta_{j,k}\, R_j\, \frac{(k-1)!}{j!}\,. \tag{33}$$

The ratio of factorials in (33) is an integer, since $j \leqslant k-1$. Thus, in order to prove the result by induction on k, it is sufficient to show that $\beta_{j,k} \in \mathbb{Z}$. Now, elementary number theory shows that $4^\ell (j + 1/2)_\ell / \ell! \in \mathbb{Z}$ for all $j, \ell \geqslant 0$. Thus, the expressions of the form $(\tau)_{m+\delta}/(m+\delta)!$ in (32) are in $\mathbb{Z}$ provided that $m + \delta \geqslant 0$. This is true as $m \geqslant k - j \geqslant 1$ and $\delta \geqslant -1$. To show that $\beta_{j,k} \in \mathbb{Z}$, it is sufficient to have $8^{2m-1} \geqslant 4^{m+2}$, which holds for all $m \geqslant 2$. In the case $m = 1$, it is easy to see that all the terms in (32) are in $\mathbb{Z}/4$, so $\beta_{m-1,k} = 8\alpha_{m-1,k} \in \mathbb{Z}$. Thus, $\beta_{j,k} \in \mathbb{Z}$ for $0 \leqslant j < k$, and the result follows by induction on k. □

Remark 19. The proof actually shows that $\beta_{j,k} \in 2\mathbb{Z}$, which implies that $R_k \in 2\mathbb{Z}$ for all $k > 0$.

7 Acknowledgements

We thank Bruno Salvy for communicating his conjecture to us. The first author was supported in part by ARC grant DP140101417.

References

[1] A. I. Aptekarev, On linear forms containing the Euler constant, arXiv:0902.1768v2, 28 Feb. 2009.

[2] P. Bala, Continued fraction for $1 - G$, entry in OEIS A073003, Oct. 11, 2012.

[3] R. P. Brent and F. Johannson, A bound for the error term in the Brent-McMillan algorithm, *Math. Comp.* **84** (2015), 2351–2359.

[4] A. Erdélyi (editor), *Higher Transcendental Functions*, Vol. 1, California Institute of Technology Bateman Manuscript Project, McGraw-Hill, 1953.

[5] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2009.

[6] G. Frobenius, Ueber die integration der linearen Differentialgleichungen durch Reihen, *J. reine angew. Math.* **76** (1873), 214–235.

[7] W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Review* **9** (1967), 24–82.

[8] E. E. Kummer, De integralibus quibusdam definitis et seriebus infinitis, *J. reine angew. Math.* **17** (1837), 228-242.

[9] J. C. Lagarias, Euler's constant: Euler's work and modern developments, *Bull. Amer. Math. Soc.* **50** (2013), 527–628.

[10] National Institute of Standards and Technology, *NIST Digital Library of Mathematical Functions*, `http://dlmf.nist.gov/`, version 1.0.19, 22 June 2018.

[11] O. Perron, Über das infinitäre Verhalten der Koeffizienten einer gewissen Potenzreihe, *Archiv d. Math. u. Phys.* (3), **22** (1914), 329–340.

[12] T. Rivoal, On the arithmetical nature of the values of the Gamma function, Euler's constant, and Gompertz's constant, *Michigan Math. J.* **61** (2012), 239–254.

[13] R. P. Stanley, Differentiably finite power series, *Europ. J. Combinatorics* **1** (1980), 175–188.

[14] N. M. Temme, Numerical evaluation of functions arising from transformations of formal series, *J. Math. Anal. Appl.* **51** (1975), 678–694.

[15] N. M. Temme, Remarks on Slater's asymptotic expansions of Kummer functions for large values of the a-parameter, *Adv. Dyn. Syst. Appl.* **8** (2013), 365–377.

[16] F. G. Tricomi, Funzioni ipergeometriche confluenti, *Edizioni Cremonese*, Rome, 1954.

[17] D. N. Tulyakov, A system of recurrence relations for rational approximations of the Euler constant, *Mat. Zametki* **85** (2009), 782–787 (in Russian). English translation in *Math. Notes* **85** (2009), 746–750.

[18] M. Wallner, A bijection of plane increasing trees with relaxed binary trees of right height at most one, *Theoret. Comput. Sci.*, to appear. Also arXiv:1706.07163v2, 11 July 2018.

[19] E. M. Wright, The coefficients of a certain power series, *J. Lond. Math. Soc.* **7** (1932), 256–262.

2010 *Mathematics Subject Classification*: Primary 34E05; Secondary 11Y55, 33C10, 33C15, 33F99.

Keywords: asymptotics, confluent hypergeometric function, D-finite, Euler-Gompertz constant, exponential integral, Hadamard product, holonomic, Kummer function, modified Bessel function, stretched exponential.